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Mathematical Modelling in Thermomechanics of Elastic Shells by Iterative-Moment Approach

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General approaches to mathematical modelling of thermomechanical processes in thin-walled elements of plate and shell type in a three-dimensional approximation are discussed. The energy balance equation, formulated in a three-dimensional statement by Lagrange approach on the extended space of parameters of local state and a corresponding choice of the function of local situation is the basis for the construction of governing equations. In this connection the mechanic energy flow, in a general case, is given by a sum of additive components in which apart of traditional characteristics (stress tensor, velocity vector) the higher order characteristics of gradientality of deformation and inertial motion and corresponding to them tensor characteristics of force action are introduced iteratively. The set of governing equations enabling to account the effects of locally-gradient and high-speed deformations was constructed for determination of the phase space of parameters of local situation. The transition to a two-dimensional analogue in governing equations is realized by the averaged characteristics of the stressed-strained state using presentation of the sought-for values by means of the expansion by a tensor base.

Key words: elastic shells, thermomechanical processes, locally-gradient and high speed deformation, averaged tensor characteristics, optimal decomposition base.

Introduction. Thin-walled elements of constructions of plate and shell types are widely used in the engineering practice. It causes a great interest and the necessity of further development and improvement of calculational models describing their mechanical behavior. It was quite natural that Professor Yaroslav Burak paid attention to investigations in this direction in his many-sided scientific activity [1].

It is known that classical postulates of Kirchhoff -Love and Tymoshenko not always sufficiently described the stressed-strained state of this type of elements of construction. The development of calculational schemes in linear and nonlinear statements is shown in [2, 3]. The analysis of the main approaches to the construction of the solutions of the boundary value problems of the theory of shells using numerical-analytical methods in the classical and refined statements is proposed in [4]. The stable method of discrete orthogonalization and an effective algorithm of its realization enabling to obtain solutions with high accuracy were taken as the basis.

As the transition from the three-dimensional statement of the problem to its two-dimensional analogue is the key problem for mathematical modelling, the method of expansion of the sought-for values by the given system of basic functions is widely used.

Though, as it was noted in the work [5], the convergence of solutions for boundary value problems for a small number of iterations is not always sufficient. On the other hand, the model presentation in the shell thermomechanical behavior study which accounts the effects of high-speed and locally-gradient deformation deserves extension. It is of special importance for pulse loadings and for materials with the clearly expressed microstructure that were investigated from other points of view in the works [6-8].

The idea of energetic approach to the construction of iterative models of the locally-gradient dynamic thermoelasticity which on a thermodynamic level describes the local gradientality and inertia of a deformational form of motion was proposed by Ya. Burak and realized in [9, 10].

In this work the approach is used for the construction of initial equations for thermoelastic shells in a three-dimensional approximation. The equations in generalized variables using the expansion of the sought-for values by the tensor basis are written for the shells of variable thickness. The method of choice of the optimal base of expansion corresponding to the boundary-value problem in the three-dimensional statement is proposed.

1. Formulation of the problem

Consider a shell with a variable thickness $2h(\alpha^1,\alpha^2)=\overset{(+)}{h}(\alpha^1,\alpha^2)+\overset{(-)}{h}(\alpha^1,\alpha^2)$ referred to a mixed orthogonal coordinate system α^1,α^2,γ, where the coordinate lines α^1,α^2 are the lines of the principal curvatures of the median (basic) surface (Σ_0); γ is a coordinate in the direction normal to (Σ_0) and $\gamma=\overset{(+)}{h}(\alpha^1,\alpha^2)$, $\gamma=-\overset{(-)}{h}(\alpha^1,\alpha^2)$ — shell face smooth surfaces. The shell is considered as a three-dimensional solid, which at the initial moment of time $\tau=\tau_o$ corresponds to the Euclidian space domain X_*. The law of the random point motion is given in the form

$$\vec{r}(\alpha_1,\alpha_2,\gamma,\tau)=\vec{r}_o+\vec{r}_{o\gamma}+\vec{u}\,, \qquad (1)$$

where $\vec{r}_o=\vec{r}_o(\alpha_1,\alpha_2,\tau)$ — radius-vector of the points of the median surface; $\vec{r}_{o\gamma}=\gamma\,\vec{э}^{\gamma}_o$, $\vec{э}^{\gamma}_o$ — basic orth in the direction to the normal to the median surface; $\vec{u}$ — vector displacement. It'll be assumed that vectors $\vec{r}$, $\vec{r}_o$, $\vec{u}$ are dimensionless i. e. normalized by some characteristic dimension and τ — normalized by some characteristic time.

Assume the full energy balance equation for the element of the shell $X\subset X_*$, built on the basis of the arbitrary derived area $\partial X^c(\tau)$ of its median surface to be an initial one

$$\frac{d}{d\tau}\int\limits_{X(\tau)}E\,dV=-\int\limits_{\partial X(\tau)}\vec{n}\cdot\vec{J}_E\,d\Sigma+\int\limits_{X(\tau)}w_E\,dV\,. \qquad (2)$$

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Here $\vec{J}_E = T\vec{J}_s + \vec{J}_A$, $w_E = \vec{f} \cdot \vec{v} + w_Q$, E — full energy density, T — absolute temperature, $\vec{J}_s$ — entropy flow, $\vec{J}_A$ — mechanical energy flow, f — body force flow, $\vec{v} = \dfrac{\partial \vec{u}}{\partial \tau}$ — velocity vector, w_Q — heat source strength density, $\vec{n}$ — outer normal, $\partial X(\tau)$ — the surface of the derived area $X(\tau)$.

