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Some properties of $E(G, W, \mathcal{F}_T G)$ and an application in the theory of splittings of groups*

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Abstract. Let us consider W a G-set and M a $\mathbb{Z}_2 G$-module, where G is a group. In this paper we investigate some properties of the cohomological the theory of splittings of groups. Namely, we give a proof of the invariant $E(G, W, M)$, defined in [5] and present related results with independence of $E(G, W, M)$ with respect to the set of G-orbit representatives in W and properties of the invariant $E(G, W, \mathcal{F}_T G)$ establishing a relation with the end of pairs of groups $\widetilde{e}(G, T)$, defined by Kropphller and Holler in [15]. The main results give necessary conditions for G to split over a subgroup T, in the cases where $M = \mathbb{Z}_2(G/T)$ or $M = \mathcal{F}_T G$.

Introduction

The theory of splittings of groups is closely related to the theory of ends of groups and pairs of groups. The first result relating splittings of groups to the theory of ends of groups was given by Stallings in [19], where it is proved that if G is finitely generated, then G splits over some finite subgroup if only if $e(G) \geqslant 2$. The number of ends $e(G)$ of a group G was introduced by Freudenthal - Hopf ([12], [13]) for finitely generated groups

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and it was motivated by theory of ends of topological spaces. Later, Specker ([18]) extended the definition to cover arbitrary groups. In the case where G is infinite, one has a cohomological formula $e(G) = 1 + \dim_{\mathbb{Z}_2} H^1(G;\mathbb{Z}_2G)$. A natural extension of $e(G)$ is the invariant end $e(G,T)$ for pairs of group, T a subgroup of G, which was introduced independently by Houghton ([14]) and Scott ([16]).

Invariant ends for pairs of groups in a more general set were studied by Kropholler and Roller, for example in [15]. The authors defined the invariant end $e(G,S,M)$ for S a subgroup of G and M, a Z_2G-module and presented an interesting study for $\widetilde{e}(G,S) := e(G,S,\mathcal{P}(S))$, where $\mathcal{P}(S)$ is the power set of all subsets of S. In an attempt to obtain a cohomological formula for $e(G,T)$, Andrade and Fanti ([2]) defined an invariant end, $E(G,\mathcal{S},M)$, where $\mathcal{S}$ is a family of subgroups of G and M is a $\mathbb{Z}_2G$-module. Afterwards, in [5] the authors adapted the definition of $E(G,\mathcal{S},M)$ by using the cohomology theory of Dicks and Dunwoody to pairs (G,W), where W is a G-set, and defining the invariant $E(G,W,M)$, they obtained general properties and results about splittings and duality of groups, particularly by considering $E(G,W,\mathbb{Z}_2)$.

In this work, we present properties of $E(G,W,M)$ and by using derivation we prove that the definition of this invariant is independent of the set of G-orbit representatives in W. We consider the case where M is the $\mathbb{Z}_2G$-module $\mathcal{F}_TG$ and we show that, under certain conditions for T, there is a relation between $E(G,W,\mathcal{F}_TG)$ and the invariant $\widetilde{e}(G,T)$. Finally, we present necessary conditions to G splits over a, non necessary finite, subgroup T where $M = \mathbb{Z}_2(G/T)$ or $M = \mathcal{F}_TG$.

1. Preliminaries

Let us recall definitions and results important for this work. We will consider $R = \mathbb{Z}$ or $\mathbb{Z}_2$.

Definition 1. *A group G is defined by generators $X = \{x_k\}$ and relations $\mathcal{R} = \{r_j = 1\}$, if $G \simeq F/H$, where F is a free group generated by X and H is the smallest normal subgroup of F generated by $\{r_j\}$. In this case, $\langle X;\mathcal{R}\rangle$ is a presentation of G.*

Definition 2. *Consider the groups G_1 and G_2 with presentations $G_k = \langle X_k;\mathcal{R}_k\rangle$, $k = 1,2$, where X_k is a set of generators and $\mathcal{R}_k$ is a set of relations for G_k.*

1) *If $T_1 \subset G_1$ and $T_2 \subset G_2$ are subgroups and $\theta : T_1 \to T_2$ is an isomorphism from T_1 into T_2, then the free product $G_1 *_T G_2$ of G_1 and*

G_2 with the amalgamated subgroup $T = T_1 = T_2$ is defined by

$$G_1 *_T G_2 := \langle X_1, X_2; \mathcal{R}_1, \mathcal{R}_2, t = \theta(t), \forall t \in T_1 \rangle .$$

2) Let G_1 be a group with presentation $G_1 = \langle X; \mathcal{R}_1 \rangle$. If T and T' are subgroups of G_1 and $\theta : T \to T'$ is an isomorphism, then the HNN-group $G_1 *_{T,\theta}$ over a base group G_1, with respect to θ and stable letter p, is given by

$$G_1 *_{T,\theta} := \left\langle X_1, p; \mathcal{R}_1, p^{-1}tp = \theta(t), \forall t \in T \right\rangle .$$

Remark 1. These classes of groups arise naturally when we calculate the fundamental group of certain spaces.

