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Endomorphisms of Clifford semigroups with injective structure homomorphisms

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Abstract. In the present paper, we study semigroups of endomorphisms on Clifford semigroups with injective structure homomorphisms, where the semilattice has a least element. We describe such Clifford semigroups having a regular endomorphism monoid. If the endomorphism monoid on the Clifford semigroup is completely regular then the corresponding semilattice has at most two elements. We characterize all Clifford semigroups $G_\alpha \cup G_\beta$ $(\alpha > \beta)$ with an injective structure homomorphism, where G_α has no proper subgroup, such that the endomorphism monoid is completely regular. In particular, we consider the case that the structure homomorphism is bijective.

1. Introduction

Inverse semigroups are a well studied class of semigroups. A completely regular inverse semigroup is called *Clifford semigroup* or also *strong semilattice of groups*. Let Y be a semilattice, i.e. an idempotent semigroup with $\alpha\beta = \beta\alpha$, for all $\alpha, \beta \in Y$. The partial order relation $\geqslant$ in Y is that obtained from the semilattice operation. Let G_ξ be a group for each $\xi \in Y$ with $G_\alpha \cap G_\beta = \varnothing$, whenever $\alpha \neq \beta$. For $\xi \in Y$, we will mean by x_ξ an element in the group G_ξ and e_ξ denotes the identity element in the group

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G_ξ. For each pair $\alpha \geqslant \beta$ of elements in Y, let $\varphi_{\alpha,\beta} : G_\alpha \to G_\beta$ be a group homomorphism such that $\varphi_{\alpha,\alpha}$ is the identity automorphism on G_α, for all $\alpha \in Y$ and if $\alpha \geqslant \beta \geqslant \gamma$ then $\varphi_{\alpha,\gamma} = \varphi_{\alpha,\beta}\varphi_{\beta,\gamma}$. We will write mappings on the right of the objects on which they act. In $S = \bigcup_{\xi \in Y} G_\xi$, we take the multiplication

$$x_\alpha * x_\beta = (x_\alpha)\varphi_{\alpha,\alpha\beta}(x_\beta)\varphi_{\beta,\alpha\beta}$$

for $\alpha, \beta \in Y$. The semigroup S is a strong semilattice Y of groups G_ξ, i.e. a Clifford semigroup and each Clifford semigroup is of this form [2]. Basic facts about Clifford semigroups can be found in any introductory book on semigroup theory, for example in [4, 7]. For each pair $\alpha \geqslant \beta \in Y$, we call $\varphi_{\alpha,\beta} : G_\alpha \to G_\beta$ the structure homomorphism or defining homomorphism. Although monoids of endomorphisms on Clifford semigroups seem a natural object of study, not so much has be done on this subject. Of course, each group is a Clifford semigroup. Semigroups of endomorphisms on groups are studied in [3, 8, 9]. Let S and T be semigroups and let $f : S \to T$ be a mapping. Then f is called a *semigroup homomorphism* if $(xy)f = (x)f(y)f$ for all $x, y \in S$. The homomorphism f is called *endomorphism* if $S = T$. The set of all semigroup homomorphisms (of all endomorphisms) is denoted by $\mathrm{Hom}(S, T)$ (by $\mathrm{End}(S) = \mathrm{Hom}(S, S)$). The image of a homomorphism $f \in \mathrm{Hom}(S, T)$ is denoted by $\mathrm{Im}(f) = \{(x)f : x \in S\}$. Endomorphisms on Clifford semigroups were studied for examples in [10, 11].

An element x of a semigroup S is called *regular* if there exists an element $y \in S$ such that $xyx = x$. In this sense, y is called *pseudoinverse* of x. An element a is called *inverse* of x if $xax = x$ and $axa = a$. The semigroup S is called *regular* if each element has at least one pseudoinverse or equivalently, each element has at least one inverse element. In the present paper, we use only the existence of a pseudoinverse for each element in order to verify the regularity of a semigroup. An element x of a semigroup S is called *completely regular* if there exists an element $y \in S$ such that $xyx = x$ and $xy = yx$. The semigroup S is called *completely regular* if all its elements are completely regular. A semigroup S is called *idempotent* or band if $xx = x$ for all $x \in S$. See for examples in [4, 7]

A semigroup S is called *endo-regular* if its monoid of endomorphisms is regular. The regularity of the endomorphism monoid for groups were studied by several authors (see for examples [1, 5]). In particular, Theorem 1.2 in [6] gives a characterization of regular endomorphisms on groups. It appears the question when each endomorphism has a pseudoinverse, i.e, when the endomorphism monoid of a group is regular. In [6], John Meldrum gives a partial answer on this question. In the present paper, we want to ask this question for Clifford semigroups. It seems almost impossible to find an answer for the class of all Clifford semigroups. Therefore, we restrict ourselves to a class of Clifford semigroups, for which we have already some important information about their endomorphisms.

Let us consider a Clifford semigroup $S = \bigcup_{\xi \in Y} G_\xi$ and let $f \in \mathrm{End}(S)$. In [10], the authors show that the restriction of an endomorphism f on S to the set $E_S = \{e_\xi : \xi \in Y\}$ of the idempotent elements in S induces an endomorphism on Y, which we denote by $\underline{f}$. In fact, $(\alpha)\underline{f} = \beta$, whenever $(e_\alpha)f = e_\beta$. This homomorphism is called the *induced index mapping.* We denote by f_ξ the restriction of f to the group G_ξ. Then the image of f_ξ is a subgroup of $G_{(\xi)\underline{f}}$, i.e. $f_\xi \in \mathrm{Hom}(G_\xi, G_{(\xi)\underline{f}})$, where $(e_\xi)f = e_{(\xi)\underline{f}}$. We suppose that the structure homomorphisms are injective and the semilattice Y has a least element ν. Then an endomorphism f on S is already determined by the endomorphism $f_\nu \in \mathrm{Hom}(G_\nu, G_{(\nu)\underline{f}})$ [10]. We will study the endomorphism monoid of such a Clifford semigroup S. In particular, we consider the case that $\mathrm{End}(S)$ is regular and completely regular, respectively. Let note that the monoid $\mathrm{End}(S)$ is related to the monoid $\mathrm{End}(Y)$ of all endomorphisms on the semilattice Y. In fact, if $\mathrm{End}(S)$ is regular (idempotent and completely regular, respectively) then $\mathrm{End}(Y)$ is also regular (idempotent and completely regular, respectively) by Theorem 3.10 in [11]. We will use this fact subsequently without referring it. In [11], the author studied the endomorphisms on a semilattice Y. In particular, the author has characterized the semilattices Y with idempotent and completely regular, respectively, endomorphism monoid $\mathrm{End}(Y)$. If Y is a semilattice then $\mathrm{End}(Y)$ is completely regular if and only if it is idempotent if and only if $|Y| \leqslant 2$.

2. Regular endomorphism monoid

In this section, we consider endo-regular Clifford semigroups $S = \bigcup_{\xi \in Y} G_\xi$ with injective structure homomorphisms $\varphi_{\alpha,\beta}$, for $\alpha > \beta \in Y$, where Y is a semilattice with a least element $\nu = \bigwedge Y$. We start with three propositions which include some already known results. In [10], it is shown that any $f \in \mathrm{End}(S)$ is determined by f_ν.

