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RESEARCH ARTICLE

Mappings preserving sum of products $a \circ b + ba^*$ on factor von Neumann algebras

J. C. M. Ferreira and M. G. B. Marietto

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Abstract. Let $\mathcal{A}$ and $\mathcal{B}$ be two factor von Neumann algebras. In this paper, we proved that a bijective mapping $\Phi : \mathcal{A} \to \mathcal{B}$ satisfies $\Phi(a \circ b + ba^*) = \Phi(a) \circ \Phi(b) + \Phi(b)\Phi(a)^*$ (where $\circ$ is the special Jordan product on $\mathcal{A}$ and $\mathcal{B}$, respectively), for all elements $a, b \in \mathcal{A}$, if and only if Φ is a $*$-ring isomorphism. In particular, if the von Neumann algebras $\mathcal{A}$ and $\mathcal{B}$ are type I factors, then Φ is a unitary isomorphism or a conjugate unitary isomorphism.

1. Introduction

Let $\mathbb{R}$ and $\mathbb{C}$ be the real and complex number fields, respectively, and $\mathcal{A}$ and $\mathcal{B}$ be two algebras over $\mathbb{C}$. If $a, b \in \mathcal{A}$ (resp., $a, b \in \mathcal{B}$), let $a \circ b = \frac{1}{2}(ab + ba)$. We call $\circ$ the *special Jordan product* on $\mathcal{A}$ (resp., *special Jordan product* on $\mathcal{B}$). We say that a mapping $\Phi : \mathcal{A} \to \mathcal{B}$ *preserves product* if $\Phi(ab) = \Phi(a)\Phi(b)$, for all elements $a, b \in \mathcal{A}$, that *preserves special Jordan product* if $\Phi(a \circ b) = \Phi(a) \circ \Phi(b)$, for all elements $a, b \in \mathcal{A}$, that *preserves Jordan product* if $\Phi(ab + ba) = \Phi(a)\Phi(b) + \Phi(b)\Phi(a)$, for all elements $a, b \in \mathcal{A}$, and that *preserves Lie product* if $\Phi(ab - ba) = \Phi(a)\Phi(b) - \Phi(b)\Phi(a)$, for all elements $a, b \in \mathcal{A}$. We say that a mapping $\Phi : \mathcal{A} \to \mathcal{B}$ is *additive* if $\Phi(a + b) = \Phi(a) + \Phi(b)$, for all elements $a, b \in \mathcal{A}$ and that is a *ring isomorphism* if Φ is an additive bijection that preserves products.

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Let $\mathcal{A}$ and $\mathcal{B}$ be two $*$-algebras over $\mathbb{C}$. We say that a mapping $\Phi : \mathcal{A} \to \mathcal{B}$ *preserves involution* if $\Phi(a^*) = \Phi(a)^*$, for all elements $a \in \mathcal{A}$, and that Φ is a *$*$-ring isomorphism* if Φ is a ring isomorphism that preserves involution.

A *von Neumann algebra* $\mathcal{A}$ is a weakly closed, self-adjoint algebra of operators on a Hilbert space $\mathcal{H}$, which are denote in this paper by lower-case Latin letters, containing the identity operator $1_{\mathcal{A}}$, called elements of $\mathcal{A}$. A von Neumann algebra $\mathcal{A}$ is called *factor* if its center contains the multiples of the identity operator only. It is well known that the factor von Neumann algebra $\mathcal{A}$ is prime, that is, if a and b are elements of $\mathcal{A}$ such that $a\mathcal{A}b = 0$, then either $a = 0$ or $b = 0$.

The study of additivity or ring isomorphism of mappings preserving products, special Jordan products, Jordan products and Lie products on operator algebras have been received fair amount of attentions. Some examples of these studies may be found in [1, 4–6]. On the other hand, many mathematicians devoted themselves to study $*$-ring isomorphisms of mappings satisfying new types of products on von Neumann algebras, for example [2, 3, 7].

Let $\mathcal{A}$ and $\mathcal{B}$ be two $*$-algebras. We say that a mapping $\Phi : \mathcal{A} \to \mathcal{B}$ *preserves sum of products $a \circ b + ba^*$* if

$$\Phi(a \circ b + ba^*) = \Phi(a) \circ \Phi(b) + \Phi(b)\Phi(a)^*, \tag{1}$$

for all elements $a, b \in \mathcal{A}$.

Inspired by the research described in [2], the aim of this paper is to prove that a bijective mapping preserving sum of products $a \circ b + ba^*$ on factor von Neumann algebras is a $*$-ring isomorphism. Our main result reads as follows.

Main Theorem. *Let $\mathcal{A}$ and $\mathcal{B}$ be two factor von Neumann algebras with $1_{\mathcal{A}}$ and $1_{\mathcal{B}}$ the identities of them, respectively. Then every bijective mapping $\Phi : \mathcal{A} \to \mathcal{B}$ satisfies $\Phi(a \circ b + ba^*) = \Phi(a) \circ \Phi(b) + \Phi(b)\Phi(a)^*$, for all elements $a, b \in \mathcal{A}$, if and only if Φ is a $*$-ring isomorphism.*

2. The proof of main theorem

In order to prove the Main Theorem we need to prove several lemmas. We begin with the following lemma.