Definition 3. *A group G splits over a subgroup T if $G = G_1 *_T G_2$ with $G_1 \neq T \neq G_2$ or $G = G_1 *_{T,\theta}$.*

Example 1. 1) $G = G_1 * G_2 = G_1 *_{\{1\}} G_2$, in particular, $\mathbb{Z} * \mathbb{Z} = \mathbb{Z} *_{\{1\}} \mathbb{Z}$.

2) $\mathbb{Z} = \{1\} *_{\{1\},\, \mathrm{id}} = \left\langle \{1\}, p, psp^{-1} = s, \forall s \in \{1\} \right\rangle = \langle p \rangle$.

3) The fundamental group G of the Torus is $\mathbb{Z} \oplus \mathbb{Z}$ which is the HNN group, $\mathbb{Z} \oplus \mathbb{Z} = \langle a \rangle \oplus \langle b \rangle =$ "$\mathbb{Z} *_{\mathbb{Z},\, \mathrm{id}}$". We can see this by considering $H = \langle a \rangle \simeq \mathbb{Z}$, b the stable letter, $T = T' = H$ and $\sigma = \mathrm{id} : T \to T$. Thus, $H *_{T,\, \mathrm{id}} = \langle a, b;\ b^{-1} \cdot a \cdot b = \sigma(a) \rangle = \langle a, b;\ a \cdot b = b \cdot a \rangle = \mathbb{Z} \oplus \mathbb{Z}$. Therefore, $\mathbb{Z} \oplus \mathbb{Z}$ splits over an infinite subgroup.

4) The fundamental group of the Bitorus is $(\mathbb{Z} * \mathbb{Z}) *_{\mathbb{Z}} (\mathbb{Z} * \mathbb{Z})$. It follows from the Seifert-van-Kampen Theorem for appropriate subspaces.

Remark 2. If $G = G_1 *_T G_2$ with $G_1 \neq T \neq G_2$ or $G = G_1 *_{T,\theta}$, then $[G : G_i] = \infty$, $i = 1, 2$ (see [3]).

Proposition 1 ([8])**.** 1) *If $G = G_1 *_T G_2$, one has the following short exact sequence of RG-modules:*

$$0 \to R(G/T) \xrightarrow{\alpha} R(G/G_1) \oplus R(G/G_2) \xrightarrow{\bar{\varepsilon}} R \to 0,$$

where α is given by $\alpha(gT) = (gG_1, -gG_2)$, $g \in G$ and $\bar{\varepsilon}(gG_1, 0) = \bar{\varepsilon}(0, gG_2) = 1$. Here $\bar{\varepsilon}$ is the augmentation map.

2) If $G = G_1 *_{T,\theta}$, one has the following short exact sequence of RG-modules:

$$0 \to R(G/T) \xrightarrow{\alpha} R(G/G_1) \xrightarrow{\tilde{\varepsilon}} R \to 0,$$

where α is given by $\alpha(gT) = gG_1 - gp^{-1}G_1$, p is the stable letter of G and $\tilde{\varepsilon}$ is the augmentation map.

Now let us introduce a relationship between the concepts of derivation groups and of the cohomology group $H^1(G; M)$ (see [10]), used to prove the independence of $E(G, W, M)$ with respect to set of G-orbit representatives in W.

Definition 4. *A derivation of a group G in an RG-module M is a map $d : G \to M$, such that, $d(gh) = d(g) + gd(h)$, for all $g, h \in G$.*

Example 2. For each $m \in M$, the map $d_m : G \to M$; $d_m(g) := gm - m$ is a derivation.

The derivations of the previous example are called principal derivations.

Proposition 2 ([10])**.** *Let G be a group and M an RG-module. Then*

$$H^1(G; M) \simeq \frac{\mathrm{Der}(G, M)}{P(G, M)},$$

where $\mathrm{Der}(G, M)$ *is the set of derivations and $P(G, M) = \{d_m,\ m \in M\}$ is the set of principal derivations of G in M.*

Proposition 3 (Shapiro's Lemma; see [10])**.** *If S is a subgroup of the group G and M is an RS-module, then $H^*(G; \mathrm{Coind}_S^G M) \simeq H^*(S; M)$.*

Example 3. Let G be a group. In $\mathcal{P}(G)$, the power set of G, considered with the operation of symmetric difference we can define an induced G-action: $G \times \mathcal{P}(G) \to \mathcal{P}(G)$; $(g, A) \mapsto g \cdot A = \{g \cdot a;\ a \in A\}$, which turns out $\mathcal{P}(G)$ a Z_2G-module. One has, by Shapiro's Lemma,

$$\begin{aligned} H^n(G; \mathcal{P}(G)) &\simeq H^n(G; \mathrm{Coind}_{\{1\}}^G \mathbb{Z}_2) \\ &\simeq H^n(\{1\}; \mathbb{Z}_2) \simeq \begin{cases} \mathbb{Z}_2, & \text{se } n = 0, \\ 0, & \text{se } n \geqslant 1. \end{cases} \end{aligned}$$