Proposition 1. [10] *Let Y be a semilattice which has a least element $\nu = \bigwedge Y$ and let $S = \bigcup_{\xi \in Y} G_\xi$ be a Clifford semigroup with injective structure homomorphisms $\varphi_{\alpha,\beta}$, for $\alpha > \beta \in Y$. If $f \in \mathrm{End}(S)$ then $f_\xi = \varphi_{\xi,\nu} f_\nu \varphi^{-1}_{(\xi)\underline{f},(\nu)\underline{f}}$, for all $\xi \in Y$.*

In particular, we have $\mathrm{Im}(\varphi_{\xi,\nu})f_\nu \subseteq \mathrm{Im}(f_{(\xi)\underline{f},(\nu)\underline{f}})$. We will use this fact subsequently without mentioning it.

On the other hand, in [11], the author constructs an endomorphism on S based on an endomorphism on G_ν. This construction does not require that the structure homomorphisms are injective.

Proposition 2. [11] *Let Y be a semilattice which has a least element $\nu = \bigwedge Y$ and let $S = \bigcup_{\xi \in Y} G_\xi$ be a Clifford semigroup. For $g \in \mathrm{End}(G_\nu)$, we define $f : S \to S$ by $(x_\xi)f = (x_\xi)f_\xi$, where*

$$f_\xi = \varphi_{\xi,\nu} g \quad \text{for } \xi \in Y.$$

Then $f \in \mathrm{End}(S)$.

In [10], Samman and Meldrum constructed an endomorphism on S starting from a semilattice homomorphism and a family of group homomorphisms.

Proposition 3. [10] *Let Y be a semilattice and let $S = \bigcup_{\xi \in Y} G_\xi$ be a Clifford semigroup. Given an endomorphism $s \in \mathrm{End}(Y)$ and a family $\{f_\xi \in \mathrm{Hom}(G_\xi, G_{(\xi)s}) : \xi \in Y\}$ satisfying*

$$\varphi_{\alpha,\beta} f_\beta = f_\alpha \varphi_{(\alpha)s,(\beta)s},$$

for all pairs $\alpha > \beta \in Y$. Then $f : S \to S$ defined by $(x_\xi)f = (x_\xi)f_\xi$, for every $\xi \in Y$, is an endomorphism on S.

As Propositions 1 and 2 show, the group G_ν plays an important part for the description of the endomorphisms on S. Hence, we expect that $\mathrm{End}(G_\nu)$ is regular, i.e. G_ν is an endo-regular group, whenever S is endo-regular (independently from the kind of the structure homomorphisms). Therefore, let us consider the set $\mathrm{End}_\nu(S) = \{f \in \mathrm{End}(S) : (\nu)\underline{f} = \nu\}$. It is easy to verify that $\mathrm{End}_\nu(S)$ is a submonoid of $\mathrm{End}(S)$.

Lemma 4. *Let Y be a semilattice which has a least element $\nu = \bigwedge Y$ and let $S = \bigcup_{\xi \in Y} G_\xi$ be a Clifford semigroup. If each $f \in \mathrm{End}_\nu(S)$ is regular in $\mathrm{End}(S)$ then $\mathrm{End}(G_\nu)$ is regular.*

Proof. Suppose that each $f \in \mathrm{End}_\nu(S)$ is regular. Let $g \in \mathrm{End}(G_\nu)$ and let $\varphi_{\alpha,\beta} : G_\alpha \to G_\beta$, for $\alpha > \beta \in Y$, be the structure homomorphisms. Then, we define a mapping $f : S \to S$ by $(x_\xi)f = (x_\xi)f_\xi$ with $f_\xi = \varphi_{\xi,\nu} g$ for $\xi \in Y$. Note that $f_\nu = g$. By Proposition 2, we can conclude that $f \in \mathrm{End}(S)$ and, in particular, $f \in \mathrm{End}_\nu(S)$. Then there is $h \in \mathrm{End}(S)$

with $fhf = f$. The restriction of f to the group G_ν provides $g = f_\nu = f_\nu h_{(\nu)\underline{f}} f_{(\nu)\underline{fh}} = f_\nu h_\nu f_{(\nu)\underline{h}} = g h_\nu \varphi_{(\nu)\underline{h},\nu} g$, i.e. $h_\nu \varphi_{(\nu)\underline{h},\nu} : G_\nu \to G_\nu$ is a pseudoinverse of g. $\square$

As already mentioned, $\mathrm{End}_\nu(S)$ is a submonoid of $\mathrm{End}(S)$. The following proposition characterizes Clifford semigroups S (of the type that we consider) such that $\mathrm{End}_\nu(S)$ is regular. We note that $\mathrm{End}_\nu(Y) = \{\underline{f} : f \in \mathrm{End}_\nu(S)\}$ is regular, whenever $\mathrm{End}_\nu(S)$ is regular. In fact, if $f \in \mathrm{End}_\nu(S)$ then there is $h \in \mathrm{End}_\nu(S)$ with $fhf = f$. Then it is easy to verify that $\underline{f}\underline{h}\underline{f} = \underline{f}$ and $\underline{h} \in \mathrm{End}(Y)$.

Proposition 5. *Let Y be a semilattice which has a least element $\nu = \bigwedge Y$ and let $S = \bigcup_{\xi \in Y} G_\xi$ be a Clifford semigroup with injective structure homomorphisms. Then $\mathrm{End}_\nu(S)$ is regular if and only if both $\mathrm{End}_\nu(Y)$ and $\mathrm{End}(G_\nu)$ are regular.*

Proof. Suppose that $\mathrm{End}_\nu(S)$ is regular. Then $\mathrm{End}(G_\nu)$ is regular by Lemma 4. Further, let $f \in \mathrm{End}_\nu(S)$. Then there is $h \in \mathrm{End}_\nu(S)$ with $fhf = f$. If we restrict both mappings fhf and f to the set E_S, we obtain $\underline{f}\underline{h}\underline{f} = \underline{f}$. Since $\underline{h} \in \mathrm{End}_\nu(Y)$, we have shown that $\mathrm{End}_\nu(Y)$ is regular.

Suppose now that both $\mathrm{End}_\nu(Y)$ and $\mathrm{End}(G_\nu)$ are regular. Let $f \in \mathrm{End}_\nu(S)$. Then $t := \underline{f} \in \mathrm{End}_\nu(Y)$ and there is $s \in \mathrm{End}_\nu(Y)$ with $tst = t$. Since $(\nu)\underline{f} = \nu$, we have $f_\nu \in \mathrm{End}(G_\nu)$. Since $\mathrm{End}(G_\nu)$ is regular, there is $g \in \mathrm{End}(G_\nu)$ with $f_\nu g f_\nu = f_\nu$. We define a mapping $h : S \to S$ by $(x_\xi)h = (x_\xi)h_\xi$ with $h_\xi = \varphi_{\xi,\nu} g \varphi^{-1}_{(\xi)s,\nu}$, for all $\xi \in Y$. Because of

$$\begin{aligned} h_\alpha \varphi_{(\alpha)s,(\beta)s} &= \varphi_{\alpha,\nu} g \varphi^{-1}_{(\alpha)s,\nu} \varphi_{(\alpha)s,(\beta)s} = \varphi_{\alpha,\nu} g \big(\varphi_{(\alpha)s,(\beta)s} \varphi_{(\beta)s,\nu}\big)^{-1} \varphi_{(\alpha)s,(\beta)s} \\ &= \varphi_{\alpha,\beta} \varphi_{\beta,\nu} g \varphi^{-1}_{(\beta)s,\nu} = \varphi_{\alpha,\beta} h_\beta, \end{aligned}$$