After the transition to the values $E_o = E\dfrac{dV}{dV_o}$, $w_E^o = w_E \dfrac{dV}{dV_o}$, $\vec{n}_o \cdot \vec{J}_E^o = \vec{n} \cdot \vec{J}_E \dfrac{d\Sigma}{d\Sigma_o}$ corresponding to the normalization by geometric characteristics of the initial configuration $X(\tau_o)$, with account of the dependence of $dV_o = (1+k_1\gamma)(1+k_2\gamma)\times \times d\gamma\, d\Sigma_o^c$ equation (2) can be written in the form

$$\frac{d}{d\tau} \int\limits_{\partial X_o^c(\tau_o)} \int\limits_{-\overset{(-)}{h}}^{\overset{(+)}{h}} E_o (1+k_1\gamma)(1+k_2\gamma) d\gamma\, d\Sigma_o^c =$$

$$= \int\limits_{\partial X_o^c(\tau_o)} \int\limits_{-\overset{(-)}{h}}^{\overset{(+)}{h}} \left[\vec{\nabla}_o \cdot \left(T\vec{J}_s^o + \vec{J}_A^o\right) + w_E^o\right](1+k_1\gamma)(1+k_2\gamma) d\gamma d\Sigma_o^c. \quad (3)$$

Here $\vec{\nabla}_o = \vec{\mathfrak{s}}_o^1 \dfrac{\partial}{\partial \alpha_1} + \vec{\mathfrak{s}}_o^2 \dfrac{\partial}{\partial \alpha_2} + \vec{\mathfrak{s}}_o^\gamma \dfrac{\partial}{\partial \gamma}$; k_1, k_2 — main curvatures of the shell, $d\Sigma_o^c$ — area of a small element of the median surface of the shell. Index «o» indicates that basic parameters are normalized relatively to the metric characteristics of the physically small subsystem at the initial moment of time.

$$\vec{J}_A = -\hat{\sigma}_o \cdot \frac{\partial \vec{u}}{\partial \tau} - \hat{\sigma}_o^{(i)} \cdot \frac{\partial \hat{e}_o^{(i-1)T}}{\partial \tau} - \hat{P}_{oj}^{(2)} \cdot \frac{\partial \vec{v}_{j-1}}{\partial \tau} - \hat{P}_{oj}^{(i)} \cdot \frac{\partial \hat{\varepsilon}_o^{(i-1)T}}{\partial \tau}. \quad (4)$$

In a general case we'll represent the mechanical energy flow $\vec{J}_A$ by a sum of the following additive components in which along with the traditional characteristics (stress tensor, velocity vector) higher order tensor characteristics of the strain process and inertial motion are introduced as well as the corresponding characteristics of the force action. Here $\hat{e}_o^{(i)} = \vec{\nabla}_o^{(i-1)} \otimes \vec{u}$, $\hat{\varepsilon}_o^{(i)} = \dfrac{\partial^j}{\partial \tau^j}\hat{e}_o^{(i)}$, $\vec{v}_j = \dfrac{\partial^j}{\partial \tau^j}\vec{v}$, $\vec{v}_o \equiv \vec{v}$, $\left(i = \overline{3,n};\quad j = \overline{1,m}\right)$; $\vec{\nabla}_o^{(i-1)}$ is $i-1$ multiple diad product of operators $\vec{\nabla}_o$; $\hat{\sigma}_o$ — Piola- Kirchhoff stress tensor; $\hat{P}_{oj}^{(2)}$ — stress tensor components that characterize material dynamic deformation; $\hat{P}_{oj}^{(i)}$ — impulses corresponding to the gradient character of the deformational motion.

By symbols « · » the operation of full scalar product and symbols «$\otimes$» the operation of tensor product are denoted. Indexes (i), $(i-1)$ refer to tensor functions valence. Indexes repetition denotes operation of summation, and «T» — transposed values.

With account of the fact that in this model only processes of heat conduction are dissipative the energetic relation (3) in a local form can be presented in the form

$$\frac{\partial \widetilde{L}_o}{\partial \tau} = \int_{\overset{(-)}{-h}}^{\overset{(+)}{h}} \left[T\frac{\partial s_o}{\partial \tau} - \vec{P}_o \cdot \frac{\partial \vec{v}}{\partial \tau} + \left(\hat{\sigma}_o + \vec{\nabla}_o \cdot \hat{\sigma}_o^{(3)}\right) \cdot \frac{\partial \hat{e}_o^T}{\partial \tau} + \left(\hat{\sigma}_o^{(i)} + \vec{\nabla}_o \cdot \hat{\sigma}_o^{(i+1)}\right) \cdot \right.$$
$$\cdot \frac{\partial \hat{e}_o^{(i)T}}{\partial \tau} + \left(\hat{P}_{o1}^{(2)} + \vec{\nabla}_o \cdot \hat{P}_{o1}^{(3)}\right) \cdot \frac{\partial \hat{\varepsilon}_o^T}{\partial \tau} + \left(\hat{P}_{oj}^{(i)} + \vec{\nabla}_o \cdot \hat{P}_{oj}^{(i+1)}\right) \cdot \frac{\partial \hat{\varepsilon}_{oj}^{(i)T}}{\partial \tau} +$$
$$\left. + \hat{\sigma}_o^{(n)} \cdot \frac{\partial \hat{e}_o^{(n)T}}{\partial \tau} + \hat{P}_{oj}^{(n)} \cdot \frac{\partial \hat{\varepsilon}_o^{(n)T}}{\partial \tau} \right] (1 + k_1 \gamma)(1 + k_2 \gamma) d\gamma \; , \tag{5}$$

