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Volume **31** (2021). Number 1, pp. 17–36
DOI:10.12958/adm1734

Spectra of locally matrix algebras

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Communicated by A. P. Petravchuk

Abstract. We describe spectra of associative (not necessarily unital and not necessarily countable-dimensional) locally matrix algebras. We determine all possible spectra of locally matrix algebras and give a new proof of Dixmier–Baranov Theorem. As an application of our description of spectra, we determine embeddings of locally matrix algebras.

Introduction

Let $\mathbb{F}$ be a ground field. Recall that an associative $\mathbb{F}$-algebra A is called a *locally matrix algebra* (see [10]) if for an arbitrary finite subset of A there exists a subalgebra $B \subset A$ containing this subset and such that B is isomorphic to a matrix algebra $M_n(\mathbb{F})$ for some $n \geqslant 1$. In what follows we will sometimes identify B and $M_n(\mathbb{F})$, that is, assume that $M_n(\mathbb{F}) \subset A$. We call a locally matrix algebra *unital* if it contains 1.

Let A be a countable-dimensional unital locally matrix algebra. In [7], J.G. Glimm defined the Steinitz number $\mathbf{st}(A)$ of the algebra A and proved that A is uniquely determined by $\mathbf{st}(A)$. J. Dixmier [5] showed that non-unital countable-dimensional locally matrix algebras over the field of complex numbers can be parameterized by pairs (s, α), where s is a Steinitz number and α is a nonnegative real number. A.A. Baranov [1] extended this parametrization to locally matrix algebras over arbitrary fields.

2020 MSC: 08A05, 16S50.

Key words and phrases: locally matrix algebra, Steinitz number, spectrum, embedding.

In [2], we defined the Steinitz number $\mathbf{st}(A)$ for a unital locally matrix algebra A of an arbitrary dimension. We showed that for a unital locally matrix algebra A of dimension $> \aleph_0$ the Steinitz number $\mathbf{st}(A)$ no longer determines A; see [3, 4]. However, it determines the universal elementary theory of A [3].

In this paper for an arbitrary (not necessarily unital and not necessarily countable-dimensional) locally matrix algebra A, we define a subset of $\mathbb{SN}$ that we call the spectrum of A and denote as $\operatorname{Spec}(A)$. We determine all possible spectra of locally matrix algebras and give a new proof of Dixmier–Baranov Theorem. As an application of our description of spectra, we determine embeddings of locally matrix algebras.

1. Spectra of locally matrix algebras

Let $\mathbb{P}$ be the set of all primes and $\mathbb{N}$ be the set of all positive integers. A *Steinitz number* (see [11]) is an infinite formal product of the form

$$\prod_{p\in\mathbb{P}} p^{r_p} ,$$

where $r_p \in \mathbb{N} \cup \{0, \infty\}$ for all $p \in \mathbb{P}$.

Denote by $\mathbb{SN}$ the set of all Steinitz numbers. Notice, that the set of all positive integers $\mathbb{N}$ is a subset of $\mathbb{SN}$. The numbers $\mathbb{SN} \setminus \mathbb{N}$ are called *infinite* Steinitz numbers.

Let A be a locally matrix algebra with a unit 1 over a field F and let $D(A)$ be the set of all positive integers n such that there is a subalgebra A', $1 \in A' \subseteq A$, $A' \cong M_n(F)$. Then the least common multiple of the set $D(A)$ is called the *Steinitz number of the algebra* A and denoted as $\mathbf{st}(A)$; see [2].

Now let A be a (not necessarily unital) locally matrix algebra. For an arbitrary idempotent $0 \neq e \in A$ the subalgebra eAe is a unital locally matrix algebra. That is why we can talk about its Steinitz number $\mathbf{st}(eAe)$. The subset

$$\operatorname{Spec}(A) = \{ \mathbf{st}(eAe) \mid e \in A, \ e \neq 0, \ e^2 = e \},$$

where e runs through all nonzero idempotents of the algebra A, is called the *spectrum* of the algebra A.

For a Steinitz number s let $\Omega(s)$ denote the set of all natural numbers $n \in \mathbb{N}$ that divide s.

For a Steinitz numbers s_1, s_2 we say that s_1 *finitely divides* s_2 if there exists $b \in \Omega(s_2)$ such that $s_1 = s_2/b$ (we denote: $s_1 \mid_{fin} s_2$).

Steinitz numbers s_1, s_2 are *rationally connected* if $s_2 = q \cdot s_1$, where q is some rational number.

We call a subset $S \subset \mathbb{SN}$ *saturated* if

1) any two Steinitz numbers from S are rationally connected;
2) if $s_2 \in S$ and $s_1 \,\big|_{fin}\, s_2$ then $s_1 \in S$;
3) if s, $ns \in S$, where $n \in \mathbb{N}$, then $is \in S$ for any i, $1 \leqslant i \leqslant n$.

Theorem 1. *For an arbitrary locally matrix algebra A its spectrum is a saturated subset of* $\mathbb{SN}$.

Let us consider examples of saturated subsets of $\mathbb{SN}$.

Example 1. For an arbitrary natural number n the set $\{1, 2, \ldots, n\}$ is saturated.

Example 2. Let s be a Steinitz number. The set

$$S(\infty, s) \ := \ \left\{ \ \frac{a}{b} \ \cdot \ s \ \middle| \ a \ \in \ \mathbb{N}, \ b \ \in \ \Omega(s) \ \right\}$$

is saturated. For an arbitrary Steinitz number $s' \in S(\infty, s)$ we have $S(\infty, s) = S(\infty, s')$. If $s \in \mathbb{N}$ then $S(\infty, s) = \mathbb{N}$.

Example 3. Let r be a real number, $1 \leqslant r < \infty$. Let s be an infinite Steinitz number. The set

$$S(r, s) \ = \ \left\{ \ \frac{a}{b} \ s \ \middle| \ a, \ b \ \in \ \mathbb{N}; \ b \ \in \ \Omega(s), \ a \ \leqslant \ rb \ \right\}$$

is saturated.

Example 4. Let s be an infinite Steinitz number and let $r = u/v$ be a rational number; $u, v \in \mathbb{N}$, $v \in \Omega\ (s)$. Then the set

$$S^{+}(r, s) = \left\{ \frac{a}{b} s \ \middle| \ a, b \in \mathbb{N}; \ b \in \Omega(s), \ a < rb \right\}$$

is saturated.

Theorem 2. *Every saturated subset of* $\mathbb{SN}$ *is one of the following sets:*

1) $\{1, 2, \ldots, n\}$, $n \in \mathbb{N}$, *or* $\mathbb{N}$;
2) $S(\infty, s)$, $s \in \mathbb{SN} \setminus \mathbb{N}$;
3) $S(r, s)$, *where* $s \in \mathbb{SN} \setminus \mathbb{N}$, $r \in [1, \infty)$;
4) $S^{+}(r, s)$, *where* $s \in \mathbb{SN} \setminus \mathbb{N}$, $r = u/v$, $u \in \mathbb{N}$, $v \in \Omega(s)$.

Remark 1. The real number r above is the inverse of the density invariant of Dixmier–Baranov.

Theorem 3. (1) *For any saturated subset* $S \subseteq \mathbb{SN}$ *there exists a countable-dimensional locally matrix algebra* A *such that* $\mathrm{Spec}(A) = S$.

(2) *If* A, B *are countable-dimensional locally matrix algebras and* $\mathrm{Spec}(A) = \mathrm{Spec}(B)$ *then* $A \cong B$.

Remark 2. The part (2) of Theorem 3 is a new proof of Dixmier–Baranov Theorem.