for all pairs $\alpha > \beta \in Y$, the mapping h is an endomorphism on S by Proposition 3. Note that $(\nu)\underline{h} = (\nu)s = \nu$. Hence, $h \in \mathrm{End}_\nu(S)$. It remains to show that h is a pseudoinverse of f, i.e. we have to show $f_\alpha = f_\alpha h_{(\alpha)t} f_{(\alpha)ts}$, for $\alpha \in Y$. Let $\alpha \in Y$. By Proposition 1, we have $f_\alpha = \varphi_{\alpha,\nu} f_\nu \varphi^{-1}_{(\alpha)t,\nu}$ and $f_{(\alpha)ts} = \varphi_{(\alpha)ts,\nu} f_\nu \varphi^{-1}_{(\alpha)tst,\nu}$. Moreover, $\varphi^{-1}_{(\alpha)t,\nu} \varphi_{(\alpha)t,\nu}$ and $\varphi^{-1}_{(\alpha)ts,\nu} \varphi_{(\alpha)ts,\nu}$ map identical on $\mathrm{Im}(\varphi_{\alpha,\nu} f_\nu)$ and $\mathrm{Im}(\varphi_{(\alpha)t,\nu} g)$, respectively. So, we get

$$\begin{aligned} f_\alpha h_{(\alpha)t} f_{(\alpha)ts} &= \varphi_{\alpha,\nu} f_\nu \varphi^{-1}_{(\alpha)t,\nu} \varphi_{(\alpha)t,\nu} g \varphi^{-1}_{(\alpha)ts,\nu} \varphi_{(\alpha)ts,\nu} f_\nu \varphi^{-1}_{(\alpha)tst,\nu} \\ &= \varphi_{\alpha,\nu} f_\nu g f_\nu \varphi^{-1}_{(\alpha)tst,\nu} = \varphi_{\alpha,\nu} f_\nu \varphi^{-1}_{(\alpha)t,\nu} = f_\alpha. \end{aligned}$$

$\square$

We turn back to the monoid $\mathrm{End}(S)$. We will now show that we only need the information about f_ν for all $f \in \mathrm{End}(S)$ and that $\mathrm{End}(G_\nu)$ is a regular group in order to verify that S is endo-regular.

Lemma 6. *Let Y be a semilattice which has a least element $\nu = \bigwedge Y$ and let $S = \bigcup_{\xi \in Y} G_\xi$ be a Clifford semigroup with injective structure homomorphisms. Then $\mathrm{End}(S)$ is regular if and only if for any $f \in \mathrm{End}(S)$ there are $t \in \mathrm{End}(Y)$ and $g \in \mathrm{End}(G_\nu)$ with $\underline{f}t\underline{f} = \underline{f}$ and $\mathrm{Im}(\varphi_{\alpha,\nu}g) \subseteq \mathrm{Im}(\varphi_{(\alpha)t,\nu})$, for any $\alpha \in Y$, such that*

$$f_\nu \varphi_{(\nu)\underline{f},\nu} g f_\nu = f_\nu.$$

Proof. Suppose that $\mathrm{End}(S)$ is regular and let $f \in \mathrm{End}(S)$. Then there is $h \in \mathrm{End}(S)$ such that $fhf = f$. We put $g^* := h_\nu$ and $t := \underline{h} \in \mathrm{End}(Y)$. Then $fhf = f$ implies $sts = s$, where $s := \underline{f}$. By Proposition 1, we have $h_{(\nu)s} = \varphi_{(\nu)s,\nu} g^* \varphi^{-1}_{(\nu)st,(\nu)t}$ and $f_{(\nu)st} = \varphi_{(\nu)st,\nu} f_\nu \varphi^{-1}_{(\nu)sts,(\nu)s} = \varphi_{(\nu)st,\nu} f_\nu \varphi^{-1}_{(\nu)s,(\nu)s} = \varphi_{(\nu)st,\nu} f_\nu$. The restriction of $fhf (= f)$ to G_ν provides

$$f_\nu = f_\nu h_{(\nu)s} f_{(\nu)st} = f_\nu \varphi_{(\nu)s,\nu} g^* \varphi^{-1}_{(\nu)st,(\nu)t} \varphi_{(\nu)st,\nu} f_\nu = f_\nu \varphi_{(\nu)s,\nu} g^* \varphi_{(\nu)t,\nu} f_\nu.$$

We have $f_\nu \varphi_{(\nu)s,\nu} g f_\nu = f_\nu$ and $g = g^* \varphi_{(\nu)t,\nu} \in \mathrm{End}(G_\nu)$. Let $\alpha \in Y$. From $h_\alpha = \varphi_{\alpha,\nu} g^* \varphi^{-1}_{(\alpha)t,(\nu)t}$ (by Proposition 1), we obtain $\mathrm{Im}(\varphi_{\alpha,\nu} g^*) \subseteq \mathrm{Im}(\varphi_{(\alpha)t,(\nu)t})$ and thus

$$\mathrm{Im}(\varphi_{\alpha,\nu} g^*) \varphi_{(\nu)t,\nu} \subseteq \mathrm{Im}(\varphi_{(\alpha)t,(\nu)t}) \varphi_{(\nu)t,\nu} = \mathrm{Im}(\varphi_{(\alpha)t,\nu}).$$

Thus, $\mathrm{Im}(\varphi_{\alpha,\nu} g) = \mathrm{Im}(\varphi_{\alpha,\nu} g^*) \varphi_{(\nu)t,\nu} \subseteq \mathrm{Im}(\varphi_{(\alpha)t,\nu})$.

Suppose now that for all $f \in \mathrm{End}(S)$, there are $t \in \mathrm{End}(Y)$ and $g \in \mathrm{End}(G_\nu)$ with $\underline{f}t\underline{f} = \underline{f}$ and $\mathrm{Im}(\varphi_{\alpha,\nu} g) \subseteq \mathrm{Im}(\varphi_{(\alpha)t,\nu})$, for all $\alpha \in Y$, such that $f_\nu \varphi_{(\nu)\underline{f},\nu} g f_\nu = f_\nu$. Let $f \in \mathrm{End}(S)$. Then there exist $t \in \mathrm{End}(Y)$ such that $sts = s$ with $s := \underline{f}$ and $g \in \mathrm{End}(G_\nu)$ such that $\mathrm{Im}(\varphi_{\alpha,\nu} g) \subseteq \mathrm{Im}(\varphi_{(\alpha)t,\nu})$, for all $\alpha \in Y$ with $f_\nu \varphi_{(\nu)s,\nu} g f_\nu = f_\nu$. Now we can define a mapping $h : S \to S$ with

$$(x_\alpha)h = (x_\alpha)h_\alpha,$$

where $h_\alpha = \varphi_{\alpha,\nu} g \varphi^{-1}_{(\alpha)t,\nu}$, for all $\alpha \in Y$. Because of

$$\mathrm{Im}(\varphi_{\alpha,\nu} g) \subseteq \mathrm{Im}(\varphi_{(\alpha)t,\nu}),$$

for all $\alpha \in Y$, the mapping h is well defined. We will show that

$$\varphi_{\alpha,\beta} h_\beta = h_\alpha \varphi_{(\alpha)s,(\beta)s},$$

for all pairs $\alpha > \beta \in Y$. We have

$$\begin{aligned}\varphi_{\alpha,\beta}h_\beta &= \varphi_{\alpha,\beta}\varphi_{\beta,\nu}g\varphi^{-1}_{(\beta)t,\nu} = \varphi_{\alpha,\nu}g\varphi^{-1}_{(\beta)t,\nu}\varphi^{-1}_{(\alpha)t,(\beta)t}\varphi_{(\alpha)t,(\beta)t}\\ &= \varphi_{\alpha,\nu}g\varphi^{-1}_{(\alpha)t,\nu}\varphi_{(\alpha)t,(\beta)t} = h_\alpha\varphi_{(\alpha)t,(\beta)t}.\end{aligned}$$

Thus, $h \in \mathrm{End}(S)$ by Proposition 3.