Proposition 4 (Mackey's Formula; see [10], Proposition III.5.6)**.** *Let S and T be subgroups of a group G and consider $\mathbb{E}$ a set of representatives for the double classes SgT. Then, for any RT-module M, there exists an RS-isomorphism*

$$\mathrm{Res}_S^G \mathrm{Ind}_T^G M \simeq \bigoplus_{g \in \mathbb{E}} \mathrm{Ind}_{S \cap gTg^{-1}}^S \mathrm{Res}_{S \cap gTg^{-1}}^{gTg^{-1}} gM.$$

In particular, if T is a normal subgroup of G and $T \subset S$, then there exists a $\mathbb{Z}_2T$-isomorphism

$$\mathrm{Res}_S^G \mathrm{Ind}_T^G M \simeq \bigoplus_{g \in \mathbb{E}} \mathrm{Ind}_T^S gM,$$

and in the case of $T = \{1\}$ *and* L *is a set representatives for the classes* Sg*, then*

$$\operatorname{Res}_S^G \operatorname{Ind}_{\{1\}}^G M \simeq \bigoplus_{g \in L} \operatorname{Ind}_{\{1\}}^S gM.$$

Definition 5 ([11])**.** *Consider* G *a group,* W *a* G*-set and* RW *the free* R*-module generated by* W*. Let* $\varepsilon : RW \to R$ *be the augmentation map,* $\Delta = \ker \varepsilon$*,* M *an* RG*-module and* $P \to \Delta$ *a projective* RG*-resolution of* Δ*. The relative cohomology group of the pair* (G, W)*, with coefficients in* M*, is defined by*

$$H^k(G, W; M) := H^{k-1}(\operatorname{Hom}_{RG}(P, M)), \textit{for all } k \in \mathbb{Z}.$$

For the relative cohomology group of pairs $(G, \mathcal{S})$, where $\mathcal{S}$ is a family of subgroups of G, we define:

Definition 6 ([9])**.** *Let* $(G, \mathcal{S})$ *be a pair with* G *a group and* $\mathcal{S} = \{S_i, i \in I\}$ *a family of subgroups of infinite index in* G*,* M *a* $\mathbb{Z}_2G$*-module,* $\varepsilon' : \mathbb{Z}_2(G/\mathcal{S}) \to \mathbb{Z}_2$*;* $\varepsilon'(gS_i) = 1$*, the augmentation map,* $\Delta_{\mathcal{S}}$ *the kernel of* ε' *and* $F \twoheadrightarrow \mathbb{Z}_2$ *a* $\mathbb{Z}_2G$*-projective resolution of the trivial* $\mathbb{Z}_2G$*-module* $\mathbb{Z}_2$*.*

The relative cohomology group of $(G, \mathcal{S})$*, with coefficients in* M*, is defined for all* n *by*

$$\begin{aligned} H^n(G, \mathcal{S}; M) &:= H^{n-1}(G; \operatorname{Hom}_{\mathbb{Z}_2}(\Delta_{\mathcal{S}}, M)) \\ &= H^{n-1}(\operatorname{Hom}_{\mathbb{Z}_2G}(F, \operatorname{Hom}_{\mathbb{Z}_2}(\Delta_{\mathcal{S}}, M)). \end{aligned}$$

The next result provides a relation between the relative cohomology groups of the pair (G, W) and the relative cohomology groups of the pair $(G, \mathcal{S})$.

Theorem 1 ([7])**.** *Let* G *be a group,* M *an* RG*-module and* W *a* G*-set. Consider* $\mathbb{E} = \{w_i, i \in I\}$ *a set of orbit representatives in* W *and let* $\mathcal{S} = \{S_i := G_{w_i}, i \in I\}$ *be the family of stabilizer subgroups of* w_i *in* $\mathbb{E}$*. Then,*

$$H^k(G, W; M) \simeq H^k(G, \mathcal{S}; M).$$

Proposition 5. *Let* (G, W) *be a pair where* G *is a group and* W *is a* G*-set. Consider* $\mathbb{E}$ *a set of* G*-orbit representatives in* W*,* G_w *the* G*-stabilizer of* w*, for each* $w \in \mathbb{E}$ *and* M *an* RG*-module. Then, we have the long exact sequence:*

$$\begin{aligned} 0 \to H^0(G; M) \to H^0(W; M) \xrightarrow{\delta} H^1(G, W; M) \xrightarrow{J} \\ \xrightarrow{J} H^1(G; M) \xrightarrow{\operatorname{res}_W^G} H^1(W; M) \to \cdots, \end{aligned}$$

where

$$\operatorname{res}^G_W \colon H^n(G;M) \to H^n(W;M) \overset{\text{not.}}{=} \prod_{w\in\mathbb{E}} H^n(G_w;M)$$
$$\operatorname{res}^G_W([f]) := \left(\operatorname{res}^G_{G_w}[f]\right)_{w\in\mathbb{E}},$$

and $\operatorname{res}^G_{G_w} : H^n(G;M) \to H^n(G_w;M)$ *is the restriction map induced by the inclusion map* $G_w \hookrightarrow G$, $w \in \mathbb{E}$.