Which spectra above correspond to unital algebras?

Theorem 4. *A locally matrix algebra* A *is unital if and only if* $\mathrm{Spec}(A) = \{1, 2, \dots, n\}$, *where* $n \in \mathbb{N}$, *or* $\mathrm{Spec}(A) = S(r, s)$, *where* $s \in \mathbb{SN} \setminus \mathbb{N}$, $r = u/v$, $u, v \in \mathbb{N}$, $v \in \Omega(s)$.

Proof of Theorem 1. In what follows, we assume that A is a locally matrix $\mathbb{F}$-algebra. Recall the partial order on the set of all idempotents of A : for idempotents e, $f \in A$ we define $e \geqslant f$ if $f \in e\,A\,e$.

We claim that for arbitrary idempotents e_1, $e_2 \in A$ there exists an idempotent $e_3 \in A$ such that $e_1 \leqslant e_3$, $e_2 \leqslant e_3$. Indeed, there exists a subalgebra $A' \subset A$ such that e_1, $e_2 \in A'$ and $A' \cong M_n(\mathbb{F})$, $n \geqslant 1$. Let e_3 be the identity element of the subalgebra A'. Then $e_1 \leqslant e_3$, $e_2 \leqslant e_3$.

Now suppose that the locally matrix algebra A is unital. Let $a \in A$. Choose a subalgebra $A' \subset A$ such that $1 \in A'$, $a \in A'$ and $A' \cong M_n(\mathbb{F})$, $n \geqslant 1$. Let r be the range of the matrix a in A'. Let

$$r(a) \;=\; \frac{r}{n}, \quad 0 \leqslant r(a) \leqslant 1.$$

V.M. Kurochkin [9] noticed that the number $r(a)$ does not depend on a choice of a subalgebra A'. We call $r(a)$ the *relative range* of the element a. In [4], we showed that if A is a unital locally matrix algebra and $e \in A$ is an idempotent, then $\mathbf{st}(eAe) = r(e) \cdot \mathbf{st}(A)$.

Now let A be a not necessarily unital locally matrix algebra. Let e_1, $e_2 \in A$ be idempotents. Choose an idempotent $e_3 \in A$ such that $e_1 \leqslant e_3$, $e_2 \leqslant e_3$, i.e. e_1, $e_2 \in e_3 A e_3$. Let q_1, q_2 be relative ranges of the idempotents e_1, e_2 in the unital locally matrix algebra $e_3 A e_3$. Then

$$\mathbf{st}(e_1\,A\,e_1) \;=\; q_1\;\mathbf{st}(e_3\,A\,e_3), \qquad \mathbf{st}(e_2\,A\,e_2) \;=\; q_2\;\mathbf{st}(e_3\,A\,e_3).$$

This implies that the Steinitz numbers $\mathbf{st}(e_1\,A\,e_1)$, $\mathbf{st}(e_2\,A\,e_2)$ are rationally connected. We have checked the condition 1) from the definition of saturated sets.

Let $0 \neq e \in A$ be an idempotent. Let $s_2 = \mathbf{st}(eAe)$, $k \in \Omega(s_2)$ and let $s_1 = s_2/k$. The unital locally matrix algebra eAe contains a subalgebra $e \in M_k(\mathbb{F}) \subset eAe$. Consider the matrix unit e_{11} of the algebra $M_k(\mathbb{F})$. The relative range of the idempotent e_{11} in the unital algebra eAe is equal to $1/k$. Hence

$$\mathbf{st}(e_{11}\, A\, e_{11}) \;=\; \frac{1}{k}\, \mathbf{st}(e\, A\, e) \;=\; s_1, \quad s_1 \in \mathrm{Spec}(A).$$

We have checked the condition 2).

Now let $n \geqslant 1$. Suppose that Steinitz numbers s and ns lie in $\mathrm{Spec}(A)$. It means that there exist idempotents $e_1, e_2 \in A$ such that $s = \mathbf{st}(e_1 A e_1)$, $ns = \mathbf{st}(e_2 A e_2)$. There exists a matrix subalgebra $M_k(\mathbb{F}) \subset A$ that contains e_1 and e_2. As above, let e_3 be the identity element of the algebra $M_k(\mathbb{F})$. Let $\mathrm{rk}(e_i)$ be the range of the idempotent e_i in the matrix algebra $M_k(\mathbb{F})$. We have

$$s = \frac{\mathrm{rk}(e_1)}{k} \cdot \mathbf{st}(e_3\, A\, e_3), \qquad n\, s \;=\; \frac{\mathrm{rk}(e_2)}{k} \cdot \mathbf{st}(e_3\, A\, e_3),$$

which implies $\mathrm{rk}(e_2) = n{\cdot}\mathrm{rk}(e_1)$. In particular, $n{\cdot}\mathrm{rk}(e_1) \leqslant k$. Let $1 \leqslant i \leqslant n$. Consider the idempotent

$$e = \mathrm{diag}\;\big(\,\underbrace{1,\ 1,\ \ldots,\ 1,}_{i\cdot \mathrm{rk}(e_1)}\ 0,\ 0,\ \ldots,\ 0\,\big)$$

in the matrix algebra $M_k(\mathbb{F})$. We have

$$\mathbf{st}(e\, A\, e) = \frac{i \cdot \mathrm{rk}(e_1)}{k} \cdot \mathbf{st}(e_3\, A\, e_3) = i \;\cdot\; \mathbf{st}(e_1\, A\, e_1) = i\ s.$$

We showed that $is \in \mathrm{Spec}(A)$. Hence $\mathrm{Spec}(A)$ is a saturated subset of $\mathbb{SN}$. It completes the proof of Theorem 1. □

2. Classification of saturated subsets of $\mathbb{SN}$

Our aim in this section is to classify all saturated subsets of $\mathbb{SN}$. We remark that if at least one Steinitz number from a saturated set S is infinite then by the condition 1) all Steinitz numbers from S are infinite.

Let S be a saturated subset of $\mathbb{SN}$. For a Steinitz number $s \in S$ and for a natural number $b \in \Omega(s)$ let

$$r_s(b) \;=\; \max\,\Big\{\, i \;\geqslant\; 1 \;\Big|\; i \;\cdot\; \frac{s}{b} \;\in\; S \,\Big\}.$$

Lemma 1. *If there exists a Steinitz number* $s_0 \in S$ *and a natural number* $b_0 \in \Omega\ (s_0)$ *such that* $r_{s_0}(b_0) = \infty$ *then for any* $s \in S$ *and any* $b \in \Omega\ (s)$ *we have* $r_s(b) = \infty$.

Proof. Let us show at first that $r_{s_0}(b) = \infty$ for any $b \in \Omega(s_0)$. Indeed, there exists a natural number $c \in \Omega(s_0)$ such that both b_0 and b divide c. Then for an arbitrary $i \geqslant 1$ we have

$$i \cdot \frac{s_0}{b_0} = \left(i \cdot \frac{c}{b_0} \right) \cdot \frac{s_0}{c} \in S.$$

This implies that $r_{s_0}(c) = \infty$. Hence,

$$i \cdot \frac{s_0}{b} = \left(i \cdot \frac{c}{b} \right) \cdot \frac{s_0}{c} \in S,$$

which proves the claim.