Finally, we show that $fhf = f$. Let $\alpha \in Y$. Using the fact that $f_\alpha = \varphi_{\alpha,\nu}f_\nu\varphi_{(\alpha)s,(\nu)s}-1$ and $f_{(\alpha)st} = \varphi_{(\alpha)st,\nu}f_\nu\varphi^{-1}_{(\alpha)sts,(\nu)s} = \varphi_{(\alpha)st,\nu}f_\nu\varphi^{-1}_{(\alpha)s,(\nu)s}$ and since $h_{(\alpha)s} \in \mathrm{Hom}(G_{(\alpha)s}, G_{(\alpha)st})$, we get

$$\begin{aligned}f_\alpha h_{(\alpha)s}f_{(\alpha)st} &= \varphi_{\alpha,\nu}f_\nu\varphi^{-1}_{(\alpha)s,(\nu)s}\varphi_{(\alpha)s,\nu}g\varphi^{-1}_{(\alpha)st,\nu}\varphi_{(\alpha)st,\nu}f_\nu\varphi^{-1}_{(\alpha)s,(\nu)s}\\ &= \varphi_{\alpha,\nu}f_\nu\varphi_{(\nu)s,\nu}gf_\nu\varphi^{-1}_{(\alpha)s,(\nu)s} = \varphi_{\alpha,\nu}f_\nu\varphi^{-1}_{(\alpha)s,(\nu)s} = f_\alpha.\end{aligned}$$

This shows that $fhf = f$ and consequently, $\mathrm{End}(S)$ is regular. □

Lemma 4 states that G_ν is necessarily an endo-regular group in case that $\mathrm{End}(S)$ is regular. If all the groups in the Clifford semigroup are pairwise isomorphic, i.e. the structure homomorphisms are bijective, then it is sufficient for S to be endo-regular that G_ν is an endo-regular group. This is an immediate consequence of Lemma 6.

Corollary 7. *Let Y be a semilattice which has a unique least element $\nu = \bigwedge Y$ and let $S = \bigcup_{\xi\in Y} G_\xi$ be a Clifford semigroup with bijective structure homomorphisms. Then* $\mathrm{End}(S)$ *is regular if and only if both* $\mathrm{End}(Y)$ *and* $\mathrm{End}(G_\nu)$ *are regular.*

Proof. Suppose that $\mathrm{End}(S)$ is regular. Then $\mathrm{End}(G_\nu)$ is regular by Lemma 4. Note that $\mathrm{End}(Y)$ is regular, whenever $\mathrm{End}(S)$ is regular.

Conversely, suppose that both $\mathrm{End}(G_\nu)$ and $\mathrm{End}(Y)$ are regular. Let $f \in \mathrm{End}(S)$. Then $s := \underline{f} \in \mathrm{End}(Y)$ and there is $t \in \mathrm{End}(Y)$ with $sts = s$. Consider the endomorphism $f_\nu\varphi_{(\nu)s,\nu} \in \mathrm{End}(G_\nu)$. Since $\mathrm{End}(G_\nu)$ is regular, there is $g \in \mathrm{End}(G_\nu)$ with $(f_\nu\varphi_{(\nu)s,\nu})g(f_\nu\varphi_{(\nu)s,\nu}) = f_\nu\varphi_{(\nu)s,\nu}$. Now, we multiply this equation with $\varphi^{-1}_{(\nu)s,\nu}$ from the right hand side. Since $\varphi_{(\nu)s,\nu}\varphi^{-1}_{(\nu)s,\nu}$ is the identity mapping on $G_{(\nu)s}$, we obtain $f_\nu\varphi_{(\nu)s,\nu}gf_\nu = f_\nu$. Moreover, for all $\alpha \in Y$, we have $\mathrm{Im}(\varphi_{\alpha,\nu}g) \subseteq G_\nu = \mathrm{Im}(\varphi_{(\alpha)t,\nu})$ since $\varphi_{(\alpha)t,\nu}$ is a bijection. Then by Lemma 6, we can conclude that $\mathrm{End}(S)$ is regular. □

Note that any idempotent endomorphism on a semigroup is regular. So, all Clifford semigroups with idempotent endomorphism monoid have a regular endomorphism monoid. In the last part of this section, we will characterize all Clifford semigroups with injective structure homomorphisms having an idempotent endomorphism monoid. We recall that $|Y| \leqslant 2$ in this case [11]. If $|Y| = 1$ then $S = G_\nu$. So, we have a group with idempotent endomorphism monoid. We focus us to the case $|Y| = 2$.

Lemma 8. *Let $Y = \{\alpha > \beta\}$ be a two-element chain and let $S = G_\alpha \cup G_\beta$ be a Clifford semigroup with an injective structure homomorphism $\varphi_{\alpha,\beta}$. If $\mathrm{End}(S)$ is a band then $\mathrm{End}(G_\beta)$ is a band.*

Proof. Suppose that $\mathrm{End}(S)$ is a band and let $g \in \mathrm{End}(G_\beta)$. We define a mapping $f : S \to S$ by $(x_\xi)f = (x_\xi)f_\xi$, for $\xi \in \{\alpha, \beta\}$, with $f_\alpha = \varphi_{\alpha,\beta} g$ and $f_\beta = g$. Then $f \in \mathrm{End}(S)$ by Proposition 2. Note that $(\beta)\underline{f} = \beta$. We observe that the restriction of $ff = f$ to G_β provides $g = f_\beta = f_\beta f_\beta = gg$, i.e. g is idempotent. Consequently, $\mathrm{End}(G_\beta)$ is a band. □

We can show that the idempotency of both monoids $\mathrm{End}(Y)$ and $\mathrm{End}(G_\beta)$ is sufficient for the idempotency of $\mathrm{End}(S)$.

Proposition 9. *Let $Y = \{\alpha > \beta\}$ be a two-element chain and let $S = G_\alpha \cup G_\beta$ be a Clifford semigroup with an injective structure homomorphism $\varphi_{\alpha,\beta}$. Then $\mathrm{End}(S)$ is a band if and only if $\mathrm{End}(G_\beta)$ is a band.*

Proof. If $\mathrm{End}(S)$ is a band then $\mathrm{End}(G_\beta)$ is a band by Lemma 8.