Lemma 1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two factor von Neumann algebras with $1_{\mathcal{A}}$ the identity of $\mathcal{A}$. Then every bijective mapping $\Phi : \mathcal{A} \to \mathcal{B}$ satisfying*

$\Phi(a \circ b + ba^*) = \Phi(a) \circ \Phi(b) + \Phi(b)\Phi(a)^*$, *for all elements* $a, b \in \mathcal{A}$, *is additive.*

We shall organize the proof of Lemma 1 in a series of claims, based on the techniques used by Li et al. [2]. Finally, we would like to mention the following well-known result that will be used throughout this paper.

Let p_1 be an arbitrary non-trivial projection of $\mathcal{A}$ and write $p_2 = 1 - p_1$. Then $\mathcal{A}$ has a Peirce decomposition $\mathcal{A} = \mathcal{A}_{11} \oplus \mathcal{A}_{12} \oplus \mathcal{A}_{21} \oplus \mathcal{A}_{22}$, where $\mathcal{A}_{ij} = p_i \mathcal{A} p_j$ $(i, j = 1, 2)$, satisfying the following multiplicative relations: $\mathcal{A}_{ij}\mathcal{A}_{kl} \subseteq \delta_{jk}\mathcal{A}_{il}$, where δ_{jk} is the *Kronecker delta function.*

We begin with the following claim.

Claim 1. Let a, b, c be elements of $\mathcal{A}$ such that $\Phi(c) = \Phi(a) + \Phi(b)$. Then hold the following identities: (i) $\Phi(t \circ c + ct^*) = \Phi(t \circ a + at^*) + \Phi(t \circ b + bt^*)$ and (ii) $\Phi(c \circ t + tc^*) = \Phi(a \circ t + ta^*) + \Phi(b \circ t + tb^*)$, for all element $t \in \mathcal{A}$.

Proof. For an arbitrary element $t \in \mathcal{A}$, we have

$$\begin{aligned}
\Phi(t \circ c + ct^*) &= \Phi(t) \circ \Phi(c) + \Phi(c)\Phi(t)^* \\
&= \Phi(t) \circ (\Phi(a) + \Phi(b)) + (\Phi(a) + \Phi(b))\Phi(t)^* \\
&= \Phi(t) \circ \Phi(a) + \Phi(a)\Phi(t)^* + \Phi(t) \circ \Phi(b) + \Phi(b)\Phi(t)^* \\
&= \Phi(t \circ a + at^*) + \Phi(t \circ b + bt^*).
\end{aligned}$$

Similarly, we obtain (ii). □

Claim 2. $\Phi(0) = 0$.

Proof. From the surjectivity of Φ there exists an element $b \in \mathcal{A}$ such that $\Phi(b) = 0$. It follows that

$$\Phi(0) = \Phi(0 \circ b + b0^*) = \Phi(0) \circ \Phi(b) + \Phi(b)\Phi(0)^* = \Phi(0) \circ 0 + 0\Phi(0)^* = 0. \quad \square$$

Claim 3. For arbitrary elements $a_{11} \in \mathcal{A}_{11}$, $b_{12} \in \mathcal{A}_{12}$, $b_{21} \in \mathcal{A}_{21}$ and $c_{22} \in \mathcal{A}_{22}$ hold: (i) $\Phi(a_{11} + b_{12}) = \Phi(a_{11}) + \Phi(b_{12})$, (ii) $\Phi(a_{11} + b_{21}) = \Phi(a_{11}) + \Phi(b_{21})$, (iii) $\Phi(b_{12} + c_{22}) = \Phi(b_{12}) + \Phi(c_{22})$ and (iv) $\Phi(b_{21} + c_{22}) = \Phi(b_{21}) + \Phi(c_{22})$.

Proof. From the surjectivity of Φ, we can choose an element $c = c_{11} + c_{12} + c_{21} + c_{22} \in \mathcal{A}$ such that $\Phi(c) = \Phi(a_{11}) + \Phi(b_{12})$. By Claim 1(i), we have

$$\Phi(p_2 \circ c + cp_2^*) = \Phi(p_2 \circ a_{11} + a_{11}p_2^*) + \Phi(p_2 \circ b_{12} + b_{12}p_2^*) = \Phi(\frac{3}{2}b_{12}).$$

This results that $p_2 \circ c + cp_2^* = \frac{3}{2}b_{12}$ which implies that $c_{12} = b_{12}$ and $c_{21} = c_{22} = 0$. Thus, $\Phi(c_{11} + b_{12}) = \Phi(a_{11}) + \Phi(b_{12})$. It follows that, for an arbitrary element $d_{12} \in \mathcal{A}_{12}$

$$\begin{aligned}&\Phi(d_{12} \circ (c_{11} + b_{12}) + (c_{11} + b_{12})d_{12}^*)\\ &\quad = \Phi(d_{12} \circ a_{11} + a_{11}d_{12}^*) + \Phi(d_{12} \circ b_{12} + b_{12}d_{12}^*)\end{aligned}$$

which implies that

$$\Phi(\frac{1}{2}c_{11}d_{12} + b_{12}d_{12}^*) = \Phi(\frac{1}{2}a_{11}d_{12}) + \Phi(b_{12}d_{12}^*).$$