Now choose an arbitrary Steinitz number $s \in S$. By the condition 1), the Steinitz numbers s and s_0 are rationally connected, i.e. there exist $a \in \mathbb{N}$, $b \in \Omega(s_0)$ such that $s = (a/b) \cdot s_0$. By the condition 2), $s_0/b \in S$. Choose a natural number $c \in \Omega(s_0/b)$. Then $c \in \Omega(s)$ and $bc \in \Omega(s_0)$. For an arbitrary $i \geqslant 1$ we have $i \cdot s/c = i \cdot a \cdot s_0/(bc) \in S$ since $r_{s_0}(bc) = \infty$. This implies $r_s(c) = \infty$ and completes the proof of the lemma. □

If a saturated set satisfies the assumptions of Lemma 1 then it is referred to as a set of *infinite type*. Otherwise, we talk about a saturated set of *finite type*.

Lemma 2. 1) *For an arbitrary Steinitz number* $s_0 \in \mathbb{SN}$ *the set*

$$S(\infty, s_0) := \left\{ \frac{a}{b} \cdot s_0 \;\middle|\; a \in \mathbb{N},\ b \in \Omega(s_0) \right\}$$

is a saturated set of infinite type.

2) *If* S *is a saturated set of infinite type, then for an arbitrary Steinitz number* $s \in S$ *we have* $S = S(\infty, s)$.

Proof. We have to show that the set $S(\infty, s_0)$ satisfies the conditions 1), 2), 3). The condition 1) is obvious. Let $s = (a/b) \cdot s_0$, $b \in \Omega(s_0)$. Without loss of generality, we assume that a and b are coprime. Let $c \in \Omega(s)$ and let $d = \gcd(c, a)$ be the greatest common divisor of a and c, $a = a'd$, $c = c'd$, the numbers a', c' are coprime. Then $a \cdot s_0/(bc) = a' \cdot s_0/(bc')$, which implies that $dc' \in \Omega(s_0)$. Hence

$$\frac{s}{c} = \frac{a}{bc} \cdot s_0 = \frac{a'}{bc'} \cdot s_0 \in S(\infty, s_0).$$

We have checked the condition 2).

Let us check the condition 3). Choose $s = (a/b) \cdot s_0 \in S(\infty, s_0)$, $b \in \Omega(s_0)$. Let $c \in \Omega(s)$. We need to check that for any $i \geqslant 1$

$$i \cdot \frac{s}{c} = \frac{ia}{bc} \cdot s_0 \in S(\infty, s_0).$$

Let $a/(bc) = a'/b'$, where the natural numbers a', b' are coprime. Since

$$\frac{a}{bc} \cdot s_0 = \frac{s}{c} \in \mathbb{SN}$$

it follows that $b' \in \Omega(s_0)$. Hence, $i \cdot (a'/b') \cdot s_0 \in S(\infty, s_0)$, which implies that $S(\infty, s_0)$ satisfies the condition 3) and, therefore, is saturated.

Let S be a saturated subset of $\mathbb{SN}$ of infinite type. Choose $s_0 \in S$. Our aim is to show that $S = S(\infty, s_0)$. Since the subset S is of infinite type it follows that $r_s(b) = \infty$ for any $s \in S$, $b \in \Omega(s)$. In particular,

$$S(\infty, s_0) = \left\{ \frac{a}{b} \cdot s_0 \;\middle|\; s \in \Omega(s_0) \right\} \subseteq S.$$

An arbitrary Steinitz number $s \in S$ is rationally connected to s_0, hence there exist a, $b \in \mathbb{N}$ such that $s = (a/b) \cdot s_0$. Without loss of generality, we assume that a and b are coprime, which implies $b \in \Omega(s_0)$. We proved that $s \in S(\infty, s_0)$. □

Now let $S \subset \mathbb{SN}$ be a saturated subset of finite type, that is, for any $s \in S$, $d \in \Omega\ (s)$ we have

$$r_s(b) = \max \left\{ i \in \mathbb{N} \;\middle|\; i \cdot \frac{s}{b} \in S \right\} < \infty.$$

By the condition 3),

$$\left\{ i \in \mathbb{N} \;\middle|\; i \cdot \frac{s}{b} \in S \right\} = [\, 1\, ,\ r_s(b)\,].$$

Since $b \cdot (s/b) \in S$ it follows that $b \leqslant r_s(b)$. Choose a Steinitz number $s \in S$ and two natural numbers b, $c \in \Omega(s)$ such that b divides c. If $i \cdot (s/b) \in S$ then $(ic/b) \cdot (s/c) \in S$. Hence $r_s(b) \cdot (c/b) \leqslant r_s(c)$. In other words,

$$\frac{r_s(b)}{b} \leqslant \frac{r_s(c)}{c}. \tag{1}$$

Let $i \in \mathbb{N}$, $s/c \in S$ and let k be a maximal nonnegative integer such that $k \cdot (c/b) \leqslant i$. By the condition 3), $k \cdot (c/b) \cdot (s/c) \in S$, hence $k \cdot (s/b) \in S$. So, $k \leqslant r_s(b)$. We proved that

$$\left[\frac{r_s(c)}{c\,/\,b} \right] \leqslant r_s(b). \tag{2}$$

The inequalities (1), (2) imply

$$\Big[\,\frac{r_s(c)}{c\,/\,b}\,\Big] \;\leqslant\; r_s(b) \;\leqslant\; \frac{r_s(c)}{c\,/\,b}.$$

Hence

$$r_s(b) \;=\; \Big[\,\frac{r_s(c)}{c\,/\,b}\,\Big]. \tag{3}$$

In particular,

$$\frac{r_s(c)}{c\,/\,b} \;-\; 1 \;<\; r_s(b), \quad \frac{r_s(c)}{c\,/\,b} \;<\; r_s(b) \;+\; 1.$$

Dividing by b, we get

$$\frac{r_s(b)}{b} \;\leqslant\; \frac{r_s(c)}{c} \;<\; \frac{r_s(b)}{b} \;+\; \frac{1}{b}. \tag{4}$$

Lemma 3. *Let $S \subset \mathbb{SN}$ be a saturated subset of finite type and let $s \in S$ be an infinite Steinitz number. Then there exists a limit*

$$r_S(s) \;=\; \lim_{\substack{b\,\in\,\Omega(s)\\ b\to\infty}} \frac{r_s(b)}{b}\,, \quad 1 \;\leqslant\; r_S(s) \;<\infty.$$

If the set S is fixed then we denote $r_S(s) = r(s)$.

Remark 3. The limit $r(s)$ is equal to the inverse of the density invariant of Dixmier–Baranov [1,5].

The proof of Lemma 3. The set $\{r_s(b)/b \mid b \in \Omega(s)\}$ is bounded from above. Indeed, choose $b_0 \in \Omega(s)$. For an arbitrary $b \in \Omega(s)$ there exists $c \in \Omega(s)$ that is a common multiple for b_0 and b. Then by (1) and (4),

$$\frac{r(b)}{b} \;\leqslant\; \frac{r(c)}{c} \;<\; \frac{r(b_0)}{b_0} \;+\; \frac{1}{b_0}.$$

Let

$$r \;=\; r(s) \;=\; \sup\;\Big\{\,\frac{r_s(b)}{b}\;\Big|\; b \;\in\; \Omega(s)\,\Big\}.$$

Clearly, $1 \leqslant r < \infty$. Choose $\varepsilon > 0$. Let $N(\varepsilon) = [2r/\varepsilon] + 1$. We will show that for any $b \in \Omega\ (s)$, $b \geqslant N(\varepsilon)$, we have $r - \varepsilon < r_s(b)/b$.