Proof. The above sequence is the exact sequence of the pair $(G, \mathcal{S})$, where $\mathcal{S}$ is a family of subgroups of G, given in [9], using the notation of Dicks and Dunwoody. □

2. Properties of $E(G, W, M)$

Let us consider G a group, W a G-set, $\mathbb{E}$ a set of G-orbit representatives in W and M a $\mathbb{Z}_2G$-module. Suppose that $[G : G_w] = \infty$, for all $w \in \mathbb{E}$. Let us recall the definition of $E(G, W, M)$ and some general properties. The main goal of this section is to show that $E(G, W, M)$ is independent of $\mathbb{E}$, using derivation of groups.

Definition 7 ([5])**.** *Define* $E(G, W, M) := 1 + \dim\ker\operatorname{res}^G_W$, *where,*

$$\operatorname{res}^G_W : H^1(G;M) \to H^1(W;M) = \prod_{w\in\mathbb{E}} H^1(G_w;M);$$

$$\operatorname{res}^G_W([f]) = \left(\operatorname{res}^G_{G_w}[f]\right)_{w\in\mathbb{E}}$$

and

$$\operatorname{res}^G_{G_w} : H^1(G;M) \to H^1(G_w;M)$$

is the restriction map induced by the inclusion map $G_w \hookrightarrow G$, $w \in \mathbb{E}$.

Remark 3. 1) In the previous definition, the requirement $[G : G_w] = \infty$ implies that G is always an infinite group.

2) The definition of $E(G, W, M)$ and other invariants for pairs of groups were, in general, motivated by the definition of $e(G)$ and $e(G, T)$, where T is a subgroup of G. For more details see [13], [16] and [17]. Since $e(G) = 0$ if G is finite and $e(G, T) = 0$ if $[G : T] < \infty$, it is natural to require $[G : G_w] = \infty$, $\forall w \in \mathbb{E}$ in the definition of $E(G, W, M)$ presented.

3) We can prove that $E(G, W, M)$ is a cohomological algebraic invariant in the sense that if $\mathcal{C}$ is the category whose objects are the pairs $((G, W), M)$ in the above conditions, and $((G, W), M)$ is isomorphic to $((G', W'), M')$ in $\mathcal{C}$ then $E(G, W, M) = E(G', W', M')$. In the category

whose objects are the pairs $((G,\mathcal{S}),M))$ it is proved in [1] that $E(G,\mathcal{S},M)$ is a cohomological algebraic invariant.

The next proposition gives us a property of $E(G,W,M)$ that will be useful to prove its independence with respect to the set of G-orbit representatives in W. It is an adaptation of the result presented in [6], for pairs $(G,\mathcal{S})$, where $\mathcal{S}$ is a family of subgroups of G.

Proposition 6. $E(G,W,M) = 1 + \dim \bigcap_{w\in E} \ker \operatorname{res}^G_{G_w}$.

Lemma 1. *Let us consider W a G-set, w and w' elements of W and G_w and $G_{w'}$ the respective isotropic subgroups. If the G-orbits $G(w) = G(w')$ and $w' = g_0 w$ with $g_0 \in G$, then $G_{w'} = g_0 G_w g_0^{-1}$.*

Proof. Suppose that $G(w') = G(w)$ and $w' = g_0 w$. Then

$$\begin{aligned} g \in G_{w'} &\iff gw' = w' \iff gg_0 w = g_0 w \iff g_0^{-1} g g_0 w = w \\ &\iff g_0^{-1} g g_0 \in G_w \iff g \in g_0 G_w g_0^{-1}. \end{aligned}$$ □

Proposition 7. *If W is a G-set, w and w' are elements of the same G-orbit and $w' = g_0 w$ with $g_0 \in G$, then by considering the maps restriction below:*

$$\operatorname{res}^G_{G_w} : H^1(G;M) \to H^1(G_w;M)$$

$$\operatorname{res}^G_{G_{w'}} : H^1(G;M) \to H^1(G_{w'};M)$$

one has $\ker \operatorname{res}^G_{G_w} = \ker \operatorname{res}^G_{G_{w'}}$.