Suppose that $\mathrm{End}(G_\beta)$ is a band. Let $f \in \mathrm{End}(S)$. Then $s := \underline{f} \in \mathrm{End}(Y)$. We note that $ss = s$ by Proposition 1, we have that $f_\gamma = \varphi_{\gamma,\beta} f_\beta \varphi^{-1}_{(\gamma)s,(\beta)s}$ for $\gamma \in \{\alpha, \beta\}$. Let $\gamma \in \{\alpha, \beta\}$. Then

$$\begin{aligned} f_\gamma f_{(\gamma)s} &= \varphi_{\gamma,\beta} f_\beta \varphi^{-1}_{(\gamma)s,(\beta)s} \varphi_{(\gamma)s,\beta} f_\beta \varphi^{-1}_{(\gamma)ss,(\beta)s} \\ &= \varphi_{\gamma,\beta}(f_\beta \varphi_{(\beta)s,\beta})(f_\beta \varphi_{(\beta)s,\beta}) \varphi^{-1}_{(\beta)s,\beta} \varphi^{-1}_{(\gamma)s,(\beta)s} \\ &= \varphi_{\gamma,\beta}(f_\beta \varphi_{(\beta)s,\beta}) \varphi^{-1}_{(\beta)s,\beta} \varphi^{-1}_{(\gamma)s,(\beta)s} = \varphi_{\gamma,\beta} f_\beta \varphi^{-1}_{(\gamma)s,(\beta)s} = f_\gamma, \end{aligned}$$

since $f_\beta \varphi_{(\beta)s,\beta}$ belongs to the idempotent monoid $\mathrm{End}(G_\beta)$, i.e. $(f_\beta \varphi_{(\beta)s,\beta})(f_\beta \varphi_{(\beta)s,\beta}) = (f_\beta \varphi_{(\beta)s,\beta})$. So, we have shown that $ff = f$ and consequently, $\mathrm{End}(S)$ is a band. □

3. Completely regular endomorphism monoid

In this section, we study Clifford semigroups with completely regular endomorphism monoid. In fact, if $S = \bigcup_{\xi \in Y} G_\xi$ is a Clifford semigroup with a completely regular endomorphism monoid then the semilattice Y

has at most two elements [11]. If Y is the trivial semilattice consisting of a singleton element then S is a group and we have only to consider groups with completely regular endomorphism monoid. So, it remains the case that Y consists of two elements, i.e. $Y = \{\alpha > \beta\}$ is a two-element chain. If $Y = \{\alpha > \beta\}$ is a two element chain then $\mathrm{End}(Y) = \{id_Y, s_\alpha, s_\beta\}$, where id_Y is the identity mapping on Y and s_α (and s_β) is the constant mapping with the image α (and β, respectively). So, each element in $\mathrm{End}(Y)$ is idempotent. In particular, $\mathrm{End}(Y)$ is completely regular. In particular, we will consider the case that the group G_α does not have nontrivial normal subgroups. The characterization of all Clifford semigroups with completely regular endomorphism monoid is still an open problem.

Lemma 10. *Let $Y = \{\alpha > \beta\}$ be a two-element chain and let $S = G_\alpha \cup G_\beta$ be a Clifford semigroup with an injective structure homomorphism $\varphi_{\alpha,\beta}$. If* $\mathrm{End}(S)$ *is completely regular then* $\mathrm{End}(G_\beta)$ *is completely regular.*

Proof. Suppose that $\mathrm{End}(S)$ is completely regular and let $f \in \mathrm{End}(G_\beta)$. Then by Proposition 2, the following mapping $h : S \to S$ is an endomorphism on S. Let $(x_\xi)h = (x_\xi)h_\xi$, for $\xi \in \{\alpha, \beta\}$ with

$$h_\alpha = \varphi_{\alpha,\beta} f \quad \text{and} \quad h_\beta = f.$$

Since $\mathrm{End}(S)$ is completely regular, there is $g \in \mathrm{End}(S)$ with $hgh = h$ and $gh = hg$. Assume that $(\beta)\underline{g} = \alpha$. Then from $gh = hg$, it follows $g_\beta h_\alpha = h_\beta g_\beta$, where $(x_\beta)g_\beta h_\alpha \in G_\beta$ and $(x_\beta)h_\beta g_\beta \in G_\alpha$, a contradiction. Therefore, we have only to consider the case that $(\beta)\underline{g} = \beta$, i.e. $g_\beta \in \mathrm{End}(G_\beta)$. Now, $hgh = h$ and $gh = hg$ implies $f g_\beta f = f$ and $g_\beta f = f g_\beta$. Hence, f is a completely regular element in $\mathrm{End}(G_\beta)$. Therefore, $\mathrm{End}(G_\beta)$ is completely regular. $\square$

Proposition 11. *Let $Y = \{\alpha > \beta\}$ be a two-element chain and let $S = G_\alpha \cup G_\beta$ be a Clifford semigroup such that G_α does not have nontrivial normal subgroups and* $\mathrm{End}(G_\beta)$ *is completely regular. Then* $\mathrm{End}(S)$ *is completely regular.*

Proof. Let $f \in \mathrm{End}(S)$ and let $t := \underline{f}$.

Case 1: Suppose that $(\beta)t = \beta$. Then $f_\beta \in \mathrm{End}(G_\beta)$. Since $\mathrm{End}(G_\beta)$ is completely regular, there is $g \in \mathrm{End}(G_\beta)$ with $f_\beta g f_\beta = f_\beta$ and $g f_\beta = f_\beta g$. Because G_α does not have nontrivial normal subgroups, we can conclude that f_α is injective or f_α is the constant mapping $c^\alpha_{(\alpha)t} : G_\alpha \to \{e_{(\alpha)t}\}$.

Case 1.1: Suppose that f_α is injective with $f_\alpha = \varphi_{\alpha,\beta} f_\beta \varphi^{-1}_{(\alpha)t,\beta}$. If $(\alpha)t = \beta$ then $\mathrm{Im}(\varphi_{\alpha,\beta} g) \subseteq G_\beta = \mathrm{Im}(\varphi_{\beta,\beta}) = \mathrm{Im}(\varphi_{(\alpha)t,\beta})$. Suppose that $(\alpha)t = \alpha$ and let $x \in \operatorname{Im}(\varphi_{\alpha,\beta}g)$. Then f_β restricted to $\operatorname{Im}(\varphi_{\alpha,\beta})$ is a bijection on $\operatorname{Im}(\varphi_{\alpha,\beta})$ and $\varphi_{\alpha,\beta}$ is injective. Since f_β restricted to $\operatorname{Im}(\varphi_{\alpha,\beta})$ is a bijection on $\operatorname{Im}(\varphi_{\alpha,\beta})$, there is $\overline{x} \in \operatorname{Im}(\varphi_{\alpha,\beta})$ such that $x = \overline{x}f_\beta f_\beta g = \overline{x}f_\beta g f_\beta = \overline{x}f_\beta \in \operatorname{Im}(\varphi_{\alpha,\beta})$. This shows $\operatorname{Im}(\varphi_{\alpha,\beta}g) \subseteq \operatorname{Im}(\varphi_{\alpha,\beta})$. We define a mapping $h : S \to S$ by $(x_\xi)h = (x_\xi)h_\xi$ for any $\xi \in Y$, where $h_\xi = \varphi_{\xi,\beta}g\varphi^{-1}_{(\xi)t,\beta}$, for any $\xi \in Y$. Since $\operatorname{Im}(\varphi_{\alpha,\beta}g) \subseteq \operatorname{Im}(\varphi_{\alpha,\beta})$ and $\operatorname{Im}(\varphi_{\beta,\beta}g) \subseteq G_\beta = \operatorname{Im}(\varphi_{\beta,\beta}) = \operatorname{Im}(\varphi_{(\beta)t,\beta})$, the mapping h is well defined. Note that $g = h_\beta$. Moreover, $h_\alpha\varphi_{(\alpha)t,\beta} = \varphi_{\alpha,\beta}g\varphi^{-1}_{(\alpha)t,\beta}\varphi_{(\alpha)t,\beta} = \varphi_{\alpha,\beta}g = \varphi_{\alpha,\beta}h_\beta$ and h is an endomorphism by Proposition 3. Now we need to show that h is a pseudoinverse of f. Note that $f_\beta = f_\beta g f_\beta = f_\beta h_\beta f_\beta$. Hence, we have still to show that $f_\alpha = f_\alpha h_{(\alpha)t} f_{(\alpha)t}$. By Proposition 1, we have $f_\alpha = \varphi_{\alpha,\beta}f_\beta\varphi^{-1}_{(\alpha)t,\beta}$ and $f_{(\alpha)t} = \varphi_{(\alpha)t,\beta}f_\beta\varphi^{-1}_{(\alpha)t,\beta}$. So, we get