Indeed, let $b \in \Omega(s)$, $b \geqslant N(\varepsilon) > 2r/\varepsilon$. Then $1/b < \varepsilon/(2r) \leqslant \varepsilon/2$. There exists a natural number $b_0 \in \Omega\ (s)$ such that $r - \varepsilon/2 < r_s(b_0)/b_0$. Let $c \in \Omega(s)$ be a common multiple of b_0 and b. Then (4) implies

$$\frac{r(b)}{b} \;>\; \frac{r(c)}{c} \;-\; \frac{1}{b} \;\geqslant\; \frac{r(b_0)}{b_0} \;-\; \frac{1}{b} \;>\; r \;-\; \frac{\varepsilon}{2} \;-\; \frac{\varepsilon}{2} \;=\; r \;-\; \varepsilon.$$

So,

$$r = \lim_{\substack{b \in \Omega(s) \\ b \to \infty}} \frac{r_s(b)}{b}$$

and this completes the proof of the lemma. □

Lemma 4. *Let s, $s' \in S$ be infinite Steinitz numbers, $s' = (a/b) \cdot s$; a, $b \in \mathbb{N}$; $b \in \Omega(s)$. Then $r(s') = (a/b) \cdot r(s)$.*

Proof. It is sufficient to show that if s, $ms \in S$, $m \in \mathbb{N}$, then $m \cdot r(ms) = r(s)$.

Suppose that $b \in \Omega(s)$ and $i \cdot (ms/b) \in S$. Then $i \cdot m \cdot (s/b) \in S$. Hence $r_{ms}(b) \cdot m \leqslant r_s(b)$ and, therefore, $r(ms) \cdot m \leqslant r(s)$.

On the other hand, if $i \cdot (s/b) \in S$ then $[i/m] \cdot m \leqslant i$ and, therefore, $[i/m] \cdot m \cdot (s/b) \in S$. We showed that

$$\left[\frac{r_s(b)}{m} \right] \leqslant r_{ms}(b), \qquad \frac{r_s(b)}{m} - 1 < r_{ms}(b),$$

$$\frac{1}{m} \cdot \frac{r_s(b)}{b} - \frac{1}{b} < \frac{r_{ms}(b)}{b}.$$

Assuming $b \to \infty$ we get $(1/m) \cdot r(s) \leqslant r(ms)$, which completes the proof of the lemma. □

In the inequality (4), let $c \to \infty$. Then

$$\frac{r_s(b)}{b} \leqslant r(s) \leqslant \frac{r_s(b)}{b} + \frac{1}{b}, \quad r_s(b) \leqslant r(s)\, b \leqslant r_s(b) + 1.$$

If the number $r(s)$ is irrational then $r_s(b) = [r(s)b]$ for all $b \in \Omega(s)$.

Now suppose that the number $r = r_s(b)$ is rational; $r = u/v$; u, v are coprime. If a number $b \in \Omega(s)$ is not a multiple of v then, as above, $r_s(b) = [(u/v) \cdot b]$. If b is a multiple of v then

$$r_s(b) = \left[\begin{array}{ll} r\, b & \text{or} \\ r\, b - 1 & . \end{array} \right.$$

Lemma 5. *If at least for one number $b_0 \in \Omega(s) \bigcap v\mathbb{N}$ we have $r_s(b_0) = rb_0$ then for all $b \in \Omega(s) \bigcap v\mathbb{N}$ we have $r_s(b) = rb$.*

Proof. Let b, $c \in \Omega(s) \bigcap v\mathbb{N}$ and b divides c. If $r_s(b) = rb$ then, by the inequality (1), we have

$$r = \frac{r_s(b)}{b} \leqslant \frac{r_s(c)}{c},$$

which implies $r_s(c) = rc$. On the other hand, if $r_s(c) = rc$ then, by the inequality (4),

$$r = \frac{r_s(c)}{c} < \frac{r_s(b)}{b} + \frac{1}{b},$$

which implies $r_s(b) > rb - 1$. Hence $r_s(b) = rb$. We showed that $r_s(b) = rb$ if and only if $r_s(c) = rc$.

Now choose b_1, $b_2 \in \Omega(s) \bigcap v\mathbb{N}$ and suppose that $r_s(b_1) = rb_1$. There exists $c \in \Omega(s) \bigcap v\mathbb{N}$ such that both b_1 and b_2 divide c. In view of the above, $r_s(b_1) = rb_1$ implies $r_s(c) = rc$, which implies $r_s(b_2) = rb_2$. This completes the proof of the lemma. □

Recall that for an infinite Steinitz number s and a real number r, $1 \leqslant r < \infty$,

$$S(r, s) = \left\{ \frac{a}{b}\, s \;\middle|\; a,\ b \in \mathbb{N};\ b \in \Omega(s),\ a \leqslant rb \right\},$$

$$S^{+}(r, s) = \left\{ \frac{a}{b}\, s \;\middle|\; a,\ b \in \mathbb{N};\ b \in \Omega(s),\ a < rb \right\}.$$

If r is an irrational number or $r = u/v$, the integers u, v are coprime and $v \notin \Omega\ (s)$ then $S(r, s) = S^{+}(r, s)$. If $r = u/v$, $v \in \Omega\ (s)$ then $S^{+}(r, s) \subsetneqq S(r, s)$.

Lemma 6. *The subsets $S(r, s)$ and $S^{+}(r, s)$ are saturated.*

Proof. The condition 1) in the definition of saturated subsets is obviously satisfied. Let us check the condition 2). Let $(a/b) \cdot s \in S(r, s)$ (respectively, $(a/b) \cdot s \in S^{+}(r, s)$), where a, b are coprime natural numbers, $b \in \Omega(s)$. Then $a \leqslant rb$ (respectively, $a < rb$). Suppose that $c \in \Omega(\frac{a}{b} s)$. We need to show that $(a \cdot s)/(b \cdot c) \in S(r, s)$ (respectively, $(a \cdot s)/(b \cdot c) \in S^{+}(r, s)$). Let $d = \gcd(a, c)$, $a = da'$, $c = dc'$. Then

$$\frac{a}{b}\frac{s}{c} = \frac{a'}{b\,c'}\, s \in \mathbb{SN}.$$

Since the number bc' is coprime with a' it follows that $bc' \in \Omega(s)$. The inequality $a' \leqslant rbc'$ (respectively, $a' < rbc'$) is equivalent to the inequality $a \leqslant rbc$ (respectively, $a < rbc$). The latter inequality follows from $a \leqslant rb$ (respectively, $a < rb$). The condition 2) is verified.

Let us check the condition 3). As above, we assume that a, b are coprime natural numbers, $b \in \Omega(s)$ and $a/b \in S(r, s)$ (respectively, $a/b \in S^{+}(r, s)$). Let $c \in \Omega((a/b) \cdot s)$, $\gcd(a, c) = d$, $a = da'$, $c = dc'$. We have shown above that $bc' \in \Omega(s)$. Let $n \in \mathbb{N}$ and $n \cdot (as/(bc)) \in S(r, s)$

(respectively, $n \cdot (as/(bc)) \in S^+(r,s)$). Then $na' \leqslant rbc'$ (respectively, $na' < rbc'$). This immediately implies that for any i, $1 \leqslant i \leqslant n$, we have $ia' \leqslant rbc'$ (respectively, $ia' < rbc'$). Hence, $i \cdot (as/b) \in S(r,s)$ (respectively, $i \cdot (as/b) \in S^+(r,s)$). □

Lemma 7. *Let $r = u/v$, where u, v are coprime natural numbers. Let s be an infinite Steinitz number and $v \in \Omega\ (s)$. Then the set $S^+(r,s)$ is not equal to any of the sets $S(r',s')$, $r' \in [1,\infty)$, $s' \in \mathbb{SN}$.*

Proof. Let $s_2 \in S(r,s_1)$ (respectively, $s_2 \in S^+(r,s_1)$). Then $s_2 = (a/b)\cdot s_1$, where a, $b \in \mathbb{N}$, $b \in \Omega(s_1)$. By Lemma 4,

$$S(r,s_1) \;=\; S\Big(r\,\frac{b}{a}, s_2\Big) \;\; \Big(\text{respectively,} \;\; S^+(r,s_1) \;=\; S^+\Big(r\,\frac{b}{a}, s_2\Big)\Big).$$

We showed that the set $S(r,s)$ (respectively, $S^+(r,s)$) is determined by any Steinitz number $s' \in S(r,s)$ (respectively, $s' \in S^+(r,s)$) with an appropriate recalibration of r.