where

$$\widetilde{L}_o = \int_{\overset{(-)}{-h}}^{\overset{(+)}{h}} L_o (1 + k_1 \gamma)(1 + k_2 \gamma) d\gamma, \; L_o = E_o - \vec{P}_o \cdot \vec{v} - \left(\vec{\nabla}_o \cdot \frac{\partial^j}{\partial \tau^j} \hat{P}_{(j+1)}^{(2)} \right) \cdot \frac{\partial \vec{v}}{\partial \tau},$$

$\vec{P}_o = \vec{P}_{o(o)} + \int_{\tau_o}^{\tau} \left\{ \vec{\nabla}_o \cdot \left[\hat{\sigma}_o + \hat{P}_{o1}^{(2)} + \frac{\partial^j}{\partial \tau^j} \hat{P}_{o(j+1)}^{(2)} \right] + \vec{f}_o \right\} d\xi$ — force impulse vector; $\vec{P}_{o(o)}$ — initial value of this vector; s_o — entropy; $\hat{\varepsilon}_o = \frac{\partial \hat{e}_o}{\partial \tau}$ — velocity deformation tensor.

It proceeds from (5) that under conditions of the potential description of the local situation the function $\widetilde{L}_o$ can be treated as a state function prescribed in the phase space of the state parameters s_o, $\vec{v}$, $\hat{e}_o$, $\left\{\hat{e}_o^{(i)}\right\}$, $\hat{\varepsilon}_o$, $\left\{\hat{\varepsilon}_o^{(i)}\right\}$.

Coupled parameters to basic ones are averaged respective by T, $\vec{P}_o$, $\left(\hat{\sigma}_o + \vec{\nabla}_o \cdot \hat{\sigma}_o^{(3)}\right)$, $\left(\hat{\sigma}_o^{(i)} + \vec{\nabla}_o \cdot \hat{\sigma}_o^{(i+1)}\right)$, $\left(\vec{P}_{o1} + \vec{\nabla}_o \cdot \vec{P}_{o1}^{(3)}\right)$, $\left(\vec{P}_o^{(i)} + \vec{\nabla}_o \cdot \vec{P}_{o1}^{(i+1)}\right)$ which will be treated as generalized forces the concretization of which will depend in the iterative model on the manner of prescription of the vector of the shell random point displacements.

We shall obtain the following equations of thermodynamic state

$$\vec{P}_o \equiv \vec{P}_{o(o)} + \int_{\tau_o}^{\tau} \left\{ \vec{\nabla}_o \cdot \left(\hat{\sigma}_o + \hat{P}_{o1}^{(2)} + \frac{\partial^j}{\partial \tau_j} \hat{P}_{o(j+1)}^{(2)} \right) + \vec{f}_o \right\} d\xi = -\frac{\partial L_o}{\partial \vec{v}} \equiv$$
$$\equiv \vec{P}_o\left(s_o, \vec{v}, \hat{e}_o, \left\{\hat{e}_o^{(i)}\right\}, \hat{\varepsilon}_o, \hat{\varepsilon}_{oj}^{(i)}\right), \; \hat{\sigma}_o + \vec{\nabla}_o \cdot \hat{\sigma}_o^{(3)} = \frac{\partial L_o}{\partial \hat{e}_o}, \; \hat{\sigma}_o^{(i)} + \vec{\nabla}_o \cdot \hat{\sigma}_o^{(i+1)} = \frac{\partial L_o}{\partial \hat{e}_o^{(i)}} \; ,$$

$$\hat{P}_{o1}^{(2)} + \vec{\nabla}_o \cdot \hat{P}_{o1}^{(3)} = \frac{\partial L_o}{\partial \hat{\varepsilon}_o},\ \hat{P}_{oj}^{(i)} + \vec{\nabla}_o \cdot \hat{P}_{oj}^{(i+1)} = \frac{\partial L_o}{\partial \hat{\varepsilon}_{oj}^{(i)}},\ \hat{\sigma}_o^{(n)} = \frac{\partial L_o}{\partial \hat{e}_o^{(n)}},\ \hat{P}_{oj}^{(m)} = \frac{\partial L_o}{\partial \hat{\varepsilon}_{oj}^{(m)}}. \tag{6}$$

It should be noted that the prescription of the mechanic energy flow in the form (4) enables to account in the model both the influence of effects of the high-speed deformation and the moment stresses in their interrelation. In the partial case for $n = 2$ only deformable form of motion is taken into account, in particular, for $m = 1$ its dependence on the velocity of deformation. For $m = 0$ we'll get the iterative model of the gradient theory of elasticity.