$$\begin{aligned} f_\alpha h_{(\alpha)t} f_{(\alpha)t} &= \varphi_{\alpha,\beta}f_\beta\varphi^{-1}_{(\alpha)t,\beta}\varphi_{(\alpha)t,\beta}g\varphi^{-1}_{(\alpha)t,\beta}\varphi_{(\alpha)t,\beta}f_\beta\varphi^{-1}_{(\alpha)t,\beta} \\ &= \varphi_{\alpha,\beta}f_\beta g f_\beta\varphi^{-1}_{(\alpha)t,\beta} = \varphi_{\alpha,\beta}f_\beta\varphi^{-1}_{(\alpha)t,\beta} = f_\alpha. \end{aligned}$$

Moreover, we have

$$\begin{aligned} h_\alpha f_{(\alpha)t} &= \varphi_{\alpha,\beta}g\varphi^{-1}_{(\alpha)t,\beta}\varphi_{(\alpha)t,\beta}f_\beta\varphi^{-1}_{(\alpha)t,\beta} \\ &= \varphi_{\alpha,\beta}g f_\beta\varphi^{-1}_{(\alpha)t,\beta} = \varphi_{\alpha,\beta}f_\beta g\varphi^{-1}_{(\alpha)t,\beta} \\ &= \varphi_{\alpha,\beta}f_\beta\varphi^{-1}_{(\alpha)t,\beta}\varphi_{(\alpha)t,\beta}g\varphi^{-1}_{(\alpha)t,\beta} = f_\alpha h_{(\alpha)t}. \end{aligned}$$

Case 1.2: Suppose now that f_α is a constant mapping $c^\alpha_{(\alpha)t}$. We define a mapping $h : S \to S$ by $(x_\xi)h = (x_\xi)h_\xi$ for any $\xi \in Y$, where $h_\alpha = c^\alpha_{(\alpha)t}$ and $h_\beta = f_\beta gg$. First, we will verify that $\varphi_{\alpha,\beta}f_\beta$ is a constant mapping. For this let $x_\alpha, \widehat{x}_\alpha \in G_\alpha$ and $x_\beta := (x_\alpha)\varphi_{\alpha,\beta}$, $\widehat{x}_\beta := (\widehat{x}_\alpha)\varphi_{\alpha,\beta}$. Then there is $a \in G_\alpha$ such that $\widehat{x}_\beta = (x_\beta)((a)\varphi_{\alpha,\beta})$ in the subgroup $\operatorname{Im}(\varphi_{\alpha,\beta})$ of G_β. We have

$$\begin{aligned} (\widehat{x}_\beta)f = (x_\beta((a)\varphi_{\alpha,\beta}))f &= ((x_\beta)\varphi_{\beta,\beta}(a)\varphi_{\alpha,\beta})f = (x_\beta * a)f \\ &= (x_\beta)f * (a)f = (x_\beta)f\varphi_{\beta,\beta}((a)f)\varphi_{(\alpha)t,\beta} \\ &= (x_\beta)f(e_{(\alpha)t})\varphi_{(\alpha)t,\beta} = (x_\beta)fe_\beta = (x_\beta)f. \end{aligned}$$

where $*$ is the operation on S. This provides $(\widehat{x}_\alpha)\varphi_{\alpha,\beta}f_\beta = (x_\alpha)\varphi_{\alpha,\beta}f_\beta$. Therefore, $\varphi_{\alpha,\beta}f_\beta$ is a constant mapping $c^\alpha_\beta : G_\alpha \to \{e_\beta\}$ and we can calculate that $\varphi_{\alpha,\beta}h_\beta = h_\alpha\varphi_{(\alpha)t,\beta} = c^\alpha_\beta$. Thus, h is an endomorphism by Proposition 3. Further, we can show that $fhf = f$ as well as $fh = hf$. In fact, we have $f_\beta h_\beta f_\beta = f_\beta f_\beta gg f_\beta = f_\beta g f_\beta g f_\beta = f_\beta$ and $f_\beta h_\beta = f_\beta f_\beta gg = f_\beta gg f_\beta = h_\beta f_\beta$. Further, it is easy to verify that $f_\alpha h_{(\alpha)t} f_{(\alpha)t} = f_\alpha h_{(\alpha)t} = h_\alpha f_{(\alpha)t} = c^\alpha_{(\alpha)t}$.

Case 2: Suppose now that $(\beta)t = \alpha$. Then $(\alpha)t = \alpha$. Since G_α does not have nontrivial normal subgroups, we conclude that f_β is onto G_α or f_β is the constant mapping $c_\alpha^\beta : G_\beta \to \{e_\alpha\}$. If $f_\beta = c_\alpha^\beta$ then $f_\alpha = \varphi_{\alpha,\beta} f_\beta \varphi^{-1}_{(\alpha)t,(\beta)t} = \varphi_{\alpha,\beta} c_\alpha^\beta \varphi^{-1}_{\alpha,\alpha} = \varphi_{\alpha,\beta} c_\alpha^\beta = c_\alpha^\alpha : G_\alpha \to \{e_\alpha\}$. This shows that f is the constant mapping $c_\alpha^S : S \to \{e_\alpha\}$. Thus, f is idempotent, i.e. f is completely regular. Finally, suppose that f_β is onto G_α. Note that $f_\beta \varphi_{\alpha,\beta} \in \mathrm{End}(G_\beta)$. Since $\mathrm{End}(G_\beta)$ is completely regular, there is $u \in \mathrm{End}(G_\beta)$ with $(f_\beta\varphi_{\alpha,\beta})u(f_\beta\varphi_{\alpha,\beta}) = f_\beta\varphi_{\alpha,\beta}$ and $(f_\beta\varphi_{\alpha,\beta})u = u(f_\beta\varphi_{\alpha,\beta})$. This implies $f_\beta\varphi_{\alpha,\beta}uf_\beta = f_\beta$ since $\varphi_{\alpha,\beta}$ is injective because G_α does not have nontrivial normal subgroups. First, we will show that $\mathrm{Im}(\varphi_{\alpha,\beta}u) \subseteq \mathrm{Im}(\varphi_{\alpha,\beta})$. For this let $x \in \mathrm{Im}(\varphi_{\alpha,\beta}u)$. Since f_β is onto G_α, there is $\overline{x} \in G_\beta$ such that $x = \overline{x} f_\beta \varphi_{\alpha,\beta} u = \overline{x} u f_\beta \varphi_{\alpha,\beta} \in \mathrm{Im}(\varphi_{\alpha,\beta})$. Then, we define a mapping $h : S \to S$ by $(x_\xi)h = (x_\xi)h_\xi$ for any $\xi \in Y$, where $h_\alpha = \varphi_{\alpha,\beta}uf_\beta\varphi_{\alpha,\beta}u\varphi^{-1}_{\alpha,\beta}$ and $h_\beta = uf_\beta\varphi_{\alpha,\beta}u\varphi^{-1}_{\alpha,\beta}$. Because of $\mathrm{Im}(\varphi_{\alpha,\beta}u) \subseteq \mathrm{Im}(\varphi_{\alpha,\beta})$, the mapping h is well defined. Further, we have $h_\alpha\varphi_{(\alpha)t,(\beta)t} = h_\alpha\varphi_{\alpha,\alpha} = h_\alpha = \varphi_{\alpha,\beta}uf_\beta\varphi_{\alpha,\beta}u\varphi^{-1}_{\alpha,\beta} = \varphi_{\alpha,\beta}h_\beta$. Thus, $h \in \mathrm{End}(S)$ by Proposition 3. We show now that h is a pseudoinverse of f. Note that $f_\alpha = \varphi_{\alpha,\beta} f_\beta \varphi_{(\alpha)t,(\beta)t} = \varphi_{\alpha,\beta} f_\beta \varphi_{\alpha,\alpha} = \varphi_{\alpha,\beta} f_\beta$. We have