Hence,

$$\begin{aligned}&\Phi(p_2 \circ (\frac{1}{2}c_{11}d_{12} + b_{12}d_{12}^*) + (\frac{1}{2}c_{11}d_{12} + b_{12}d_{12}^*)p_2^*)\\ &\quad = \Phi(p_2 \circ (\frac{1}{2}a_{11}d_{12}) + (\frac{1}{2}a_{11}d_{12})p_2^*) + \Phi(p_2 \circ (b_{12}d_{12}^*) + (b_{12}d_{12}^*)p_2^*)\end{aligned}$$

which yields $\Phi(\frac{3}{4}c_{11}d_{12}) = \Phi(\frac{3}{4}a_{11}d_{12})$. This shows that $c_{11}d_{12} = a_{11}d_{12}$ which leads to $c_{11} = a_{11}$.

Similarly, we prove the cases (ii), (iii) and (iv). $\square$

Claim 4. For arbitrary elements $a_{12} \in \mathcal{A}_{12}$ and $b_{21} \in \mathcal{A}_{21}$ holds $\Phi(a_{12} + b_{21}) = \Phi(a_{12}) + \Phi(b_{21})$.

Proof. Choose an element $c = c_{11} + c_{12} + c_{21} + c_{22} \in \mathcal{A}$ such that $\Phi(c) = \Phi(a_{12}) + \Phi(b_{21})$. Hence, for an arbitrary element $d_{12} \in \mathcal{A}_{12}$, we have

$$\begin{aligned}\Phi(d_{12} \circ c + cd_{12}^*) &= \Phi(d_{12} \circ a_{12} + a_{12}d_{12}^*) + \Phi(d_{12} \circ b_{21} + b_{21}d_{12}^*)\\ &= \Phi(a_{12}d_{12}^*) + \Phi(d_{12} \circ b_{21})\end{aligned}$$

which yields

$$\begin{aligned}&\Phi(p_2 \circ (d_{12} \circ c + cd_{12}^*) + (d_{12} \circ c + cd_{12}^*)p_2^*)\\ &\quad = \Phi(p_2 \circ (a_{12}d_{12}^*) + (a_{12}d_{12}^*)p_2^*) + \Phi(p_2 \circ (d_{12} \circ b_{21}) + (d_{12} \circ b_{21})p_2^*)\\ &\quad = \Phi(b_{21}d_{12}).\end{aligned}$$

This results that $p_2 \circ (d_{12} \circ c + cd_{12}^*) + (d_{12} \circ c + cd_{12}^*)p_2^* = b_{21}d_{12}$ which implies that $c_{11} = c_{22} = 0$ and $c_{21} = b_{21}$. Thus,

$$\Phi(c_{12} + b_{21}) = \Phi(a_{12}) + \Phi(b_{21}).$$

It follows that, for an arbitrary element $d_{21} \in \mathcal{A}_{21}$, we have

$$\begin{aligned}
&\Phi(d_{21} \circ (c_{12} + b_{21}) + (c_{12} + b_{21})d_{21}^*) \\
&\qquad = \Phi(d_{21} \circ a_{12} + a_{12}d_{21}^*) + \Phi(d_{21} \circ b_{21} + b_{21}d_{21}^*) \\
&\qquad = \Phi(d_{21} \circ a_{12}) + \Phi(b_{21}d_{21}^*).
\end{aligned}$$

This implies that

$$\begin{aligned}
&\Phi(p_1 \circ (d_{21} \circ c_{12} + b_{21}d_{21}^*) + (d_{21} \circ c_{12} + b_{21}d_{21}^*)p_1^*) \\
&\quad = \Phi(p_1 \circ (d_{21} \circ a_{12}) + (d_{21} \circ a_{12})p_1^*) + \Phi(p_1 \circ (b_{21}d_{21}^*) + (b_{21}d_{21}^*)p_1^*) \\
&\quad = \Phi(a_{12}d_{21}).
\end{aligned}$$

It follows that $p_1 \circ (d_{21} \circ c_{12} + b_{21}d_{21}^*) + (d_{21} \circ c_{12} + b_{21}d_{21}^*)p_1^* = a_{12}d_{21}$ which yields $c_{12}d_{21} = a_{12}d_{21}$. This allows us to conclude that $c_{12} = a_{12}$. □

Claim 5. For arbitrary elements $a_{11} \in \mathcal{A}_{11}$, $b_{12} \in \mathcal{A}_{12}$, $c_{21} \in \mathcal{A}_{21}$ and $d_{22} \in \mathcal{A}_{22}$ hold: (i) $\Phi(a_{11} + b_{12} + c_{21}) = \Phi(a_{11}) + \Phi(b_{12}) + \Phi(c_{21})$ and (ii) $\Phi(b_{12} + c_{21} + d_{22}) = \Phi(b_{12}) + \Phi(c_{21}) + \Phi(d_{22})$.