Putting $n = 2$, $m = 0$ in (4) we'll get a classical model of nonlinear thermoelasticity and the generalized equation of impulse conservation.

2. Equations of elastic shells in generalized variables

Thus, if one represents displacement vector $\vec{u}$ and the velocity vector $\vec{v}$ by the following invariant approximations

$$\vec{u}\left(\vec{r}_o + \vec{r}_{o\gamma}, \tau\right) = \hat{F}^{(l-1)}\left(\vec{r}_{o\gamma}\right) \cdot \hat{u}^{(l)}\left(\vec{r}_o, \tau\right),$$

$$\vec{v}\left(\vec{r}_o + \vec{r}_{o\gamma}, \tau\right) = \hat{F}^{(l-1)}\left(\vec{r}_{o\gamma}\right) \cdot \hat{v}^{(l)}\left(\vec{r}_o, \tau\right), \qquad \left(l = \overline{1, N}\right) \tag{7}$$

and restricts oneself by the first component only in the expression (4), then for conditions of isothermal deformation relation (5) will have the form

$$\frac{\partial \widetilde{L}_o}{\partial \tau} = -\frac{\partial \hat{v}^{(l)}}{\partial \tau} \cdot \hat{Q}_{o1}^{(l)} + \frac{\partial e_k^{(l+1)}}{\partial \tau} \cdot \hat{Q}_{2k}^{(l+1)} + \frac{\partial e_\gamma^{(l+1)}}{\partial \tau} \cdot \hat{Q}_{2\gamma}^{(l+1)},\ (k = 1,2). \tag{8}$$

Here

$$\hat{Q}_{o1}^{(l)} = \int_{\overset{(-)}{-h}}^{\overset{(+)}{h}} \vec{P}_o \otimes \hat{F}^{(l-1)} (1 + k_1\gamma)(1 + k_2\gamma) d\gamma,\ \hat{Q}_{2k}^{(l+1)} =$$

$$= \int_{\overset{(-)}{-h}}^{\overset{(+)}{h}} \hat{\sigma}_o \otimes \hat{F}^{(l-1)} (1 + k_1\gamma)(1 + k_2\gamma) d\gamma,$$

$$\hat{Q}_{2\gamma}^{(l+1)} = \int_{\overset{(-)}{-h}}^{\overset{(+)}{h}} \hat{\sigma}_o \otimes \frac{\partial \hat{F}^{(l-1)}}{\partial \gamma} (1 + k_1\gamma)(1 + k_2\gamma) d\gamma,$$

$$\hat{e}_k^{(l+1)} = \frac{\partial \hat{u}^{(l)}}{\partial \alpha^k} \otimes \vec{\ni}_k^o,\quad \hat{e}_\gamma^{(l+1)} = \hat{u}^{(l)} \otimes \vec{\ni}_\gamma^o .$$

A general form of the equations of thermodynamic state will be

$$\hat{Q}_{o1}^{(l)} = -\frac{\partial \widetilde{L}_o}{\partial \hat{v}^{(l)}} \equiv \hat{Q}_{o1}^{(l)}\left(\left\{\hat{v}^{(l)}\right\}, \left\{\hat{e}_k^{(l+1)T}\right\}, \left\{\hat{e}_\gamma^{(l+1)T}\right\}\right),$$

$$\hat{Q}_{2k}^{(l+1)} = \frac{\partial \widetilde{L}_o}{\partial \hat{e}_k^{(l+1)}} \equiv \hat{Q}_{2k}^{(l+1)}\left(\left\{\hat{v}^{(l)}\right\}, \left\{\hat{e}_k^{(l+1)T}\right\}, \left\{\hat{e}_\gamma^{(l+1)T}\right\}\right),$$

$$\hat{Q}_{2\gamma}^{(l+1)} = \frac{\partial \widetilde{L}_o}{\partial \hat{e}_\gamma^{(l+1)}} \equiv \hat{Q}_{2\gamma}^{(l+1)}\left(\left\{\hat{v}^{(l)}\right\}, \left\{\hat{e}_k^{(l+1)T}\right\}, \left\{\hat{e}_\gamma^{(l+1)T}\right\}\right). \quad (9)$$

In order to determine the unknown functions $\hat{u}^{(l)}(\vec{r}_o, \tau)$ it is necessary to concentrate the state function structure $\widetilde{L}_o$, to choose the system of basic functions $\hat{F}^{(l-1)}(\vec{r}_{o\gamma})$, and to formulate the initial boundary conditions, which correspond to the initial ones for three-dimensional statement of the problem.