$$
\begin{aligned}
f_\beta h_\alpha f_\alpha &= f_\beta\varphi_{\alpha,\beta}uf_\beta\varphi_{\alpha,\beta}u\varphi^{-1}_{\alpha,\beta}\varphi_{\alpha,\beta}f_\beta = f_\beta\varphi_{\alpha,\beta}uf_\beta\varphi_{\alpha,\beta}\varphi^{-1}_{\alpha,\beta}\\
&= f_\beta\varphi_{\alpha,\beta}\varphi^{-1}_{\alpha,\beta} = f_\beta,\\
f_\alpha h_\alpha f_\alpha &= \varphi_{\alpha,\beta}f_\beta\varphi_{\alpha,\beta}uf_\beta\varphi_{\alpha,\beta}u\varphi^{-1}_{\alpha,\beta}\varphi_{\alpha,\beta}f_\beta = \varphi_{\alpha,\beta}f_\beta\varphi_{\alpha,\beta}uf_\beta\varphi_{\alpha,\beta}\varphi^{-1}_{\alpha,\beta}\\
&= \varphi_{\alpha,\beta}f_\beta\varphi_{\alpha,\beta}\varphi^{-1}_{\alpha,\beta} = \varphi_{\alpha,\beta}f_\beta = f_\alpha,
\end{aligned}
$$

and

$$
\begin{aligned}
h_\beta f_\alpha &= uf_\beta\varphi_{\alpha,\beta}u\varphi^{-1}_{\alpha,\beta}\varphi_{\alpha,\beta}f_\beta = uf_\beta\varphi_{\alpha,\beta}uf_\beta\varphi_{\alpha,\beta}\varphi^{-1}_{\alpha,\beta}\\
&= f_\beta\varphi_{\alpha,\beta}uf_\beta\varphi_{\alpha,\beta}u\varphi^{-1}_{\alpha,\beta} = f_\beta h_\alpha
\end{aligned}
$$

as well as $f_\alpha h_\alpha = \varphi_{\alpha,\beta} f_\beta h_\alpha = \varphi_{\alpha,\beta} h_\beta f_\alpha = h_\alpha f_\alpha$. Consequently, f is a completely regular element of $\mathrm{End}(S)$. □

So, we can state the main result of this section concerning Clifford semigroups with completely regular endomorphism monoid.

Theorem 12. *Let $Y = \{\alpha > \beta\}$ be a two-element chain and let $S = G_\alpha \cup G_\beta$ such that G_α does not have nontrivial normal subgroups. Then* $\mathrm{End}(S)$ *is completely regular if and only if* $\mathrm{End}(G_\beta)$ *is completely regular.*

Proof. Suppose that $\mathrm{End}(S)$ is completely regular. The fact that $\mathrm{End}(G_\beta)$ is completely regular follows from Lemma 10. The converse direction is given by Proposition 11. □

Now, we drop the condition that G_α has no proper normal subgroups. We consider Clifford semigroups $S = G_\alpha \cup G_\beta$ $(\alpha > \beta)$ such that G_β is the direct product of G_α and a group. Note that such Clifford semigroups can be constructed whenever G_β is an abelian group by the fundamental theorem of abelian groups.

Proposition 13. *Let $Y = \{\alpha > \beta\}$ be a two-element chain and let $S = G_\alpha \cup G_\beta$ be a Clifford semigroup with an injective structure homomorphism $\varphi_{\alpha,\beta}$, where G_β is the direct product of the group G_α and a group with the identity element e such that $(x_\alpha)\varphi_{\alpha,\beta} = (x_\alpha, e)$. If $\mathrm{End}(S)$ is completely regular then $\mathrm{End}(G_\alpha)$ is completely regular.*

Proof. Suppose that $\mathrm{End}(S)$ is completely regular. Then $\mathrm{End}(G_\beta)$ is completely regular by Lemma 10. We put $A := \mathrm{Im}(\varphi_{\alpha,\beta}) = \{(x_\alpha, e) : x_\alpha \in G_\alpha\}$. Clearly, A is isomorphic to G_α. We will show that $\mathrm{End}(A)$ is completely regular. For this let $f \in \mathrm{End}(A)$. We define a mapping $h : G_\beta \to G_\beta$ by

$$(x_\alpha, x)h = (x_\alpha, e)f \cdot (e_\alpha, x) \quad \text{for all } (x_\alpha, x) \in G_\beta,$$

where $\cdot$ is the multiplication on G_β. We have to show that h is an endomorphism on G_β. It is easy to verify that

$$(x_\alpha, e) \cdot (e_\alpha, x) = (e_\alpha, x) \cdot (x_\alpha, e).$$

Let $(x_\alpha, x), (\widetilde{x}_\alpha, \widetilde{x}) \in G_\beta$. Then we obtain

$$\begin{aligned} ((x_\alpha, x) \cdot (\widetilde{x}_\alpha, \widetilde{x}))h = ((x_\alpha\widetilde{x}_\alpha, x\widetilde{x}))h &= (x_\alpha\widetilde{x}_\alpha, e)f \cdot (e_\alpha, x\widetilde{x}) \\ &= (x_\alpha, e)f \cdot (\widetilde{x}_\alpha, e)f \cdot (e_\alpha, x) \cdot (e_\alpha, \widetilde{x}) \\ &= (x_\alpha, e)f \cdot (e_\alpha, x) \cdot (\widetilde{x}_\alpha, e)f \cdot (e_\alpha, \widetilde{x}) \\ &= (x_\alpha, x)f \cdot (\widetilde{x}_\alpha, \widetilde{x})f. \end{aligned}$$