Let $S = S(r_1,s_1) = S^+(r_2,s_2)$. Choosing an arbitrary Steinitz number $s \in S$ we get $S(r'_1,s) = S^+(r'_2,s)$ for some r'_1, $r'_2 \in [1,\infty)$. The number $r'_2 = u/v$ is rational, $\gcd(u,v) = 1$ and $v \in \Omega(s)$.

The number r is uniquely determined by a saturated subset S and a choice of $s \in S$. Hence $r'_1 = r'_2$. Now it remains to notice that for a rational number $r = u/v$, $\gcd(u,v) = 1$, and an infinite Steinitz number s such that $v \in \Omega(s)$ we have $S(r,s) \neq S^+(r,s)$. This completes the proof of the lemma. □

Lemma 8. *Let $S \subset \mathbb{SN}\diagdown\mathbb{N}$ be a saturated subset of finite type, $s \in S$, $r = r_S(s) \in [1,\infty)$. Then $S = S(r,s)$ or $S = S^+(r,s)$.*

Proof. Recall that for a natural number $b \in \Omega(s)$ we defined

$$r_s(b) \;=\; \max\,\Big\{\; i \;\in\; \mathbb{N} \;\Big|\; i\,\frac{s}{b} \;\in\; S \;\Big\}.$$

We showed that if r is an irrational number or $r = u/v$; u, v are coprime and $v \notin \Omega\ (s)$, then $r_s(b) = [rb]$ for an arbitrary $b \in \Omega(s)$.

An arbitrary Steinitz number $s' \in S$ is representable as $s' = (a/b)\cdot s$, where a, b are coprime natural numbers. Clearly, $b \in \Omega(s)$ and $a \leqslant r_s(b) = [rb]$. That is why in the case when r is irrational or $r = u/v$, $\gcd(u,v) = 1$, $v \notin \Omega(s)$, we have $S = S(r,s) = S^+(r,s)$.

Suppose now that $r = u/v$, $\gcd(u,v) = 1$, $v \in \Omega(s)$. If $b \in \Omega(s)\diagdown v\mathbb{N}$ then as above $r_s(b) = [rb]$. By Lemma 5, either for all $b \in \Omega(s) \cap v\mathbb{N}$ we have $r_s(b) = rb$ or for all $b \in \Omega(s) \cap v\mathbb{N}$ we have $r_s(b) = rb - 1$. In the first case $S = S(r,s)$, in the second case $S = S^+(r,s)$. □

Lemma 9. *Let $S \subseteq \mathbb{N}$ be a saturated subset. Then either $S = \{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$ or $S = \mathbb{N}$.*

Proof. First, notice that the subsets $\{1, 2, \dots, n\}$ and $\mathbb{N}$ are saturated.

Now let $S \subseteq \mathbb{N}$ be a saturated subset. If $n \in S$ then $n \in \Omega(n)$ and $n \cdot (n/n) \in S$. By the condition 3), all natural numbers $i = i \cdot (n/n)$, $1 \leqslant i \leqslant n$, lie in S. This implies the assertion of the lemma. □

Now, Theorem 2 follows from Lemmas 8, 9.

3. Countable-dimensional locally matrix algebras

For an algebra A and an idempotent $0 \neq e \in A$ we call the subalgebra eAe a *corner* of the algebra A.

Let $A_1 \subset A_2 \subset \cdots$ be an ascending chain of unital locally matrix algebras, A_i is a corner of the algebra A_{i+1}, $i \geqslant 1$,

$$A = \bigcup_{i=1}^{\infty} A_i.$$

Clearly, $\mathrm{Spec}(A_1) \subseteq \mathrm{Spec}(A_2) \subseteq \cdots$.

Lemma 10.

$$\mathrm{Spec}(A) = \bigcup_{i=1}^{\infty} \mathrm{Spec}(A_i).$$

Proof. For an arbitrary idempotent $e \in A_i$ we have $eA_ie = eAe$, hence $\mathrm{Spec}(A_i) \subseteq \mathrm{Spec}(A)$. On the other hand, an arbitrary idempotent $e \in A$ lies in one of the subalgebras A_i. Hence $\mathbf{st}(eAe) = \mathbf{st}(eA_ie) \in \mathrm{Spec}(A_i)$. □

Proof of Theorem 3 (1). To start with we notice that $\{1, 2, \dots, n\} = \mathrm{Spec}(M_n(\mathbb{F}))$. Let s be a Steinitz number. In [2], we showed that there exists a unital locally matrix algebra A, $\dim_{\mathbb{F}} A \leqslant \aleph_0$, such that $\mathbf{st}(A) = s$. Consider the algebra $M_\infty(A)$ of infinite $\mathbb{N} \times \mathbb{N}$-matrices, having finitely many nonzero entries. The algebra $M_n(A)$ of $n \times n$-matrices over A is embedded in $M_\infty(A)$ as a north-west corner,

$$M_1(A) \subset M_2(A) \subset \cdots, \qquad M_\infty(A) = \bigcup_{n=1}^{\infty} M_n(A).$$

In particular, it implies that $M_\infty(A)$ is a locally matrix algebra. We will show that

$$\operatorname{Spec}\big(M_\infty(A)\big) \;=\; S(\infty, s). \tag{5}$$

Indeed, by Lemma 10,

$$\operatorname{Spec}\big(M_\infty(A)\big) \;=\; \bigcup_{n=1}^{\infty} \operatorname{Spec}\big(M_n(A)\big).$$

We have $\mathbf{st}(M_n(A)) = ns$. In [4], we showed that

$$\operatorname{Spec}\big(\ M_n(A)\ \big) \;=\; \Big\{\ \frac{a}{b}\ n\ s\ \Big|\ b \in \Omega(ns),\ a, b \in \mathbb{N};\ 1 \leqslant a \leqslant b\Big\}.$$

This implies $\operatorname{Spec}(M_n(A)) \subseteq S(\infty, s)$. A Steinitz number $(a/b)\cdot s$, $b \in \Omega(s)$, lies in $\operatorname{Spec}(M_n(A))$ provided that $a/b \leqslant n$. This completes the proof of (5). In particular, $\operatorname{Spec}(M_\infty(\mathbb{F})) = \mathbb{N}$.