Proof. By Proposition 2, we have

$$H^1(G;M) \simeq \frac{\operatorname{Der}(G,M)}{P(G,M)} \quad \text{and} \quad H^1(G_w;M) \simeq \frac{\operatorname{Der}(G_w,M)}{P(G_w,M)}.$$

We will indicate an element $d + P(G,M)$ of $H^1(G,M)$ by $[d]$. Then, we have $\operatorname{res}^G_{G_w} : H^1(G;M) \to H^1(G_w;M)$, $[d] \mapsto [d|_{G_w}]$. Thus,

$$[d] \in \ker \operatorname{res}^G_{G_w} \iff \operatorname{res}^G_{G_w}([d]) = 0 \iff [d|_{G_w}] = 0 \iff d|_{G_w} \in P(G_w,M).$$

Therefore, there exists $m \in M$ such that $d|_{G_w} = d_m : G_w \to M$. Similarly, $[d] \in \ker \operatorname{res}^G_{G_{w'}}$ if there exists $m' \in M$ such that $d|_{G_{w'}} = d_{m'} : G_{w'} \to M$. Let us show that $\ker \operatorname{res}^G_{G_w} \subset \ker \operatorname{res}^G_{G_{w'}}$. Suppose $[d] \in \ker \operatorname{res}^G_{G_w}$ and let $m \in M$ such that $d|_{G_w} = d_m$.

By the previous lemma, since $w' = g_0 w$, we have $G_{w'} = g_0 G_w g_0^{-1}$. Let $m' := g_0 m - d(g_0)$. We can see that $[d] \in \ker \operatorname{res}^G_{G_{w'}}$. Indeed if $g' \in G_{w'}$, then $g' = g_0 g g_0^{-1}$, for some $g \in G_w$. Thus, since in a derivation $d(g) = -gd(g^{-1})$, one has $d|_{G_{w'}}(g') = d_{m'}(g')$. Hence, $\ker \operatorname{res}^G_{G_w} \subset \ker \operatorname{res}^G_{G_{w'}}$. Analogously, we have $\ker \operatorname{res}^G_{G_{w'}} \subset \ker \operatorname{res}^G_{G_w}$. Therefore, $\ker \operatorname{res}^G_{G_w} = \ker \operatorname{res}^G_{G_{w'}}$. □

Corollary 1. *The definition of $E(G,W,M)$ is independent of G-orbit representatives set in W.*

Proof. Let W be a G-set and let $\mathbb{E}$, $\mathbb{E}'$ be sets of G-orbit representatives in W. Note that if $w' = g_0 w$, with $w \in \mathbb{E}$ and $w' \in \mathbb{E}'$, then

$$[G : G_w] = \infty \iff [G : G_{w'}] = \infty,$$

because $[G : G_w] = |G(w)| = |G(w')| = [G : G_{w'}]$. Hence, the result follows by the previous proposition and Proposition 6. □

We note that the previous result was proved in [7] using different arguments.

Proposition 8 ([5])**.** *If the $\mathbb{Z}_2$-vector spaces: $H^0(G;M)$, $H^0(W;M) = \prod_{w\in E} H^0(G_w;M)$and $H^1(G,W;M)$ have finite dimensions, then*

$$E(G,W,M) = 1 + \dim H^0(G;M) - \dim H^0(W;M) + \dim H^1(G,W;M).$$

Proposition 9. *Let G be an infinity group and W a G-set. If the G-action in W is free and $\mathbb{E}$ is a set of G-orbit representatives in W, then $E(G,W,M) = 1 + \dim H^1(G;M)$.*

Proof. Since the action is free, we have $G_w = \{g \in G;\, gw = w\} = \{1\}$ for all $w \in \mathbb{E}$, so, $H^1(G_w;M) = H^1(\{1\};M) = 0$ for all $w \in \mathbb{E}$ and $\ker \operatorname{res}^G_W = H^1(G;M)$. Therefore, $E(G,W,M) = 1 + \dim H^1(G;M)$. □

The above result was presented in [5] for the particular invariant $E(G,W,\mathbb{Z}_2)$ but, as we have shown, it is true for any $\mathbb{Z}_2G$-module M.

The next proposition is similar to that introduced in [2], for pairs of groups (G,S), where S is a subgroup of the group G with $[G : S] = \infty$ and it is adapted here for pairs (G,W).

Proposition 10. *Let G be a group and let W be a G-set with $[G : G_w] = \infty$, $\forall w \in W$. Consider $\mathbb{E}$ a set of G-orbit representatives in W and let N and M be $\mathbb{Z}_2G$-modules. If there exists a $\mathbb{Z}_2G$-homomorphism $\phi : N \to M$ such that the induced map $\phi^* : H^1(G;N) \to H^1(G;M)$ is a monomorphism, then $E(G,W,N) \leqslant E(G,W,M)$.*

3. $E(G, W, \mathcal{F}_T G)$ and splittings of groups

In this section, we initially introduce some notations, briefly recalling the definition of $\widetilde{e}(G,T)$, where T is a subgroup of G (for more details see [15]), and we present some properties of the invariant $E(G,W,\mathcal{F}_TG)$ establishing a relation with the end of the pair $\widetilde{e}(G,T)$. Then, we prove the two main results of this work, which give applications in the theory of splittings of groups.