In the partial case for a shell of constant thickness $2h_o$ the set of equations of motion will be a set of *N* tensor equations

$$\vec{\nabla}_{o\alpha} \cdot \hat{Q}_{2\alpha}^{(l+1)} - \vec{\ni}_\gamma^o \cdot \hat{Q}_{2\gamma}^{(l+1)} + \hat{\Phi}_o^{(l)} + \hat{\sigma}_{o\gamma}^+ \otimes \hat{F}_+^{(l-1)}(1 + k_1 h_o)(1 + k_2 h_o) -$$

$$- \hat{\sigma}_{o\gamma}^- \otimes \hat{F}_-^{(l-1)}(1 - k_1 h_o)(1 - k_2 h_o) = \frac{\partial \hat{Q}_{o1}^{(l)}}{\partial \tau}, \quad (10)$$

which within the accuracy of the approximate equation corresponds to the equation in the three-dimensional statement of the problem.
Here

$$\vec{\nabla}_{o\alpha} = \vec{\ni}_k^o \frac{\partial}{\partial \alpha_k}, \quad \vec{\nabla}_{o\gamma} = \vec{\ni}_\gamma^o \frac{\partial}{\partial \gamma}, \quad \vec{\sigma}_{o\gamma}^\pm = \vec{\ni}_\gamma^o \cdot \hat{\sigma}_o^\pm,$$

$$\hat{\Phi}_o^{(l)} = \int_{-h_o}^{h_o} \vec{f}_o \otimes \hat{F}^{(l-1)}(1 + k_1\gamma)(1 + k_2\gamma)d\gamma,$$

index «$\pm$» denotes boundary values of the corresponding observables for $\gamma = \pm h_o$.

3. Optimal basic functions

Consider a functional

$$\Pi[\hat{\sigma}, \vec{u}] = -\int_{(V)} \left[\frac{1}{2}\hat{\sigma} : (\hat{b} : \hat{\sigma} + 2\alpha\theta\hat{I}) + \vec{u} \cdot \left(\vec{\nabla} \cdot \hat{\sigma} + \vec{F}\right)\right] dV +$$

$$+ \int_{(\Sigma)} \vec{u} \cdot (\hat{\sigma} \cdot \vec{n} - \vec{p}_n) d\Sigma, \quad (11)$$

the steady-state conditions of which are described by equations of linear thermoelasticity

$$\vec{\nabla}\cdot\hat{\sigma}+\vec{F}=0 \text{ in } (V);\;\; \hat{\sigma}:\hat{b}+\alpha\theta\,\hat{I}-\frac{1}{2}\left(\vec{\nabla}\vec{u}+\vec{u}\vec{\nabla}\right)=0 \text{ in } (V);$$

$$\hat{\sigma}\cdot\vec{n}-\vec{p}_n=0 \text{ on } (\Sigma). \qquad (12)$$

Here $\theta=T-T_0$; $\vec{p}_n$ are surface forces; $\hat{\sigma}$ is the stress tensor; $\vec{u}$ is the displacement vector; $\hat{b}$ is the tensor of elastic material pliability; α is the coefficient of linear thermal expansion; $\hat{I}$ is the metric tensor; $\vec{n}$ is the unit vector directed normally outside the surface (Σ); $\vec{\nabla}$ is the Hamilton operator in an actual configuration; the dot and colon denote, respectively, scalar product and double scalar product; T_0 is the temperature of the shell in its natural state; (V) is the region occupied by the shell; (Σ) is the shell surface.

Let us present the components of stress tensor and displacement vector in the form

$$\sigma^{ij}\left(\alpha^1,\alpha^2,\gamma\right)=M_m^{(ij)}\left(\alpha^1,\alpha^2\right)\varphi_m^{(ij)}(\gamma), \qquad (13)$$

$$u_i\left(\alpha^1,\alpha^2,\gamma\right)=U_{(i)m}\left(\alpha^1,\alpha^2\right)\psi_{(i)m}(\gamma) \quad \left(i=\overline{1,3};\; m=\overline{0,N}\right). \qquad (14)$$

Here, according to the indices which repeat and are not taken in brackets, the summation holds. Both moments $M_m^{ij}\left(\alpha^1,\alpha^2\right)$, $U_{im}\left(\alpha^1,\alpha^2\right)$ and basic functions $\varphi_m^{ij}(\gamma)$, $\psi_{im}(\gamma)$ are taken as the sought values, which allows to obtain an adequate mathematical model with a small quantity of terms in expansions.

Making use of expansions (13), (14) and the formula of reintegration, the steady-state condition of functional (11)

$$\int\limits_{(V)}\left\{\left[\frac{1}{2}\left(\vec{\nabla}\vec{u}+\vec{u}\vec{\nabla}\right)-\hat{\sigma}:\hat{b}-\alpha\theta\hat{I}\right]:\delta\hat{\sigma}-\left(\vec{\nabla}\cdot\hat{\sigma}+\vec{F}\right)\cdot\delta\vec{u}\right\}dV+$$

$$+\int\limits_{(\Sigma)}\left(\hat{\sigma}\cdot\vec{n}-\vec{p}_n\right)\cdot\delta\vec{u}\,d\Sigma=0$$

is transformed into the form

$$\int\limits_{(\Sigma_0)}\int\limits_{\overset{(-)}{-h}}^{\overset{(+)}{h}}\left\{\left[\frac{1}{2}\left(\vec{\nabla}\vec{u}+\vec{u}\vec{\nabla}\right)-\hat{\sigma}:\hat{b}-\alpha\theta\hat{I}\right]:\left(M_m^{ij}\delta\varphi_m^{ij}+\varphi_m^{ij}\delta M_m^{ij}\right)\vec{э}_i\vec{э}_j-\right.$$

