GROUP ANALYSIS OF MOTION EQUATION OF CHARGED PARTICLE PLACED IN CROSSED FIELDS

Yu.N. Yeliseyev

National Science Center "Kharkov Institute of Physics and Technology", Institute of Plasma Physics, Kharkiv, Ukraine

E-mail: eliseev2004@rambler.ru

The transformation of motion equation of a charged particle placed in crossed longitudinal magnetic field and radial electric one is considered under transition from the laboratory frame of reference to a uniformly rotating coordinate system. It is shown that the transformation of equation in a plane transversal to the magnetic field admits a group. Using the method of group analysis, the transformation invariant and canonical variables are found.

PACS: 02.30.Hq; 52.20.Dq

INTRODUCTION

The method of group analysis is the only general method that allows solving analytically nonlinear differential equations [1]. Among mathematicians, interest in group analysis of particle motion equations in electromagnetic fields and other plasma physics equations (hydrodynamic, kinetic) has not decreased in recent years (see, for example, [2-5]). The results of their research are strongly "mathematized". This does not allow readers unprepared in this area to analyze understand and use them. Probably, for this reason, the group analysis method has not found proper application in the theory of non-neutral plasma, although new powerful methods of group analysis are developed and await application.

Many methods of solving differential equations use the substitution of variables (dependent and independent) that transform a given differential equation into another equation with known properties. Since the class of linear equations is the simplest and most studied class of equations, it is expedient to transform this differential equation into a linear one. This problem is called the problem of linearization. It is a special case of the equivalence problem. Two differential equations are equivalent if there is a transformation that converts one equation into another. The problem of equivalence includes a number of related problems, such as defining a class of transformations, finding the invariants of these transformations, obtaining equivalence criteria, and constructing a transformation.

In [6], the transformation of motion equation of a charged particle placed in crossed homogeneous magnetic field and radial electric one was considered under transition from a laboratory coordinate system to a uniformly rotating one. It was shown that the motion equation in fields E, H in the laboratory system has in a rotating system the form of an equation of motion in other fields E', H'. The invariant of the rotation transformation was heuristically found. A problem was also considered that, in the author's opinion, generalizes the problem considered by Larmor: is there a rotating coordinate system in which the equation of particle motion in fields E_2, H_2 in the plane transverse to the magnetic field coincides with the equation of particle motion in the fields E_1, H_1 in the laboratory system? The frequency of rotation ω_{rot} and the criteria were found under which the coincidence is possible. The consideration was carried in a "traditional" way. This problem is related to the problem of the equivalence of differential equations.

In the present paper the results of [6] are obtained using the group analysis of motion equations: a group generator, an invariant of the transformation, and canonical variables are determined.

1. PROBLEM FORMULATION

A charged particle moves in a uniform magnetic field $\vec{H}$ and a centrally symmetric electric field $\vec{E}$ ($E = E(r)$). The origin of the coordinate system O is aligned with the center of symmetry of the electric field. The axis Oz is directed along the magnetic field. In the laboratory system the motion equation in a plane transverse to the magnetic field can be represented in a complex form as follows [6]

$$\ddot{u} + i\omega_c \dot{u} - (e/m)(E/r)u = 0 . \qquad (1)$$

Here $u = x + iy$ is a complex radius vector in the transverse plane, $r = \left(x^2 + y^2 + z^2\right)^{1/2}$ is a three-dimensional radius, $\omega_c = eH/mc$ is a cyclotron frequency, e, m are charge and mass of a particle.

The motion equation in Oz direction can be written in the form: $\ddot{z} - (e/m)(E/r)z = 0$.

When the electric field has a cylindrical symmetry, the motion equation in the transverse plane has the same form (1) with the radius $r = \left(x^2 + y^2\right)^{1/2}$. The motion equation in Oz direction has the form $\ddot{z} = 0$ in the laboratory and in any rotating around the axis Oz coordinate system.

The motion equation of the same particle in the same fields *E*, *H*, but observed in a coordinate system rotating with frequency ω_{rot}, has the form in the transverse plane

$$\ddot{u}' + i\omega_c' \dot{u}' - \left[eE'/(mr)\right]u' = 0 . \qquad (2)$$

Here $u' = x' + y'$ is a complex radius vector in the rotating system. It is related with the radius u in the laboratory system by the transformation $u = u' \exp(i\omega_{rot} t)$. (3)

In Eq. (2) the notations are introduced

$$eE'/(mr) \equiv \omega_{rot}^2 + \omega_c \omega_{rot} + eE/(mr), \quad (4)$$

$$\omega'_c \equiv 2\omega_{rot} + \omega_c, \quad (5)$$

(in Eq. (5) $\omega'_c = eH'/mc$). Comparison of Eqs. (2) and (1) shows that the motion equation in fields E and H in a coordinate system rotating with the frequency ω_{rot} in the transverse plane x', y' looks like the motion equation of the particle with the same value of ratio e/m in the fields E' and H' (4), (5) in the laboratory system [6].