The following subsets are $\mathbb{Z}_2G$-submodules of $\mathcal{P}(G)$

$$\begin{aligned}
\mathcal{F}G &:= \{F \subset G;\ F \text{ is finite}\} \subset \mathcal{P}(G),\\
\mathcal{F}_TG &:= \{A \in \mathcal{P}(G);\ A \subset F.T, \text{ for some } F \in \mathcal{F}G\},\\
\mathcal{Q}G &:= \{A \subset G;\ A + g\cdot A \in \mathcal{F}G,\ \forall g \in G\},
\end{aligned}$$

where G, "+" denotes the operation of symmetric difference in $\mathcal{P}(G)$ and $g \cdot A := \{ga,\ a \in A\}$.

Remark 4. If T is a subgroup of the group G, then:

1) $\mathcal{F}_TG = \mathcal{P}(G)$ when $[G:T] < \infty$.
2) If S is a subgroup of G, with $T \subset S$ and $[S:T] < \infty$, then $\mathcal{F}_SG = \mathcal{F}_TG$.
3) $\mathcal{F}_TG \simeq \operatorname{Ind}_T^G \mathcal{P}(T)$.
4) If T is a normal subgroup of G, then $g\mathcal{P}(T) \simeq \mathcal{P}(T)$.
5) If T is a finite subgroup of G, then $\mathcal{F}_TG = \mathcal{F}G \simeq \mathbb{Z}_2G \simeq \operatorname{Ind}_{\{1\}}^G \mathbb{Z}_2$.

Definition 8 ([15])**.** *Let T be a subgroup of the group G. Then*

$$\widetilde{e}(G,T) := \dim H^0(G;\mathcal{P}(G)/\mathcal{F}_TG) = \dim(\mathcal{P}(G)/\mathcal{F}_TG)^G.$$

Lemma 2 ([15])**.** *If $[G:T] = \infty$, then $\widetilde{e}(G,T) = 1 + \dim H^1(G;\mathcal{F}_TG)$.*

Proposition 11. *If $[G:T] < \infty$, then $E(G,W,\mathcal{F}_TG) = 1$.*

Proof. Using known results and the hypothesis of the proposition, we obtain

$$\begin{aligned}
H^1(G;\mathcal{F}_TG) &\overset{\text{Remark 4.3}}{\simeq} H^1(G;\operatorname{Ind}_T^G \mathcal{P}(T)) \overset{[G:T]<\infty}{\simeq} H^1(G;\operatorname{Coind}_T^G \mathcal{P}(T))\\
&\overset{\text{Shapiro}}{\simeq} H^1(T;\mathcal{P}(T)) \overset{\text{Example 3}}{=} 0.
\end{aligned}$$

Thus, $\ker \operatorname{res}^G_{W,\mathcal{F}_TG} = 0$.

Therefore, $E(G,W,\mathcal{F}_TG) = 1 + \dim\ker\operatorname{res}^G_{W,\mathcal{F}_TG} = 1$. □

Proposition 12. *Let G be a group and let W be a G-set. If the G-action in W is transitive and G_w, $w \in W$ is finitely generated and normal subgroup* *of G, with $[G : G_w] = \infty$, then*

$$E(G, W, \mathcal{F}_{G_w} G) = 1 + \dim H^1(G; \mathcal{F}_{G_w} G) = \widetilde{e}(G, G_w).$$

Proof. From Remark 4.3, Proposition 4 and Remark 4.4 one has

$$\begin{aligned} H^1(G_w; \mathcal{F}_{G_w} G) &\overset{\text{Remark } 4.3}{\simeq} H^1(G_w; \operatorname{Ind}_{G_w}^G \mathcal{P}(G_w)) \\ &\overset{\text{Prop. } 4}{\simeq} H^1(G_w; \bigoplus_{g \in G/G_w} g\mathcal{P}(G_w)) \simeq \bigoplus_{g \in G/G_w} H^1(G_w; g\mathcal{P}(G_w)) \\ &\overset{\text{Remark } 4.4}{\simeq} \bigoplus_{g \in G/G_w} H^1(G_w; \mathcal{P}(G_w)) \overset{\text{Example } 3}{\simeq} 0. \end{aligned}$$

Hence, $\operatorname{res}_W^G : H^1(G; \mathcal{F}_{G_w} G) \to H^1(G_w; \mathcal{F}_{G_w} G)$ is the null map and $\ker \operatorname{res}_W^G = H^1(G; \mathcal{F}_{G_w} G)$. Therefore,

$$E(G, W, \mathcal{F}_{G_w} G) = 1 + \dim H^1(G; \mathcal{F}_{G_w} G) \overset{\text{Lemma } 2}{=} \widetilde{e}(G, G_w). \qquad \square$$

Proposition 13. *Let (G, W) be a pair where G is a group and W is a G-set. Consider $\mathbb{E}$ a set of G-orbit representatives in W such that $[G : G_w] = \infty$, $\forall w \in \mathbb{E}$ and let T be a subgroup of G with $[G : T] = \infty$. Then, $E(G, W, \mathbb{Z}_2(G/T)) \leqslant E(G, W, \mathcal{F}_T G)$.*