Consider now a saturated subset $S = S(r, s)$ or $S = S^{+}(r, s)$, $1 \leqslant r < \infty$, where s is an infinite Steinitz number. Choose a sequence b_1, b_2, $\ldots \in \Omega(s)$ such that b_i divides b_{i+1}, $i \geqslant 1$, and s is the least common multiple of b_i, $i \geqslant 1$. There exists a unique (up to isomorphism) unital countable-dimensional locally matrix algebra A_{s/b_i} such that $\mathbf{st}(A_{s/b_i}) = s/b_i$. Let $A_i = M_{r_s(b_i)}(A_{s/b_i})$. We have

$$\mathbf{st}(A_{s/b_i}) \;=\; \mathbf{st}\Big(\big(M_{b_{i+1}/b_i}(A_{s/b_{i+1}})\big)\Big) \;=\; s/b_{i+1}.$$

Hence, by Glimm's Theorem, $A_{s/b_i} \cong M_{b_{i+1}/b_i}(A_{s/b_{i+1}})$ and, therefore,

$$A_i \;=\; M_{r_s(b_i)}\ \big(\ A_{s/b_i}\ \big) \;\cong\; M_{r_s(b_i)\cdot\frac{b_{i+1}}{b_i}}\ \big(\ A_{s/b_{i+1}}\ \big).$$

By the inequality (1), $r_s(b_i) \cdot (b_{i+1}/b_i) \leqslant r_s(b_{i+1})$. Hence, the algebra A_i is embeddable in the algebra A_{i+1} as a north-west corner. Let

$$A \;=\; \bigcup_{i=1}^{\infty} A_i.$$

We will show that $\operatorname{Spec}(A) = S$. Let $0 \neq e \in A$ be an idempotent. Then $e \in A_i$ for some $i \geqslant 1$. In [4], we showed that

$$\mathbf{st}(eA_ie) \;=\; \frac{a}{b}\ \mathbf{st}(A_i),$$

where a, $b \in \mathbb{N}$; a, b are coprime natural numbers; $b \in \Omega(\mathbf{st}(A_i))$, $a \leqslant b$. Furthermore,

$$\mathbf{st}(A_i) \;=\; r_s(b_i)\,\frac{s}{b_i}, \qquad \mathbf{st}(eA_ie) \;=\; \frac{a}{b}\,r_s(b_i)\,\frac{s}{b_i}.$$

Let $d = \gcd(b, r_s(b_i))$, $b = db'$, $r_s(b_i) = d \cdot r_s(b_i)'$. So,

$$\mathbf{st}(eA_ie) \;=\; \frac{a \cdot r_s(b_i)'}{b'} \;\cdot\; \frac{s}{b_i} \;\in\; \mathbb{SN}.$$

This implies that $b' \in \Omega(s/b_i)$. Therefore, $b'b_i \in \Omega(s)$. To show that $\mathbf{st}(eA_ie)$ lies in $S(r,s)$ (respectively, $S^+(r,s)$) we need to verify that $a\cdot r_s(b_i)' \leqslant r\,b'\,b_i$ (respectively, $a\cdot r_s(b_i)' < r\,b'\,b_i$). Multiplying both sides of the inequality by d we get $a\cdot r_s(b_i) \leqslant r\,b\,b_i$ (respectively, $a\cdot r_s(b_i) < r\,b\,b_i$). This inequality holds since $a \leqslant b$ and $r_s(b_i) \leqslant r\cdot b_i$ (respectively, $a \leqslant b$ and $r_s(b_i) < r\cdot b_i$). We proved that $\mathrm{Spec}(A) \subseteq S$.

Let us show that $S \subseteq \mathrm{Spec}(A)$. Consider a Steinitz number $(a/b)\cdot s \in S$, where a, $b \in \mathbb{N}$; $b \in \Omega(s)$, $a \leqslant r\,b$ in the case $S = S(r,s)$ or $a < r\,b$ in the case $S = S^+(r,s)$.

There exists a member of our sequence b_i such that b divides b_i, $b_i = k\cdot b$, $k \in \mathbb{N}$. Then $(a/b)\cdot s = (a\,k/b_i)\cdot s$.

We will show that $a\,k \leqslant r_s(b_i)$. Indeed, multiplying both sides of the inequality by b we get $ab_i \leqslant r_s(b_i)b$. Let $S = S(r,s)$. Then $a \leqslant rb$. Since $a \in \mathbb{N}$ it implies $a \leqslant [r\,b]$. Furthermore, $r_s(b_i) = [rb_i] = [rbk]$. So, it is sufficient to show that $[rb]k \leqslant [rbk]$. This inequality holds since $[rb]k$ is an integer and $[rb]k \leqslant rbk$.

Now suppose that $S = S^+(r,s)$. Then $a < rb$,

$$r_s(b_i) = \begin{cases} [r\;b_i], & \text{if} \quad r\;b_i\;\notin\;\mathbb{N}, \\ r\;b_i\;-\;1, & \text{if} \quad r\;b_i\;\in\;\mathbb{N}. \end{cases}$$

There are three possibilities:

1) $r\,b \in \mathbb{N}$ and, therefore, $r\,b_i \in \mathbb{N}$. In this case $a \leqslant rb-1$, $r_s(b_i) = rb_i - 1$. We have $ab_i \leqslant (rb-1)b_i \leqslant (rb_i - 1)b = r_s(b_i)b$;
2) $r\,b \notin \mathbb{N}$, but $r\,b_i \in \mathbb{N}$. In this case $a \leqslant [rb]$, $r_s(b_i) = rb_i - 1$, we have $ab_i \leqslant [rb]b_i$, $r_s(b_i)b = (rb_i-1)b$. Hence, it is sufficient to show that $[rb]k \leqslant rb_i - 1 = rbk - 1$. The number $[r\,b]\,k$ is an integer and $[rb]k < rbk$ since $[rb] < rb$. This implies the claimed inequality;
3) $r\,b_i \notin \mathbb{N}$ and, therefore, $r\,b \notin \mathbb{N}$. In this case $ab_i \leqslant [rb]bk$, $r_s(b_i)b = [rbk]b$ and it remains to notice that $[r\,b]\,k \leqslant [r\,b\,k]$.

We showed that both for $S = S(r, s)$ and for $S = S^{+}(r, s)$ there holds the inequality $ak \leqslant r_s(b_i)$.

Recall that $A_i = M_{r_s(b_i)}(A_{s/b_i})$. Consider the north-east corner $M_{ak}\left(A_{s/b_i}\right)$ of the algebra A_i. We have

$$\mathbf{st}\left(M_{ak}\left(A_{s/b_i}\right)\right) = a\,k \cdot \frac{s}{b_i} = \frac{a}{b}\,s,$$

and, therefore, $S \subseteq \operatorname{Spec}(A)$. This completes the proof of Theorem 3 (1). □

For the proof of Theorem 3 (2) we will need several lemmas on extensions of isomorphisms.

Lemma 11. *Let A be a locally matrix algebra and let A_1 be a subalgebra of A such that $A_1 \cong M_n(\mathbb{F})$. Then every automorphism of the algebra A_1 extends to an automorphism of the algebra A.*

Proof. Let e be the identity element of the subalgebra A_1. Then the corner eAe is a unital locally matrix algebra. Let C be the centralizer of the subalgebra A_1 in eAe. By Wedderbern's Theorem (see [6, 8]), we have $e\,A\,e = A_1 \otimes_{\mathbb{F}} C$. An arbitrary automorphism φ of the subalgebra A_1 is inner, that is, there exists an invertible element x of the subalgebra A_1 such that $\varphi(a) = x^{-1}ax$ for all elements $a \in A_1$. The conjugation by the element $x \otimes e$ extends φ to an automorphism of the algebra eAe. Consider the Peirce decomposition

$$A = eAe + eA(1-e) + (1-e)Ae + (1-e)A(1-e),$$

and the mapping

$$\tilde{\varphi}\colon\ A \ni a \mapsto x^{-1}ax + x^{-1}a(1-e) + (1-e)ax + (1-e)a(1-e).$$

The mapping $\tilde{\varphi}$ extends φ and $\tilde{\varphi} \in \operatorname{Aut}(A)$. This completes the proof of the lemma. □

Lemma 12. *Let A be a unital locally matrix algebra with an idempotent $e \neq 0$. Then an arbitrary automorphism of the corner eAe extends to an automorphism of the algebra A.*

Proof. Suppose at first that an automorphism φ of the algebra eAe is inner, and there exists an element $x_e \in eAe$ that is invertible in the algebra eAe such that $\varphi(a) = x_e^{-1}ax_e$ for an arbitrary element $a \in e\,A\,e$. The element $x = x_e + (1 - e)$ is invertible in the algebra A. So, conjugation by the element x extends φ.