This shows that $h \in \mathrm{End}(G_\beta)$. Thus, there is $g \in \mathrm{End}(G_\beta)$ with $hgh = h$ and $hg = gh$ since $\mathrm{End}(G_\beta)$ is completely regular. We show now that the image of g restricted to A belongs to A. For this, let $(x_\alpha, e) \in A$. Assume that $(x_\alpha, e)g \notin A$. Then by $hg = gh$ and by the definition of h, we obtain that $(x_\alpha, e)hgh = (x_\alpha, e)ghh \notin A$. On the other hand, we have $(x_\alpha, e)h = (x_\alpha, e)f \in A$. This contradicts to $hgh = h$. Consequently, the image of g restricted to A is a subset of A. Note that $\varphi_{\alpha,\beta}^{-1}\varphi_{\alpha,\beta}$ is the identity mapping on A. Then $\delta\varphi_{\alpha,\beta}^{-1}\varphi_{\alpha,\beta} = \delta$ for any mapping δ with $\mathrm{Im}(\delta) \subseteq A$ and $\varphi_{\alpha,\beta}^{-1}\varphi_{\alpha,\beta}h = f$. Moreover, the mapping $p : A \to A$ defined by

$$p = \varphi_{\alpha,\beta}^{-1}\varphi_{\alpha,\beta}g$$

is an endomorphism on A. Thus, we obtain $fpf = f\varphi_{\alpha,\beta}^{-1}\varphi_{\alpha,\beta}gf = fgf = \varphi_{\alpha,\beta}^{-1}\varphi_{\alpha,\beta}hg\varphi_{\alpha,\beta}^{-1}\varphi_{\alpha,\beta}h = \varphi_{\alpha,\beta}^{-1}\varphi_{\alpha,\beta}hgh = \varphi_{\alpha,\beta}^{-1}\varphi_{\alpha,\beta}h = f$ as well as $fp = f\varphi_{\alpha,\beta}^{-1}\varphi_{\alpha,\beta}g = fg = \varphi_{\alpha,\beta}^{-1}\varphi_{\alpha,\beta}hg = \varphi_{\alpha,\beta}^{-1}\varphi_{\alpha,\beta}gh = ph = p\varphi_{\alpha,\beta}^{-1}\varphi_{\alpha,\beta}h = pf$. This shows that f is a completely regular element in $\mathrm{End}(A)$. So, the proof is done. $\square$

We finish this section with the case that the structure homomorphism $\varphi_{\alpha,\beta}$ is bijective. Such Clifford semigroups with completely regular endomorphism monoid can be characterized by the fact that $\mathrm{End}(G_\beta)$ is completely regular (and thus, $\mathrm{End}(G_\alpha)$ is it also). Here, we can drop any restriction to the groups G_α and G_β.

Theorem 14. *Let $Y = \{\alpha > \beta\}$ be a two-element chain and let $S = G_\alpha \cup G_\beta$ be a Clifford semigroup with a bijective structure homomorphism $\varphi_{\alpha,\beta}$. Then* $\mathrm{End}(S)$ *is completely regular if and only if* $\mathrm{End}(G_\beta)$ *is completely regular.*

Proof. Suppose that $\mathrm{End}(S)$ is completely regular. Then Lemma 10 shows that $\mathrm{End}(G_\beta)$ is completely regular.

Suppose now that $\mathrm{End}(G_\beta)$ is completely regular. Then $\mathrm{End}(G_\alpha)$ is also completely regular since G_α is isomorphic to G_β. Let $f \in \mathrm{End}(S)$ with $t := \underline{f}$. Then it is obvious that $(\alpha)t = \alpha$ or $(\beta)t = \beta$. We consider the case that $(\alpha)t = \alpha$. If $(\beta)t = \beta$ then the proof is similar. From $(\alpha)t = \alpha$, it follows $f_\alpha \in \mathrm{End}(G_\alpha)$. Since $\mathrm{End}(G_\alpha)$ is completely regular, there is $g \in \mathrm{End}(G_\alpha)$ with $f_\alpha g f_\alpha = f_\alpha$ and $f_\alpha g = g f_\alpha$. We define a mapping $h : S \to S$ by $(x_\xi)h = (x_\xi)h_\xi$ for $\xi \in \{\alpha, \beta\}$ with

$$h_\alpha = g \quad \text{and} \quad h_\beta = \varphi_{\alpha,\beta}^{-1} g \varphi_{\alpha,\beta}.$$

Since $\varphi_{\alpha,\beta}$ is bijective, h is well defined. In particular, we have $\varphi_{\alpha,\beta}h_\beta = \varphi_{\alpha,\beta}\varphi_{\alpha,\beta}^{-1}g\varphi_{\alpha,\beta} = g\varphi_{\alpha,\beta} = h_\alpha\varphi_{\alpha,\beta}$. By Proposition 3, h is an endomorphism on S. Note that $f_\beta = \varphi_{\alpha,\beta}^{-1} f_\alpha \varphi_{\alpha,(\beta)t}$ follows from Proposition 1, since $\varphi_{\alpha,\beta}$ is bijective. Moreover, it holds $f_\alpha = \varphi_{\alpha,\alpha}^{-1} f_\alpha \varphi_{\alpha,\alpha}$ and $h_\alpha = \varphi_{\alpha,\alpha}^{-1} g \varphi_{\alpha,\alpha}$. Then, we obtain $f_\alpha h_\alpha f_\alpha = f_\alpha g f_\alpha = f_\alpha$ and $f_\alpha h_\alpha = f_\alpha g = g f_\alpha = h_\alpha f_\alpha$.

On the other hand, we can calculate

$$\begin{aligned} f_\beta h_{(\beta)t} f_{(\beta)t} &= \varphi_{\alpha,\beta}^{-1} f_\alpha \varphi_{\alpha,(\beta)t} \varphi_{\alpha,(\beta)t}^{-1} g \varphi_{\alpha,(\beta)t} \varphi_{\alpha,(\beta)t}^{-1} f_\alpha \varphi_{\alpha,(\beta)tt} \\ &= \varphi_{\alpha,\beta}^{-1} f_\alpha g f_\alpha \varphi_{\alpha,(\beta)tt} = \varphi_{\alpha,\beta}^{-1} f_\alpha \varphi_{\alpha,(\beta)t} = f_\beta. \end{aligned}$$

To obtain $(\beta)t = (\beta)tt$, we used that $\mathrm{End}(Y)$ is idempotent, whenever Y is a two-element chain. Finally, we can compute that

$$\begin{aligned} f_\beta h_{(\beta)t} &= \varphi_{\alpha,\beta}^{-1} f_\alpha \varphi_{\alpha,(\beta)t} \varphi_{\alpha,(\beta)t}^{-1} g \varphi_{\alpha,(\beta)t} = \varphi_{\alpha,\beta}^{-1} f_\alpha g \varphi_{\alpha,(\beta)t} \\ &= \varphi_{\alpha,\beta}^{-1} g f_\alpha \varphi_{\alpha,(\beta)t} = \varphi_{\alpha,\beta}^{-1} g \varphi_{\alpha,\beta} \varphi_{\alpha,\beta}^{-1} f_\alpha \varphi_{\alpha,(\beta)t} = h_\beta f_\beta. \end{aligned}$$

Altogether, we have shown that $fhf = f$ and $fh = hf$. Consequently, f is a completely regular element of $\mathrm{End}(S)$. □

Open problem. The characterization of all Clifford semigroups with completely regular endomorphism monoid is still an open problem if the structure homomorphisms $\varphi_{\alpha,\beta}$ are not surjective.

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