Proof. Consider the short exact sequence of $\mathbb{Z}_2 T$-modules:

$$0 \to \mathbb{Z}_2 \simeq \{\varnothing, T\} \to \mathcal{P}(T) \to Q = \frac{\mathcal{P}(T)}{\{\varnothing, T\}} \to 0.$$

Since $\mathbb{Z}_2 G$ is a free $\mathbb{Z}_2 T$-module, we obtain the following exact sequence:

$$0 \to \mathbb{Z}_2 G \otimes_{\mathbb{Z}_2 T} \mathbb{Z}_2 \xrightarrow{\phi} \mathbb{Z}_2 G \otimes_{\mathbb{Z}_2 T} \mathcal{P}(T) \xrightarrow{\pi} \mathbb{Z}_2 G \otimes_{\mathbb{Z}_2 T} Q \to 0.$$

This sequence induces the long exact sequence in cohomology groups:

$$\begin{aligned} 0 &\to H^0(G; \operatorname{Ind}_T^G \mathbb{Z}_2) \to H^0(G; \operatorname{Ind}_T^G \mathcal{P}(T)) \to H^0(G; \operatorname{Ind}_T^G Q) \\ &\to H^1(G; \operatorname{Ind}_T^G \mathbb{Z}_2) \xrightarrow{\phi^*} H^1(G; \operatorname{Ind}_T^G \mathcal{P}(T)) \to H^1(G; \operatorname{Ind}_T^G Q) \to \cdots \end{aligned}$$

By hypothesis $[G : T] = \infty$, then

$$H^0(G; \operatorname{Ind}_T^G \mathbb{Z}_2) = H^0(G; \operatorname{Ind}_T^G \mathcal{P}(T)) = H^0(G; \operatorname{Ind}_T^G Q) = 0.$$

So, $\phi^* : H^1(G; \operatorname{Ind}_T^G \mathbb{Z}_2) \to H^1(G; \operatorname{Ind}_T^G \mathcal{P}(T))$ is a monomorphism, hence by Proposition 10, $E(G, W, \operatorname{Ind}_T^G \mathbb{Z}_2) \leqslant E(G, W, \operatorname{Ind}_T^G \mathcal{P}(T))$.

Since $\mathbb{Z}_2(G/T) \simeq \operatorname{Ind}_T^G \mathbb{Z}_2$, and $\mathcal{F}_T G \simeq \operatorname{Ind}_T^G \mathcal{P}(T)$, it follows that $E(G, W, \mathbb{Z}_2(G/T)) \leqslant E(G, W, \mathcal{F}_T G)$. $\square$

In the following main theorems, we will give necessary conditions for G to split over a, not necessarily finite, subgroup T, considering $M = \mathbb{Z}_2(G/T)$ or $M = \mathcal{F}_T G$.

Theorem 2. *Let (G, W) be a pair where G is a group that splits over a subgroup T in G. Suppose that $[G : T] = \infty$ and let W be a G-set. Consider $\mathbb{E}$ a set of G-orbit representatives in W and suppose that $H^0(T; \mathbb{Z}_2(G/T)) \simeq \mathbb{Z}_2$, then if $G = G_1 *_T G_2$ and $\mathbb{E} = \{w_1, w_2\}$ with $G_{w_i} = G_i$, $i = 1, 2$, or $G = G_1 *_{T,\theta}$ and $\mathbb{E} = \{w\}$, one has*

$$E(G, W, \mathbb{Z}_2(G/T)) \leqslant 2.$$

Proof. Suppose that $G = G_1 *_T G_2$. Using the short exact sequence given in Proposition 1.1, we have $\Delta = \ker \varepsilon \simeq \mathbb{Z}_2(G/T)\,(*)$. Thus,

$$\begin{aligned} &H^1(G, W; \mathbb{Z}_2(G/T)) \simeq H^1(G, \{G_1, G_2\}; \mathbb{Z}_2(G/T)) \\ &\quad := H^0(G; \mathrm{Hom}_{\mathbb{Z}_2}(\Delta, \mathbb{Z}_2(G/T)) \overset{(*)}{\simeq} H^0(G; \mathrm{Hom}_{\mathbb{Z}_2}(\mathbb{Z}_2(G/T), \mathbb{Z}_2(G/T))) \\ &\quad \simeq H^0(G; \mathrm{Coind}_T^G(\mathrm{Res}_T^G\ \mathbb{Z}_2(G/T))) \overset{\text{Shapiro}}{\simeq} H^0(T; \mathbb{Z}_2(G/T)) \overset{\text{Hyp.}}{\simeq} \mathbb{Z}_2. \end{aligned}$$