Generally speaking, Eqs. (1), (2) can be non-linear ones. Relations (2) – (5) are valid for an arbitrary frequency of rotation ω_{rot}, for an arbitrary dependence of the electric field E on the radius r, for a finite and infinite motion of the particle along the radius. They do not contain an approximation of weak magnetic or electric fields. The quantities E and E', H and H' can turn out to be of different signs. When $E/r = const$ Eqs. (1) and (2) are linear equations with constant coefficients.

Eqs. (1), (2) are equivalent in function [1], because one is reduced to another with the help of linear transformation u into u' (3).

Eqs. (4), (5) specify the transformation $E, H \to E', H'$ associated with the parameter $a \equiv \omega_{rot}$. This equivalence transformation admits a one–parameter group on the plane of variables

$$x \equiv eE(r)/(mr), \qquad y \equiv \omega_c. \quad (6)$$

Below, under x, y, and $x' \equiv eE'(r)/(mr)$, $y' \equiv \omega'_c$ we mean exactly these variables, but not the spatial coordinates. In this notations, the transformation (4), (5) takes the form

$$x' = a^2 + ya + x, \quad (7)$$

$$y' = 2a + y. \quad (8)$$

The transformation has a unite element ($a \equiv \omega_{rot} = 0$), an inverse element ($-\omega_{rot}$), a sequential application of two transformations (3) with frequencies ω_{rot1} and ω_{rot2} is also a transformation with a frequency ($\omega_{rot1} + \omega_{rot2}$).

This allows us to apply the Lie method of group analysis to the search for invariants of the transformation [1].

2. ANALYSIS OF TRANSFORMATION

1. The operator having the form:

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} \quad (9)$$

is called the generator of the transformation group [1], where $\xi(x, y)$ and $\eta(x, y)$ are the coefficients of expansion of the right-hand sides of Eqs. (7), (8) in a Taylor series with respect to a small parameter $a \equiv \omega_{rot}$:

$$x' \approx x + \xi(x, y)a, \qquad \xi(x, y) = \partial x'/\partial a\,|_{a=0}, \quad (10)$$

$$y' \approx y + \eta(x, y)a, \qquad \eta(x, y) = \partial y'/\partial a\,|_{a=0}. \quad (11)$$

In the case under consideration they are equal

$$\xi(x, y) = y, \qquad \eta(x, y) = 2. \quad (12)$$

2. The Lie equations [1]:

$$\frac{dx'}{da} = \xi(x', y'), \quad x'|_{a=0} = x, \quad (13)$$

$$\frac{dy'}{da} = \eta(x', y'), \quad y'|_{a=0} = y \quad (14)$$

take the form

$$\frac{dx'}{da} = y', \quad x'|_{a=0} = x, \quad (15)$$

$$\frac{dy'}{da} = 2, \quad y'|_{a=0} = y. \quad (16)$$

The transformation of the group (7), (8) is restored by the integration of the system of ordinary differential Eqs. (15), (16).

3. A function $F(x, y)$ is an invariant of the transformation group if for all admissible x, y, a the equality is fulfilled :

$$F(x, y) = F(x', y'). \quad (17)$$

A function $F(x, y)$ is an invariant of a group if and only if it is a solution of a first-order partial differential equation [1]

$$XF \equiv \xi(x, y)\frac{\partial F}{\partial x} + \eta(x, y)\frac{\partial F}{\partial y} = 0. \quad (18)$$

In the case under consideration, this equation takes the form

$$y\frac{\partial F}{\partial x} + 2\frac{\partial F}{\partial y} = 0. \quad (19)$$

Its solution can be found by the method of characteristics. The equation of characteristics of equation (19) has the form

$$\frac{dx}{\xi(x, y)} = \frac{dy}{\eta(x, y)} \quad \text{or} \quad \frac{dx}{y} = \frac{dy}{2}. \quad (20)$$

From here we find the invariant of the transformation (4), (5):

$$h = x - y^2/4 \quad (21)$$

In physical variables it has the form

$$h = \frac{eE}{mr} - \frac{\omega_c^2}{4} = -\frac{\Omega^2}{4}. \quad (22)$$

Any one-parameter group of transformations on the plane has one independent invariant [1], which can be chosen as the left-hand side of any first integral $h(x, y) = C$ of the equation of characteristics of Eqs. (18), (19). Any other invariant F is a function of h. In [6], the invariant (22) was found heuristically.