Proof. Choose an element $f = f_{11} + f_{12} + f_{21} + f_{22} \in \mathcal{A}$ such that $\Phi(f) = \Phi(a_{11}) + \Phi(b_{12}) + \Phi(c_{21})$. By Claim 4 we have

$$\begin{aligned}
&\Phi(p_2 \circ f + fp_2^*) \\
&\qquad = \Phi(p_2 \circ a_{11} + a_{11}p_2^*) + \Phi(p_2 \circ b_{12} + b_{12}p_2^*) + \Phi(p_2 \circ c_{21} + c_{21}p_2^*) \\
&\qquad = \Phi(\frac{3}{2}b_{12}) + \Phi(\frac{1}{2}c_{21}) = \Phi(\frac{3}{2}b_{12} + \frac{1}{2}c_{21}).
\end{aligned}$$

This results that $p_2 \circ f + fp_2^* = \frac{3}{2}b_{12} + \frac{1}{2}c_{21}$ which leads to $f_{12} = b_{12}$, $f_{21} = c_{21}$ and $f_{22} = 0$. It follows that,

$$\Phi(f_{11} + b_{12} + c_{21}) = \Phi(a_{11}) + \Phi(b_{12}) + \Phi(c_{21}).$$

Next, for an arbitrary element $d_{12} \in \mathcal{A}_{12}$, we have

$$\begin{aligned}
&\Phi(d_{12} \circ (f_{11} + b_{12} + c_{21}) + (f_{11} + b_{12} + c_{21})d_{12}^*) \\
&\quad = \Phi(d_{12} \circ a_{11} + a_{11}d_{12}^*) + \Phi(d_{12} \circ b_{12} + b_{12}d_{12}^*) + \Phi(d_{12} \circ c_{21} + c_{21}d_{12}^*)
\end{aligned}$$

which implies that

$$\Phi(d_{12} \circ (f_{11} + c_{21}) + b_{12}d_{12}^*) = \Phi(\frac{1}{2}a_{11}d_{12}) + \Phi(b_{12}d_{12}^*) + \Phi(d_{12} \circ c_{21}).$$

As consequence, we get

$$\begin{aligned}&\Phi(p_2\circ(d_{12}\circ(f_{11}+c_{21})+b_{12}d_{12}^*)+(d_{12}\circ(f_{11}+c_{21})+b_{12}d_{12}^*)p_2^*)\\&\quad=\Phi(p_2\circ(\frac{1}{2}a_{11}d_{12})+(\frac{1}{2}a_{11}d_{12})p_2^*)+\Phi(p_2\circ(b_{12}d_{12}^*)+(b_{12}d_{12}^*)p_2^*)\\&\qquad\quad+\Phi(p_2\circ(d_{12}\circ c_{21})+(d_{12}\circ c_{21})p_2^*)\end{aligned}$$

which results

$$\Phi(\frac{3}{4}f_{11}d_{12}+c_{21}d_{12})=\Phi(\frac{3}{4}a_{11}d_{12})+\Phi(c_{21}d_{12})=\Phi(\frac{3}{4}a_{11}d_{12}+c_{21}d_{12}).$$

This shows that $\frac{3}{4}f_{11}d_{12}+c_{21}d_{12}=\frac{3}{4}a_{11}d_{12}+c_{21}d_{12}$ which yields $f_{11}=a_{11}$.

Similarly, we prove the case (ii). □

Claim 6. For arbitrary elements $a_{11}\in\mathcal{A}_{11}$, $b_{12}\in\mathcal{A}_{12}$, $c_{21}\in\mathcal{A}_{21}$ and $d_{22}\in\mathcal{A}_{22}$ holds $\Phi(a_{11}+b_{12}+c_{21}+d_{22})=\Phi(a_{11})+\Phi(b_{12})+\Phi(c_{21})+\Phi(d_{22})$.

Proof. Choose an element $f=f_{11}+f_{12}+f_{21}+f_{22}\in\mathcal{A}$ such that $\Phi(f)=\Phi(a_{11})+\Phi(b_{12})+\Phi(c_{21})+\Phi(d_{22})$. By Claim 5(i) we have

$$\begin{aligned}\Phi(p_1\circ f+fp_1^*)&=\Phi(p_1\circ a_{11}+a_{11}p_1^*)+\Phi(p_1\circ b_{12}+b_{12}p_1^*)\\&\qquad+\Phi(p_1\circ c_{21}+c_{21}p_1^*)+\Phi(p_1\circ d_{22}+d_{22}p_1^*)\\&=\Phi(2a_{11})+\Phi(\frac{1}{2}b_{12})+\Phi(\frac{3}{2}c_{21})=\Phi(2a_{11}+\frac{1}{2}b_{12}+\frac{3}{2}c_{21}).\end{aligned}$$