Now let φ be an arbitrary automorphism of the corner eAe. Let $A_1 \subseteq A$ be a subalgebra such that $1, e \in A_1$ and $A_1 \cong M_m(\mathbb{F})$ for some $m \geqslant 1$. Consider $A_2 \subseteq A$ such that $A_1 \subseteq A_2$, $\varphi(e\,A_1\,e) \subseteq A_2$ and $A_2 \cong M_n(\mathbb{F})$ for some $n \geqslant 1$. Consider the embedding

$$\varphi\colon\ e\,A_1\,e\ \to\ \varphi\,(e\,A_1\,e)\ \subseteq\ e\,A_2\,e$$

that preserves the identity element e. By Skolem–Noether Theorem (see [6]), there exists an invertible element $x_e \in eA_2e$ such that $\varphi(a) = x_e^{-1} a x_e$ for an arbitrary element $a \in e\,A_1\,e$.

As noticed above, there exists an automorphism ψ of the algebra A that extends the automorphism $eAe \to eAe$, $a \mapsto x_e^{-1} a x_e$. The composition $\psi^{-1} \circ \varphi$ leaves all elements of the algebra eA_1e fixed. Since it is sufficient to prove that the automorphism $\psi^{-1} \circ \varphi \in \operatorname{Aut}(eAe)$ extends to an automorphism of A we will assume without loss of generality that the automorphism $\varphi \in \operatorname{Aut}(eAe)$ fixes all elements of eA_1e.

Let C be the centralizer of the subalgebra A_1 in A. Then $A = A_1 \otimes_{\mathbb{F}} C$ and $eAe = eA_1e \otimes_{\mathbb{F}} C$. Since the subalgebra $e \otimes_{\mathbb{F}} C$ is the centralizer of $eA_1e \otimes_{\mathbb{F}} C$ in the algebra eAe it follows that $e \otimes_{\mathbb{F}} C$ is invariant with respect to the automorphism φ. Hence, there exists an automorphism $\theta \in \operatorname{Aut}(C)$ such that $\varphi(a \otimes c) = a \otimes \theta(c)$ for all elements $a \in eAe$, $c \in C$. So, the automorphism $\tilde{\varphi}(a \otimes c) = a \otimes \theta(c)$, $a \in A_1$, $c \in C$, extends φ. This completes the proof of the lemma. □

Lemma 13. *Let A be a unital locally matrix algebra with nonzero idempotents e_1, e_2. An arbitrary isomorphism $\varphi\colon e_1Ae_1 \to e_2Ae_2$ extends to an automorphism of the algebra A.*

Proof. There exists a subalgebra $A_1 \subseteq A$ such that $1, e_1, e_2 \in A_1$ and $A_1 \cong M_n(\mathbb{F})$ for some $n \geqslant 1$. Let r_i be the matrix range of the idempotent e_i in A_1, $i = 1, 2$. In [4], it was shown that

$$\mathbf{st}\,(e_1\ A\ e_1)\ =\ \frac{r_1}{n}\ \cdot\ \mathbf{st}(A), \quad \mathbf{st}\,(e_2\ A\ e_2)\ =\ \frac{r_2}{n}\ \cdot\ \mathbf{st}(A).$$

Since $e_1Ae_1 \cong e_2Ae_2$ it follows that $r_1 = r_2$. In the matrix algebra $M_n(\mathbb{F})$ any two idempotents of the same range are conjugate via an automorphism. Hence, the idempotents e_1, e_2 are conjugate via an automorphism of A_1. By Lemma 11, an arbitrary automorphism of A_1 extends to an automorphism of the algebra A. Now the assertion of the lemma follows from Lemma 12. □

Lemma 14. *Let A, B be isomorphic unital locally matrix algebras with nonzero idempotents $e \in A$, $f \in B$. An arbitrary isomorphism $eAe \to fBf$ extends to an isomorphism $A \to B$.*

Proof. Let $\varphi : A \to B$, $\psi : eAe \to fBf$ be isomorphisms. Then

$$\varphi^{-1} \circ \psi \; : \; e \; A \; e \;\; \to \;\; \varphi^{-1}(f) \; A \; \varphi^{-1}(f)$$

is an isomorphism of two corners of the algebra A. By Lemma 13, $\varphi^{-1} \circ \psi$ extends to an automorphism χ of the algebra A, the isomorphism $\varphi \circ \chi$ extends ψ. □

Lemma 15. *Let A be a unital locally matrix algebra and let s_1, s_2 be Steinitz numbers from* Spec(A). *Suppose that $s_2/s_1 > 1$. Let $e_1 \in A$ be an idempotent such that $\mathbf{st}(e_1 A e_1) = s_1$. Then there exists an idempotent $e_2 > e_1$ such that $\mathbf{st}(e_2 A e_2) = s_2$.*

Proof. Since $s_2 \in \operatorname{Spec}(A)$ there exists an idempotent $e' \in A$ such that $\mathbf{st}(e' A e') = s_2$. Choose a subalgebra $A_1 \subseteq A$ such that e_1, $e' \in A_1$ and $A_1 \cong M_n(\mathbb{F})$.

Let r_1, r_2 be the matrix ranges of e_1, e' in $M_n(\mathbb{F})$, respectively. In [4], it was shown that

$$\mathbf{st}(e_1 \; A \; e_1) \;\; = \;\; s_1 \;\; = \;\; \frac{r_1}{n} \, \mathbf{st}(A), \quad \mathbf{st}(e' \; A \; e') \;\; = \;\; s_2 \;\; = \;\; \frac{r_2}{n} \, \mathbf{st}(A).$$

Hence $r_2 > r_1$. Since every idempotent in the algebra $M_n(\mathbb{F})$ is diagonalizable there exist automorphisms φ, ψ of the algebra A_1 such that $\psi(e') > \varphi(e_1)$. By Lemma 11, the automorphisms φ, ψ extend to automorphisms $\widetilde{\varphi}$, $\widetilde{\psi}$ of the algebra A, respectively.

Let $e_2 = \widetilde{\varphi}^{\,-1} \left(\psi \left(e' \right) \right)$. Then $e_2 > e_1$ and $\mathbf{st}(e_2 A e_2) = s_2$, which completes the proof of the lemma. □

Proof of Theorem 3 (2). Let A, B be countable-dimensional locally matrix algebras, $\operatorname{Spec}(A) = \operatorname{Spec}(B)$. Choose bases $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ in the algebras A, B, respectively.

We will construct ascending chains of corners $\{0\} = A_0 \subset A_1 \subset A_2 \subset \cdots$ in the algebra A and $\{0\} = B_0 \subset B_1 \subset B_2 \subset \cdots$ in the algebra B, such that

$$\bigcup_{i=0}^{\infty} A_i \;=\; A, \qquad \bigcup_{i=0}^{\infty} B_i \;=\; B$$

and $a_1, \ldots, a_i \in A_i$, $b_1, \ldots, b_i \in B_i$, $\mathbf{st}(A_i) = \mathbf{st}(B_i)$ for all $i \geqslant 1$.