$$\left.-\left(\vec{\nabla}\cdot\hat{\sigma}+\vec{F}\right)\cdot\left(U_{im}\delta\psi_{im}+\psi_{im}\delta U_{im}\right)\vec{э}^i\right\}H_1\,d\gamma\,d\Sigma_0+$$

$$+\int_{(\Sigma_0)}\left[\Phi_\sigma(\delta U_{im},\delta\psi_{im})\Big|_{\alpha^3=\overset{(+)}{h}}\overset{(+)}{H}_3+\Phi_\sigma(\delta U_{im},\delta\psi_{im})\Big|_{\alpha^3=-\overset{(-)}{h}}\overset{(-)}{H}_3\right]d\Sigma_0+$$

$$+\int_{(\Gamma_0)}\int_{-\overset{(-)}{h}}^{\overset{(+)}{h}}\Phi_\sigma(\delta U_{im},\delta\psi_{im})H_2\,d\gamma\,d\Gamma_0=0,$$

where

$$\Phi_\sigma(\delta U_{im},\,\delta\psi_{im})=(\hat{\sigma}\cdot\vec{n}-\vec{p}_n)\cdot(U_{im}\,\delta\psi_{im}+\psi_{im}\delta U_{im})\vec{\ni}^i;\; H_1=(1+k_1\gamma)(1+k_2\gamma);$$

$$H_2=\left[1+2\gamma(k_1\cos^2\chi+k_2\sin^2\chi)+(\gamma)^2(k_1^2\cos^2\chi+k_2^2\sin^2\chi)\right]^{1/2};$$

$$\overset{(\pm)}{H}_3=\sqrt{\left(1\pm k_1\overset{(\pm)}{h}\right)^2\left(1\pm k_2\overset{(\pm)}{h}\right)^2+A_1^{-2}\left(1\pm k_2\overset{(\pm)}{h}\right)^2\left(\frac{\partial\overset{(\pm)}{h}}{\partial\alpha^1}\right)^2+A_2^{-2}\left(1\pm k_1\overset{(\pm)}{h}\right)^2\left(\frac{\partial\overset{(\pm)}{h}}{\partial\alpha^2}\right)^2}\;.$$

(Γ_0) is a contour limiting (Σ_0); $\vec{\ni}_i,\vec{\ni}^i$ are the covariant and countervariant basic vector systems of coordinates α^1,α^2,γ in an actual configuration; k_1,k_2 are principal curvatures of the surface (Σ_0); χ is the angle between the coordinate line α^1 and contour (Γ_0); A_1,A_2 are coefficients of the first quadratic form of the basic surface.

From condition (15), we obtain equations

$$a_{ms}^{(\kappa\omega)}\frac{\partial U_{(\kappa)s}}{\partial\alpha^{(\omega)}}+a_{ms}^{(\omega\kappa)}\frac{\partial U_{(\omega)s}}{\partial\alpha^{(\kappa)}}-2c_{ms}^{\kappa\omega k}U_{ks}-2b_{klms}^{\kappa\omega}M_s^{kl}-2\vartheta_m^{\kappa\omega}=0,$$

$$a_{ms}^{3(\kappa)}\frac{\partial U_{3s}}{\partial\alpha^{(\kappa)}}+d_{ms}^{(\kappa)}U_{(\kappa)s}-2c_{ms}^{\kappa 3k}U_{ks}-2b_{klms}^{\kappa 3}M_s^{kl}=0,$$

$$d_{ms}^{3}U_{3s}-c_{ms}^{33k}U_{ks}-b_{klms}^{33}M_s^{kl}-\vartheta_m^{33}=0,$$

$$a_{sm}^{(i)\kappa}\frac{\partial M_s^{(i)\kappa}}{\partial\alpha^\kappa}+e_{(i)ms}M_s^{(i)3}+g_{(i)jms}M_s^{(i)j}+p_{jkms}^{i}M_s^{jk}-$$

$$-q_{(i)jms}M_s^{(i)j}+q_m^{*i}+f_m^i=0; \qquad (16)$$

$$A_{ms}^{(\kappa)\omega}\psi_{(\kappa)s}+A_{ms}^{(\omega)\kappa}\psi_{(\omega)s}-2C_{ms}^{\kappa\omega k}\psi_{ks}-2B_{klms}^{\kappa\omega}\varphi_s^{kl}-2\theta_m^{\kappa\omega}=0,$$

$$D_{ms}^{(\kappa)}\frac{d\psi_{(\kappa)s}}{d\alpha^3}+A_{ms}^{3\kappa}\psi_{3s}-2C_{ms}^{\kappa 3k}\psi_{ks}-2B_{klms}^{\kappa 3}\varphi_s^{kl}=0,$$

$$D_{ms}^{3}\frac{d\psi_{3s}}{d\alpha^3}-C_{ms}^{33k}\psi_{ks}-B_{klms}^{33}\varphi_s^{kl}-\theta_m^{33}=0,$$

$$D_{sm}^{(i)}\frac{d\varphi_s^{(i)3}}{d\alpha^3}+E_{(i)\kappa ms}\varphi_s^{(i)\kappa}+G_{(i)jms}\varphi_s^{(i)j}+P_{jkms}^{i}\varphi_s^{jk}-$$

$$-Q_{(i)jms}\varphi_s^{(i)j}+Q_m^{*i}+F_m^i=0 \qquad (17)$$

and boundary conditions

$$\overset{(\pm)}{V}_{(i)jms}\left[\varphi_s^{(i)j}\left(\pm\overset{(\pm)}{h}\right)\right]=\overset{(\pm)*i}{V}_{m}; \qquad (18)$$

$$v_{(i)jms}M_s^{(i)j}=v_m^{*i}\ \text{ on } \left(\Gamma_0\right). \qquad (19)$$