It follows that $p_1\circ f+fp_1^*=2a_{11}+\frac{1}{2}b_{12}+\frac{3}{2}c_{21}$ which implies that $f_{11}=a_{11}$, $f_{12}=b_{12}$ and $f_{21}=c_{21}$. Next, by Claim 5(ii) we have

$$\begin{aligned}\Phi(p_2\circ f+fp_2^*)&=\Phi(p_2\circ a_{11}+a_{11}p_2^*)+\Phi(p_2\circ b_{12}+b_{12}p_2^*)\\&\quad+\Phi(p_2\circ c_{21}+c_{21}p_2^*)+\Phi(p_2\circ d_{22}+d_{22}p_2^*)\\&=\Phi(\frac{3}{2}b_{12})+\Phi(\frac{1}{2}c_{21})+\Phi(2d_{22})=\Phi(\frac{3}{2}b_{12}+\frac{1}{2}c_{21}+2d_{22}).\end{aligned}$$

This results that $p_2\circ f+fp_2^*=\frac{3}{2}b_{12}+\frac{1}{2}c_{21}+2d_{22}$ which yields $f_{22}=d_{22}$. □

Claim 7. For arbitrary elements $a_{12},b_{12}\in\mathcal{A}_{12}$ and $a_{21},b_{21}\in\mathcal{A}_{21}$ hold: (i) $\Phi(a_{12}+b_{12})=\Phi(a_{12})+\Phi(b_{12})$ and (ii) $\Phi(a_{21}+b_{21})=\Phi(a_{21})+\Phi(b_{21})$.

Proof. First, we note that the following identity holds

$$\begin{aligned}&(p_1+a_{12})\circ(p_2+b_{12})+(p_2+b_{12})(p_1+a_{12})^*\\&\qquad=\frac{1}{2}a_{12}+\frac{1}{2}b_{12}+a_{12}^*+b_{12}a_{12}^*.\end{aligned}$$

Hence, by Claim 6 we have

$$
\begin{aligned}
\Phi(\frac{1}{2}a_{12}+\frac{1}{2}b_{12})+\Phi(a_{12}^*)+\Phi(b_{12}a_{12}^*) &= \Phi(\frac{1}{2}a_{12}+\frac{1}{2}b_{12}+a_{12}^*+b_{12}a_{12}^*)\\
&= \Phi((p_1+a_{12})\circ(p_2+b_{12})+(p_2+b_{12})(p_1+a_{12})^*)\\
&= \Phi(p_1+a_{12})\circ\Phi(p_2+b_{12})+\Phi(p_2+b_{12})\Phi(p_1+a_{12})^*\\
&= (\Phi(p_1)+\Phi(a_{12}))\circ(\Phi(p_2)+\Phi(b_{12}))\\
&\quad + (\Phi(p_2)+\Phi(b_{12}))(\Phi(p_1)^*+\Phi(a_{12})^*)\\
&= \Phi(p_1)\circ\Phi(p_2)+\Phi(p_2)\Phi(p_1)^*+\Phi(p_1)\circ\Phi(b_{12})+\Phi(b_{12})\Phi(p_1)^*\\
&\quad + \Phi(a_{12})\circ\Phi(p_2)+\Phi(p_2)\Phi(a_{12})^*+\Phi(a_{12})\circ\Phi(b_{12})+\Phi(b_{12})\Phi(a_{12})^*\\
&= \Phi(p_1\circ p_2+p_2p_1^*)+\Phi(p_1\circ b_{12}+b_{12}p_1^*)+\Phi(a_{12}\circ p_2+p_2a_{12}^*)\\
&\quad + \Phi(a_{12}\circ b_{12}+b_{12}a_{12}^*)\\
&= \Phi(\frac{1}{2}b_{12})+\Phi(\frac{1}{2}a_{12})+\Phi(a_{12}^*)+\Phi(b_{12}a_{12}^*).
\end{aligned}
$$

This allows us to conclude that $\Phi(a_{12}+b_{12})=\Phi(a_{12})+\Phi(b_{12})$.

Similarly, we prove the case (ii) using the identity

$$
\begin{aligned}
&(p_2+a_{21})\circ(p_1+b_{21})+(p_1+b_{21})(p_2+a_{21})^*\\
&\qquad = \frac{1}{2}a_{21}+\frac{1}{2}b_{21}+a_{21}^*+b_{21}a_{21}^*.
\end{aligned}
$$

□

Claim 8. For arbitrary elements $a_{11},b_{11}\in\mathcal{A}_{11}$ and $a_{22},b_{22}\in\mathcal{A}_{22}$ hold: (i) $\Phi(a_{11}+b_{11})=\Phi(a_{11})+\Phi(b_{11})$ and (ii) $\Phi(a_{22}+b_{22})=\Phi(a_{22})+\Phi(b_{22})$.