4. Any one-parameter transformation group with the generator X (9) with the help of a suitable change of variables $t = t(x, y)$, $u = u(x, y)$ can be reduced to the group of displacements [1]:

$$t' = t + a, \; u' = u, \quad (23)$$

with generator $X = \partial/\partial t$. The variables t and u are called canonical variables. They are determined by the following system of equations [1]:

$$X(t) \equiv \xi(x,y)\frac{\partial t}{\partial x} + \eta(x,y)\frac{\partial t}{\partial y} = 1, \quad (24)$$

$$X(u) \equiv \xi(x,y)\frac{\partial u}{\partial x} + \eta(x,y)\frac{\partial u}{\partial y} = 0. \quad (25)$$

Eq. (25) coincides with equation (18) for determining the invariant. From this and also from the second Eq. (23) it follows that one canonical variable u is the invariant (21), (22) of the transformations (23) and (4), (5) :

$$u = h = x - \frac{y^2}{4} \quad \text{or} \quad u = h = \frac{eE}{mr} - \frac{\omega_c^2}{4} = -\frac{\Omega^2}{4}. \quad (26)$$

We define the canonical variable t, assuming that it depends only on $y : t = t(y)$. Eq. (24) takes the form

$$2\frac{dt}{dy} = 1. \quad (27)$$

Its solution gives an expression for the canonical variable t :

$$t = \frac{y}{2} \quad \text{or} \quad t = \frac{\omega_c}{2}. \quad (28)$$

The transformation (23) expressed in terms of the variable (28) takes the form

$$\frac{\omega_c'}{2} = \frac{\omega_c}{2} + a. \quad (29)$$

It repeats the Eq. (5).

In the variables t, u the trajectories along which the points on plane move when going from the laboratory system to the rotating system (or from one rotating system to another rotating one) are horizontal straight lines. In the variables (6) these trajectories are parabolas [6].

REFERENCES

1. N.H. Ibragimov. *A Practical Course in Differential Equations and Mathematical Modelling*. Karlskrona (Sweden), ALGA Publications Blekinge Institute of Technology, 2006, 341 p. [Nizhnij Novgorod, 2007, 421 p. (in Russian)].
2. N. Kallinikos. *Dynamics of Charged Particles in Electromagnetic Fields with Applications in Fusion Devices* / Ph.D. Thesis, Aristotle University of Thessaloniki, 2016.
3. N. Kallinikos. Group classification of charged particle motion in stationary electromagnetic fields *// Journal of Mathematical Physics.* 2017, v. 58, № 11, p. 113506.
4. A.A. Gainetdinova, R.K. Gazizov. Integrability of systems of two second-order ordinary differential equations admitting four-dimensional Lie algebras // *Proc. R. Soc.* A473:20160461, 2017.
5. S.V. Meleshko, S. Moyo, C. Muriel, J.L. Romero, P. Guha, A.G. Choudhury. On first integrals of second-order ordinary differential equations // *Journ. of Engineering Mathematics.* 2013, v. 82, № 1, p. 17-30.
6. Yu.N. Yeliseyev. Properties of charged particle motion equations in crossed fields and Larmor's theorem *// Problems of Atomic Science and Technology.* S*eries "Plasma Physics".* 2018, № 6*(118)*, p. 90-63.

Article received 18.10.2018

ГРУППОВОЙ АНАЛИЗ УРАВНЕНИЙ ДВИЖЕНИЯ ЗАРЯЖЕННОЙ ЧАСТИЦЫ В СКРЕЩЕННЫХ ПОЛЯХ

Ю.Н. Елисеев

Рассмотрено преобразование уравнения движения заряженной частицы в поперечной к магнитному полю плоскости при переходе от лабораторной к равномерно вращающейся системе координат. Показано, что это преобразование образует группу. С применением метода группового анализа найден инвариант преобразования и канонические переменные.

ГРУПОВИЙ АНАЛІЗ РІВНЯНЬ РУХУ ЗАРЯДЖЕНОЇ ЧАСТКИ В СХРЕЩЕНИХ ПОЛЯХ

Ю.М. Єлісеєв

Розглянуто перетворення рівняння руху зарядженої частки в поперечній до магнітного поля площини при переході від лабораторної до системи координат, що рівномірно обертається. Показано, що це перетворення утворює групу. Із застосуванням методу групового аналізу знайдений інваріант перетворення й канонічні змінні.