Here $i,j,k,l=\overline{1,3}$; $\kappa,\omega=1,2$; $m,s=\overline{0,N}$ and

$$a_{ms}^{i\kappa}=\int_{-\overset{(-)}{h}}^{\overset{(+)}{h}}\varphi_m^{(i)\kappa}\psi_{(i)s}H_1d\gamma;\quad b_{klms}^{ij}=\int_{-\overset{(-)}{h}}^{\overset{(+)}{h}}\varphi_m^{(ij)}\varphi_s^{(kl)}b_{(ijkl)}H_1d\gamma;$$

$$c_{ms}^{ijk}=\int_{-\overset{(-)}{h}}^{\overset{(+)}{h}}\varphi_m^{(ij)}\psi_{(k)s}\Gamma_{(ij)}^{(k)}H_1d\gamma;\quad d_{ms}^{i}=\int_{-\overset{(-)}{h}}^{\overset{(+)}{h}}\varphi_m^{(i)3}\frac{d\psi_{(i)s}}{d\alpha^3}H_1d\gamma;$$

$$e_{ims}=\int_{-\overset{(-)}{h}}^{\overset{(+)}{h}}\frac{d\varphi_s^{(i)3}}{d\alpha^3}\psi_{(i)m}H_1d\gamma;\quad f_m^i=\int_{-\overset{(-)}{h}}^{\overset{(+)}{h}}F^{(i)}\psi_{(i)m}H_1d\gamma;$$

$$g_{ijms}=\int_{-\overset{(-)}{h}}^{\overset{(+)}{h}}\varphi_s^{(ij)}\psi_{(i)m}\Gamma_{(j)k}^{k}H_1d\gamma;\quad p_{jkms}^{i}=c_{sm}^{jki};$$

$$q_{ijms}=\overset{(+)}{q}_{ijms}+\overset{(-)}{q}_{ijms};\quad q_m^{*i}=\overset{(+)}{q}{}_m^{*i}+\overset{(-)}{q}{}_m^{*i};$$

$$\overset{(\pm)}{q}_{ijms}=\varphi_s^{(ij)}\psi_{(i)m}n_{(j)}\Big|_{\alpha^3=\pm\overset{(\pm)}{h}}\overset{(\pm)}{H}_3;\quad \overset{(\pm)}{q}{}_m^{*i}=\psi_{(i)m}p_n^{(i)}\Big|_{\alpha^3=\pm\overset{(\pm)}{h}}\overset{(\pm)}{H}_3;$$

$$v_{ijms} = \int\limits_{\overset{(-)}{-h}}^{\overset{(+)}{h}} \varphi_s^{(ij)} \psi_{(i)m} n_{(j)} H_2 d\gamma\,; \quad v_m^{*i} = \int\limits_{\overset{(-)}{-h}}^{\overset{(+)}{h}} \psi_{(i)m} p_n^{(i)} H_2 d\gamma;$$

$$\vartheta_m^{ij} = \int\limits_{\overset{(-)}{-h}}^{\overset{(+)}{h}} \alpha\theta I_{(ij)} \varphi_m^{(ij)} H_1 d\gamma\,;$$

$$A_{ms}^{i\kappa} = \int\limits_{(\Sigma_0)} M_m^{(i\kappa)} \frac{\partial U_{(i)s}}{\partial \alpha^{(\kappa)}} H_1 d\Sigma_0; \quad B_{klms}^{ij} = \int\limits_{(\Sigma_0)} M_m^{(ij)} M_s^{(kl)} b_{(ijkl)} H_1 d\Sigma_0;$$

$$C_{ms}^{ijk} = \int\limits_{(\Sigma_0)} M_m^{(ij)} U_{(k)s} \Gamma_{(ij)}^{(k)} H_1 d\Sigma_0; \quad D_{ms}^{i} = \int\limits_{(\Sigma_0)} M_m^{(i)3} U_{(i)s} H_1 d\Sigma_0;$$

$$E_{i\kappa ms} = \int\limits_{(\Sigma_0)} \frac{\partial M_s^{(i\kappa)}}{\partial \alpha^{(\kappa)}} U_{(i)m} H_1 d\Sigma_0; \quad F_m^{i} = \int\limits_{(\Sigma_0)} F^{(i)} U_{(i)m} H_1 d\Sigma_0;$$

$$G_{ijms} = \int\limits_{(\Sigma_0)} M_s^{(ij)} U_{(i)m} \Gamma_{(j)k}^{k} H_1 d\Sigma_0; \quad P_{jkms}^{i} = C_{sm}^{jki}\,;$$