Proof. Choose an element $c=c_{11}+c_{12}+c_{21}+c_{22}\in\mathcal{A}$ such that $\Phi(c)=\Phi(a_{11})+\Phi(b_{11})$. By Claim 7(i), for an arbitrary element $d_{12}\in\mathcal{A}_{12}$ we have

$$
\begin{aligned}
\Phi(d_{12}\circ c+cd_{12}^*) &= \Phi(d_{12}\circ a_{11}+a_{11}d_{12}^*)+\Phi(d_{12}\circ b_{11}+b_{11}d_{12}^*)\\
&= \Phi(\frac{1}{2}a_{11}d_{12})+\Phi(\frac{1}{2}b_{11}d_{12})=\Phi(\frac{1}{2}(a_{11}+b_{11})d_{12}).
\end{aligned}
$$

This shows that $d_{12}\circ c+cd_{12}^*=\frac{1}{2}(a_{11}+b_{11})d_{12}$ which implies $c_{11}=a_{11}+b_{11}$ and $c_{12}=c_{21}=c_{22}=0$.

Similarly, we prove the case (ii). □

Claim 9. Φ is an additive mapping.

Proof. The result is an immediate consequence of Claims 6, 7 and 8. □

Lemma 2. *Let $\mathcal{B}$ be a factor von Neumann algebra with identity $1_{\mathcal{B}}$ and an element $a \in \mathcal{B}$. If $\mathcal{B}$ satisfies the condition $a \circ b + ba^* = 0$, for all element $b \in \mathcal{B}$, then $a \in i\mathbb{R}1_{\mathcal{B}}$ (i is the imaginary number unit).*

Proof. When $b = 1_{\mathcal{B}}$, we get that $a + a^* = 0$. Replacing this last result in given condition, we obtain $ab - ba = 0$, for all element $b \in \mathcal{B}$. This implies that a belongs to the center of $\mathcal{B}$ which results that $a \in i\mathbb{R}1_{\mathcal{B}}$. □

In the remainder of this paper, all lemmas satisfy the conditions of the main theorem.

Lemma 3. $\Phi(i\mathbb{R}1_{\mathcal{A}}) = i\mathbb{R}1_{\mathcal{B}}$ *and* $\Phi(\mathbb{C}1_{\mathcal{A}}) = \mathbb{C}1_{\mathcal{B}}$.

Proof. For arbitrary elements $\lambda \in \mathbb{R}$ and $a \in \mathcal{A}$ we have

$$\begin{aligned} 0 &= \Phi((i\lambda 1_{\mathcal{A}})a + (i\lambda 1_{\mathcal{A}})^* a) = \Phi((i\lambda 1_{\mathcal{A}}) \circ a + a(i\lambda 1_{\mathcal{A}})^*) \\ &= \Phi(i\lambda 1_{\mathcal{A}}) \circ \Phi(a) + \Phi(a)\Phi(i\lambda 1_{\mathcal{A}})^*. \end{aligned}$$

By Lemma 2 results that $\Phi(i\lambda 1_{\mathcal{A}}) \in i\mathbb{R}1_{\mathcal{B}}$. Since λ is an arbitrary real number, then we conclude that $\Phi(i\mathbb{R}1_{\mathcal{A}}) \subseteq i\mathbb{R}1_{\mathcal{B}}$. Note that Φ^{-1} has the same properties that Φ. Thus, applying a similar argument to the above we obtain that $i\mathbb{R}1_{\mathcal{B}} \subseteq \Phi(i\mathbb{R}1_{\mathcal{A}})$. Consequently, $\Phi(i\mathbb{R}1_{\mathcal{A}}) = i\mathbb{R}1_{\mathcal{B}}$.

Let $a \in \mathcal{A}$ such that $a^* = -a$ (called an *anti-self-adjoint element*). Then $0 = \Phi((i1_{\mathcal{A}})a + (i1_{\mathcal{A}})a^*) = \Phi(a \circ (i1_{\mathcal{A}}) + (i1_{\mathcal{A}})a^*) = \Phi(a) \circ \Phi(i1_{\mathcal{A}}) + \Phi(i1_{\mathcal{A}})\Phi(a)^*$ which implies that $\Phi(a) = -\Phi(a)^*$. Since Φ^{-1} has the same properties of Φ, then applying a similar reasoning we prove that $\Phi(a) = -\Phi(a)^*$ implies $a = -a^*$. Consequently, $a = -a^*$ if and only if $\Phi(a) = -\Phi(a)^*$ for any such element $a \in \mathcal{A}$. From this fact, for arbitrary elements $\lambda \in \mathbb{C}$ and $a \in \mathcal{A}$ such that $a^* = -a$, we have $0 = \Phi((\lambda 1_{\mathcal{A}})a + (\lambda 1_{\mathcal{A}})a^*) = \Phi(a \circ (\lambda 1_{\mathcal{A}}) + (\lambda 1_{\mathcal{A}})a^*) = \Phi(a) \circ \Phi(\lambda 1_{\mathcal{A}}) + \Phi(\lambda 1_{\mathcal{A}})\Phi(a)^*$ which implies that $\Phi(a)\Phi(\lambda 1_{\mathcal{A}}) = \Phi(\lambda 1_{\mathcal{A}})\Phi(a)$. It follows that $b\Phi(\lambda 1_{\mathcal{A}}) = \Phi(\lambda 1_{\mathcal{A}})b$, for any element $b \in \mathcal{B}$ such that $b^* = -b$. As consequence, we have $b\Phi(\lambda 1_{\mathcal{A}}) = \Phi(\lambda 1_{\mathcal{A}})b$, for an arbitrary element $b \in \mathcal{B}$, since we can write any element $b \in \mathcal{B}$ in the form $b = b_1 + ib_2$, where $b_1 = \frac{b-b^*}{2}$ and $b_2 = \frac{b+b^*}{2i}$ are anti-self-adjoint elements of $\mathcal{B}$. This results that $\Phi(\lambda 1_{\mathcal{A}}) \in \mathbb{C}1_{\mathcal{B}}$. It follows that $\Phi(\mathbb{C}1_{\mathcal{A}}) \subseteq \mathbb{C}1_{\mathcal{B}}$. Since Φ^{-1} has the same properties of Φ, then applying a similar argument we obtain $\mathbb{C}1_{\mathcal{B}} \subseteq \Phi(\mathbb{C}1_{\mathcal{A}})$. Consequently, $\Phi(\mathbb{C}1_{\mathcal{A}}) = \mathbb{C}1_{\mathcal{B}}$. □