Suppose that corners $\{0\} = A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_n$, $\{0\} = B_0 \subset B_1 \subset B_2 \subset \cdots \subset B_n$ have already been selected, $n \geqslant 0$. There exist corners $A' \subset A$, $B' \subset B$ in the algebras A, B, respectively, such that $A_n \subset A'$, $a_{n+1} \in A'$ and $B_n \subset B'$, $b_{n+1} \in B'$. The Steinitz numbers $\mathbf{st}(A')$, $\mathbf{st}(B')$ lie in the same saturated subset of $\mathbb{SN}$, therefore, they are rationally connected.

Suppose that $\mathbf{st}(B') \geqslant \mathbf{st}(A')$. Let $= e'$ be an idempotent of the algebra A such that $A' = e'Ae'$. The Steinitz number $\mathbf{st}(B')$ lies in $\mathrm{Spec}(A)$. Hence, by Lemma 15, there exists an idempotent $e \in A$ such that $e \geqslant e'$ and $\mathbf{st}(eAe) = \mathbf{st}(B')$. Choose $A_{n+1} = eAe$, $B_{n+1} = B'$. The chains $\{0\} = A_0 \subset A_1 \subset A_2 \subset \cdots$ and $\{0\} = B_0 \subset B_1 \subset B_2 \subset \cdots$ have been constructed.

By Lemma 14, every isomorphism $A_i \to B_i$ extends to an isomorphism $A_{i+1} \to B_{i+1}$. This gives rise to a sequence of isomorphisms $\varphi_i : A_i \to B_i$, $i \geqslant 0$, where each φ_{i+1} extends φ_i. Taking the union $\cup_{i \geqslant 0} \varphi_i$ we get an isomorphism from the algebra A to the algebra B. This completes the proof of the theorem. □

Proof of Theorem 4. It is easy to see that a locally matrix algebra A is a unital if and only if the set of idempotents of A has a largest element: an identity. This is equivalent to $\mathrm{Spec}(A)$ containing a largest Steinitz number. Among saturated sets of Steinitz numbers only $\{1, 2, \ldots, n\}$ and $S(r, s)$, $s \in \mathbb{SN} \diagdown \mathbb{N}$, $r = u/v$; u and v are coprime natural numbers, $v \in \Omega(s)$, satisfy this assumption. □

4. Embeddings of locally matrix algebras

Lemma 16. *Let S_1, S_2 be saturated sets of Steinitz numbers. Then either $S_1 \bigcap S_2 = \varnothing$ or one of the sets S_1, S_2 contains the other one.*

Proof. Let $s \in S_1 \bigcap S_2$. If $s \in \mathbb{N}$ then, by Lemma 9, each set S_i is either a segment $[1, n]$, $n \geqslant 1$, or the whole $\mathbb{N}$. In this case the assertion of the lemma is obvious.

Suppose that the number s is infinite. Then by Theorem 2, $S_i = S(r_i, s)$ or $S_i = S^+(r_i, s)$, where $r_i = r_{S_i}(s) \in [1, \infty) \cup \{\infty\}$, $i = 1, 2$. Clearly, if $r_{S_1}(s) < r_{S_2}(s)$ then $S_1 \subsetneqq S_2$. If $r_{S_1}(s) = r_{S_2}(s)$ then

$$S_1,\ S_2 \ = \left[\begin{array}{l} S(r, s) \\ S^+(r, s) \\ S(\infty, s) \end{array}\right.$$

and $S^+(r,s) \subseteq S(r,s) \subset S(\infty,s)$ for any $r \in [1,\infty)$. This completes the proof of the lemma. □

Let A be a locally matrix algebra. A subalgebra $B \subseteq A$ is called an *approximative corner* of A if B is the union of an increasing chain of corners. In other words, there exist idempotents e_0, e_1, e_2, … such that

$$e_0\ A\ e_0 \ \subseteq\ e_1\ A\ e_1\ \subseteq\ e_2\ A\ e_2\ \subseteq\ \cdots, \qquad B\ =\ \bigcup_{i=0}^{\infty} e_i\ A\ e_i.$$

It is easy to see that an approximative corner of a locally matrix algebra is a locally matrix algebra.

Theorem 5. *Let A, B be countable-dimensional locally matrix algebras. Then B is embeddable in A as an approximative corner if and only if* $\mathrm{Spec}(B) \subseteq \mathrm{Spec}(A)$.

Proof. If B is an approximative corner of A then every corner of B is a corner of A, hence $\mathrm{Spec}(B) \subseteq \mathrm{Spec}(A)$.

Suppose now that $\mathrm{Spec}(B) \subseteq \mathrm{Spec}(A)$. If the algebra B is unital then it embedds in the algebra A as a corner. Indeed, the embedding $\mathrm{Spec}(B) \subseteq \mathrm{Spec}(A)$ implies that there exists an idempotent $e \in A$ such that $\mathbf{st}(B) = \mathbf{st}(eAe)$. By Glimm's Theorem [7], we have $B \cong eAe$.

Suppose now that the algebra B is not unital. Then there exists a sequence of idempotents $0 = f_0$, f_1, f_2, … of algebra B such that

$$\{0\} =\ f_0\ B\ f_0\ \subsetneqq\ f_1\ B\ f_1\ \subsetneqq\ f_2\ B\ f_2\ \subsetneqq\ \cdots, \quad \bigcup_{i=0}^{\infty} f_i\ B\ f_i\ =\ B.$$

We will construct a sequence of idempotents e_0, e_1, e_2, … in the algebra A such that

$$e_0\ A\ e_0\ \subsetneqq\ e_1\ A\ e_1\ \subsetneqq\ e_2\ A\ e_2\ \subsetneqq\ \cdots, \quad \mathbf{st}(f_i\ B\ f_i)\ =\ \mathbf{st}(e_i\ A\ e_i)$$

for an arbitrary $i \geqslant 0$. Let $e_0 = 0$. Suppose that we have already selected idempotents e_0, e_1, …, $e_n \in A$ such that $e_0Ae_0 \subset e_1Ae_1 \subset \cdots \subset e_nAe_n$ and $\mathbf{st}(e_iAe_i) = \mathbf{st}(f_iBf_i)$, $0 \leqslant i \leqslant n$. We have

$$\mathbf{st}(f_{n+1}\ B\ f_{n+1})\ >\ \mathbf{st}(f_n\ B\ f_n)\ =\ \mathbf{st}(e_n\ A\ e_n)$$

and $\mathbf{st}(f_{n+1}Bf_{n+1}) \in \mathrm{Spec}(A)$. By Lemma 15, there exists an idempotent $e_{n+1} \in A$ such that $e_nAe_n \subset e_{n+1}Ae_{n+1}$ and $\mathbf{st}(e_{n+1}Ae_{n+1}) = \mathbf{st}(f_{n+1}Bf_{n+1})$, which proves existence of a sequence e_0, e_1, e_2, … .

The union

$$A' = \bigcup_{i=0}^{\infty} e_i \; A \; e_i$$

is an approximative corner of the algebra A. By Glimm's Theorem [7], $e_i A e_i \cong f_i B f_i$, $i \geqslant 1$. By Lemma 10,

$$\mathrm{Spec}(A') = \bigcup_{i=1}^{\infty} \mathrm{Spec}(e_i \; A \; e_i) \quad \text{and} \quad \mathrm{Spec}(B) = \bigcup_{i=1}^{\infty} \mathrm{Spec}(f_i B f_i).$$

Hence $\mathrm{Spec}(B) = \mathrm{Spec}(A')$. By Theorem 3 (2), we have $A' \cong B$, which completes the proof of the theorem. □

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Received by the editors: 08.12.2020
and in final form 15.02.2021.