$$Q_{ijms} = \int\limits_{(\Gamma_0)} M_s^{(ij)} U_{(i)m} n_{(j)} H_2 d\Gamma_0; \quad Q_m^{*i} = \int\limits_{(\Gamma_0)} U_{(i)m} p_n^{(i)} H_2 d\Gamma_0;$$

$$\overset{(\pm)}{V}_{(i)jms}\left[\varphi_s^{(i)j}\left(\pm \overset{(\pm)}{h}\right)\right] = \int\limits_{(\Sigma_0)} M_s^{(i)j} \varphi_s^{(i)j}\left(\pm \overset{(\pm)}{h}\right) U_{(i)m} \overset{(\pm)}{n}_j \overset{(\pm)}{H}_3 d\Sigma_0;$$

$$\overset{(\pm)}{V}{}_m^{*i} = \int\limits_{(\Sigma_0)} U_{(i)m} \overset{(\pm)}{p}{}_n^{(i)} \overset{(\pm)}{H}_3 d\Sigma_0; \quad \theta_m^{ij} = \int\limits_{(\Sigma_0)} \alpha\theta I_{(ij)} M_m^{(ij)} H_1 d\Sigma_0;$$

$\overset{(\pm)}{n}_i = n_i\big|_{\gamma=\pm \overset{(\pm)}{h}}$; $\overset{(\pm)}{p}{}_n^i = p_n^i\big|_{\gamma=\pm \overset{(\pm)}{h}}$; b_{ijkl}, I_{ij} are tensor components, respectively, $\hat{b}$, $\hat{I}$; Γ_{ij}^k are Christoffel's symbols of the second kind; F^i, p_n^i, n_i are vector components, respectively, $\vec{F}$, $\vec{p}_n$, $\vec{n}$.

The system of equations (16), (17) and boundary conditions (18), (19) make it possible to find solutions to static problems of thermoelastic shells with variable thickness.

The obtained boundary-value problem, (16-19), enables one to define all components of the stress tensor and consider boundary conditions of the face surfaces, which is particularly essential in regions of abrupt changes in load as well as geometrical and mechanical parameters of the shell. A solution of nonlinear system of equations (16-19) is proposed in [11].

Conclusions. General equations of iterative models of thermoelastic shells which take into account the effects of locally-gradient and high-speed deformations were obtained in the three-dimensional approximation. The method of choice of the optimal basis of expansion functions at the transition from the three-dimensional boundary-value problems to their two-dimensional analogues.

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Математичне моделювання в термомеханіці пружних оболонок за ітераційно-моментним підходом

Юрій Зозуляк

Обговорюються загальні підходи до математичного моделювання термомеханічних процесів у тонкостінних елементах типу пластин і оболонок у тривимірному наближенні. Вихідним для побудови визначальних співвідношень є рівняння балансу енергії, сформульоване в тривимірній постановці за підходом Лагранжа на розширеному просторі параметрів локального стану та відповідного вибору функції локальної ситуації. У зв'язку з цим потік механічної енергії в загальному випадку подається сумою адитивних складових, в яких поруч з традиційними характеристиками (тензором напружень і вектором швидкості) ітераційним шляхом вводяться вищого порядку характеристики градієнтності деформування й інерційного руху та відповідні їм тензорні характеристики силової дії. Для встановленого на цій основі фазового простору параметрів локальної ситуації будується система *визначальних співвідношень, які дозволяють враховувати ефекти локально-градієнтного і високошвидкісного деформування. Перехід до двовимірного аналогу у визначальних співвідношеннях здійснюється через осереднені характеристики напружено-деформованого стану, використовуючи розвинення шуканих величин у ряди за тензорною базою. Пропонується варіант вибору оптимальної бази розкладу, яка дозволяє здійснювати найадекватніший перехід від тривимірних крайових задач до їх двовимірних аналогів.*

Математическое моделирование в термомеханике упругих оболочек с помощью итерационно-моментного подхода

Юрий Зозуляк

Обсуждаются общие подходы к математическому моделированию термомеханических процессов в тонкостенных элементах типа пластин и оболочек в трехмерном приближении. Исходным при построении определяющих соотношений является уравнение баланса энергии, сформулированное в трехмерной постановке за подходом Лагранжа на расширенном пространстве параметров локального состояния и соответственного выбора функций локальной ситуации. В этой связи поток механической энергии в общем случае представляется суммой аддитивных слагаемых, в которых наряду с традиционными характеристиками (тензором напряжений и вектором скорости) итерационным путем вводятся высшей степени характеристики градиентности деформирования и инерционного движения, а также соответствующие им тензорные характеристики силового воздействия. Для определенного на этом основании фазового пространства параметров локальной ситуации строится система определяющих соотношений, позволяющих учитывать эффекты локально-градиентного и высокоскоростного деформирования. Переход к двумерному аналогу в определяющих уравнениях реализуется посредством осредненных тензорных характеристик напряженно-деформированного состояния с использованием представления искомых величин разложением в ряды по тензорном базисе. Предлагается вариант выбора оптимального базиса разложения, позволяющий осуществлять наиболее адекватный переход от трехмерных краевых задач к их двумерным аналогам.

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