Lemma 4. $\Phi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$ *and there is a real number ν satisfying $\nu^2 = 1$ such that $\Phi(ia) = i\nu\Phi(a)$, for all element $a \in \mathcal{A}$.*

Proof. By Lemma 3, there is a nonzero element $\lambda \in \mathbb{C}$ such that $\Phi(1_{\mathcal{A}}) = \lambda 1_{\mathcal{B}}$. Fix a nontrivial projection $p \in \mathcal{A}$. It follows that $2\Phi(p) = \Phi(2p) = \Phi(p \circ 1_{\mathcal{A}} + 1_{\mathcal{A}} p^*) = \Phi(p) \circ \Phi(1_{\mathcal{A}}) + \Phi(1_{\mathcal{A}})\Phi(p)^* = \lambda(\Phi(p) + \Phi(p)^*)$. This allows us to write $\Phi(p) = \lambda q$, where $q = \frac{1}{2}(\Phi(p) + \Phi(p)^*)$. Also, $2\Phi(p) = \Phi(2p) = \Phi(1_{\mathcal{A}} \circ p + p1_{\mathcal{A}}^*) = \Phi(1_{\mathcal{A}}) \circ \Phi(p) + \Phi(p)\Phi(1_{\mathcal{A}})^* = (\lambda + \overline{\lambda})\Phi(p)$ which results that $\lambda + \overline{\lambda} = 2$. Yet, $2\Phi(p) = \Phi(2p) = \Phi(p \circ p + pp^*) = \Phi(p) \circ \Phi(p) + \Phi(p)\Phi(p)^* = \Phi(p)\Phi(p) + \Phi(p)\Phi(p)^* = (\lambda^2 + \lambda\overline{\lambda})q^2 = 2\lambda q^2$ which implies that $q = q^2$. This show that q is a non-trivial projection of $\mathcal{B}$.

Let a be an element of $\mathcal{A}$ such that $s = pap^{\perp}$ is a nonzero element. Then $\frac{1}{2}\Phi(s) = \Phi(\frac{1}{2}s) = \Phi(p \circ s + sp^*) = \Phi(p) \circ \Phi(s) + \Phi(s)\Phi(p)^* = (\lambda q) \circ \Phi(s) + \overline{\lambda}\Phi(s)q$ which implies $q\Phi(s)q = 0$, $q^{\perp}\Phi(s)q^{\perp} = 0$, $(1-\lambda)q\Phi(s)q^{\perp} = 0$ and $(1+\overline{\lambda})q^{\perp}\Phi(s)q = 0$. If $\lambda \neq \pm 1$, then $q\Phi(s)q^{\perp} = 0$ and $q^{\perp}\Phi(s)q = 0$ which implies that $\Phi(s) = 0$ resulting in $s = 0$ which is a contradiction. Thus we must have $\lambda = 1$ which results $\Phi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$, $q^{\perp}\Phi(s)q = 0$ and $\Phi(s) = q\Phi(s)q^{\perp}$. Let ν be a nonzero real number such that $\Phi(i1_{\mathcal{A}}) = i\nu 1_{\mathcal{B}}$, by Lemma 3. Then $2\Phi(ip) = \Phi(p \circ (i1_{\mathcal{A}}) + (i1_{\mathcal{A}})p^*) = \Phi(p) \circ \Phi(i1_{\mathcal{A}}) + \Phi(i1_{\mathcal{A}})\Phi(p)^* = 2i\nu q$ which implies that $\Phi(ip) = i\nu q$. It follows that $\frac{1}{2}\Phi(is) = \Phi(\frac{1}{2}is) = \Phi((ip) \circ s + s(ip)^*) = \Phi(ip) \circ \Phi(s) + \Phi(s)\Phi(ip)^* = (i\nu q) \circ \Phi(s) + \Phi(s)(i\nu q)^* = \frac{1}{2}i\nu\Phi(s)$ which yields $\Phi(is) = i\nu\Phi(s)$. Moreover, as $-\frac{1}{2}\Phi(s) = \Phi(-\frac{1}{2}s) = \Phi((ip) \circ (is) + (is)(ip)^*) = \Phi(ip) \circ \Phi(is) + \Phi(is)\Phi(ip)^* = (i\nu q) \circ (i\nu\Phi(s)) + (i\nu\Phi(s))(i\nu q)^* = -\frac{1}{2}\nu^2\Phi(s)$ we still get that $\nu^2 = 1$.

Let t be an element of $\mathcal{A}$ such that $t = t^*$. Then $2\Phi(t) = \Phi(2t) = \Phi(t \circ 1_{\mathcal{A}} + 1_{\mathcal{A}}t^*) = \Phi(t) \circ \Phi(1_{\mathcal{A}}) + \Phi(1_{\mathcal{A}})\Phi(t)^* = \Phi(t) + \Phi(t)^*$ which implies that $\Phi(t) = \Phi(t)^*$. This results that $2\Phi(it) = \Phi(2it) = \Phi(t \circ (i1_{\mathcal{A}}) + (i1_{\mathcal{A}})t^*) = \Phi(t) \circ \Phi(i1_{\mathcal{A}}) + \Phi(i1_{\mathcal{A}})\Phi(t)^* = \Phi(t) \circ (i\nu 1_{\mathcal{B}}) + (i\nu 1_{\mathcal{B}})\Phi(t)^*$ which allows us to conclude that $\Phi(it) = i\nu\Phi(t)$. Therefore, for an arbitrary element $a \in \mathcal{A}$, let us write $a = a_1 + ia_2$, where $a_1 = \frac{a+a^*}{2}$ and $a_2 = \frac{a-a^*}{2i}$ are self-adjoint elements of $\mathcal{A}$. Then we conclude that $\Phi(ia) = \Phi(ia_1 - a_2) = \Phi(ia_1) - \Phi(a_2) = i\nu\Phi(a_1) + i\nu\Phi(ia_2) = i\nu\Phi(a_1 + a_2) = i\nu\Phi(a)$. $\square$

Lemma 5. *$\Phi : \mathcal{A} \to \mathcal{B}$ is a $*$-ring isomorphism.*

Proof. For arbitrary elements $a, b \in \mathcal{A}$, we have

$$\begin{aligned}\Phi(a \circ b - ba^*) &= \Phi((ia) \circ (-ib) + (-ib)(ia)^*)\\ &= \Phi(ia) \circ \Phi(-ib) + \Phi(-ib)\Phi(ia)^*\\ &= (i\nu\Phi(a)) \circ (-i\nu\Phi(b)) + (-i\nu\Phi(b))(i\nu\Phi(a))^*\\ &= \Phi(a) \circ \Phi(b) - \Phi(b)\Phi(a)^*. \qquad (2)\end{aligned}$$

Thus, from (1) and (2) we get $\Phi(a\circ b) = \Phi(a)\circ\Phi(b)$ and $\Phi(ba^*) = \Phi(b)\Phi(a)^*$ and, as consequence, we deduce that $\Phi(a^*) = \Phi(1_{\mathcal{A}}a^*) = \Phi(1_{\mathcal{A}})\Phi(a)^* = 1_{\mathcal{B}}\Phi(a)^* = \Phi(a)^*$ and $\Phi(ab) = \Phi(a(b^*)^*) = \Phi(a)\Phi(b^*)^* = \Phi(a)(\Phi(b)^*)^* = \Phi(a)\Phi(b)$. This allows us to conclude that Φ is a $*$-ring isomorphism.

The theorem is proved. $\square$

From the main theorem and the fact that every ring isomorphism between type I factor von Neumann algebras is spatial, we have the following corollary.

Corollary 1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two type I factor von Neumann algebras acting on complex Hilbert spaces H and K, respectively. Then every bijective mapping $\Phi : \mathcal{A} \to \mathcal{B}$ satisfies $\Phi(a\circ b + ba^*) = \Phi(a)\circ\Phi(b) + \Phi(b)\Phi(a)^*$, for all elements $a, b \in \mathcal{A}$, if and only if there exists a unitary or conjugate unitary operator $U : H \to K$ such that $\Phi(a) = UaU^*$, for all element $a \in \mathcal{A}$.*

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Contact information

João Carlos da Motta Ferreira, Maria das Graças Bruno Marietto

Center for Mathematics, Computation and Cognition, Federal University of ABC, Avenida dos Estados, 5001, 09210-580, Santo André, Brazil

E-Mail(s): joao.cmferreira@ufabc.edu.br, graca.marietto@ufabc.edu.br

Web-page(s): http://www.ufabc.edu.br

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