Furthermore, $H^0(G; \mathbb{Z}_2(G/T)) \simeq (\mathrm{Ind}_T^G)^G \overset{[G:T]=\infty}{=} 0$. Then, by Proposition 5, we have the long exact sequence:

$$\begin{aligned} &0 \to H^0(G_1; \mathbb{Z}_2(G/T)) \oplus H^0(G_2; \mathbb{Z}_2(G/T)) \\ &\quad \xrightarrow{\delta} H^1(G, W; \mathbb{Z}_2(G/T)) \simeq \mathbb{Z}_2 \xrightarrow{J} H^1(G; \mathbb{Z}_2(G/T)) \to \cdots \end{aligned}$$

Therefore, δ is a monomorphism and

$$H^0(W; \mathbb{Z}_2(G/T)) = H^0(G_1; \mathbb{Z}_2(G/T)) \oplus H^0(G_2; \mathbb{Z}_2(G/T))$$

is a submodule of $H^1(G, W; \mathbb{Z}_2(G/T)) \simeq \mathbb{Z}_2$. Thereby, we have the following possibilities: $H^0(G_i; \mathbb{Z}_2(G/T)) = 0$, $i = 1, 2$, or $H^0(G_i; \mathbb{Z}_2(G/T)) = 0$ and $H^0(G_j; \mathbb{Z}_2(G, T)) \simeq \mathbb{Z}_2$, $i, j \in \{1, 2\}$, $i \neq j$. In other words, $\dim H^0(G_1; \mathbb{Z}_2(G/T)) + \dim H^0(G_2; \mathbb{Z}_2(G/T))$ can be 0 or 1, $(\dim H^0(W; \mathbb{Z}_2(G/T)) \leqslant 1)$. Therefore, by Proposition 8 one has

$$E(G, W; \mathbb{Z}_2(G/T)) = 1 - \dim H^0(W; \mathbb{Z}_2(G/T)) + 1 \leqslant 2.$$

If $G = G_1 *_{T,\theta}$, the proof is analogous. □

Theorem 3. *Let (G, W) be a pair where G is a group and W is a G-set. Consider $\mathbb{E}$ a set of G-orbit representatives in W and T a, not necessarily finite, normal finitely generated subgroup of G. If $G = G_1 *_T G_2$, $\mathbb{E} =$*

$\{w_1, w_2\}$ with $G_{w_i} = G_i$ and $[G_i : T] = \infty$, $i = 1, 2$, or $G = G_1 *_{T,\theta}$, $\mathbb{E} = \{w\}$ with $G_w = G_1$ and $[G_1 : T] = \infty$, then $E(G, W, \mathcal{F}_TG) = \infty$.

Proof. Consider $G = G_1 *_T G_2$. Note that $[G : G_i] \overset{\text{Remark 2}}{=} \infty$ and $[G : T] = [G : G_i][G_i : T] = \infty$. Thus, we have

$$H^0(G; \mathcal{F}_TG) \overset{\text{Remark 4.3}}{\simeq} H^0(G; \operatorname{Ind}_T^G \mathcal{P}(T)) \simeq (\operatorname{Ind}_T^G \mathcal{P}(T))^G \overset{[G:T]=\infty}{=} 0.$$

Consider L a transversal for the coset G_igT, $i = 1, 2$. Then,

$$\begin{aligned} H^0(G_i; \mathcal{F}_TG) &\overset{\text{Remark 4.3}}{\simeq} H^0(G_i; \operatorname{Ind}_T^G \mathcal{P}(T)) \\ &\overset{\text{Prop. 4}}{\simeq} H^0(G_i; \bigoplus_{g\in L} \operatorname{Ind}_T^{G_i} g\mathcal{P}(T)) \overset{\text{Remark 4.4}}{\simeq} H^0(G_i; \bigoplus_{g\in L} \operatorname{Ind}_T^{G_i} \mathcal{P}(T)) \\ &\simeq \bigoplus_{g\in L} H^0(G_i; \operatorname{Ind}_T^{G_i} \mathcal{P}(T)) \simeq \bigoplus_{g\in L} (\operatorname{Ind}_T^{G_i} \mathcal{P}(T))^{G_i} \overset{[G_i:T]=\infty}{=} 0. \end{aligned}$$

Thus, $H^0(W; \mathcal{F}_TG) = H^0(G_{w_1}; \mathcal{F}_TG) \oplus H^0(G_{w_2}, \mathcal{F}_TG) = 0$ and the long exact sequence given in Proposition 5 takes the following form:

$$0 \to H^1(G, W; \mathcal{F}_TG) \overset{J}{\to} H^1(G; \mathcal{F}_TG) \overset{\operatorname{res}_W^G}{\to} H^1(W; \mathcal{F}_TG) \to \cdots$$

Since J is injective and $\ker \operatorname{res}_W^G = \operatorname{Im} J$ it follows that

$$\dim H^1(G, W; \mathcal{F}_TG) = \dim \operatorname{Im} J = \dim \ker \operatorname{res}_W^G. \tag{$*$}$$

On the other hand, from the short exact sequence given in Proposition 1.1 we have $\Delta = \ker \varepsilon = \operatorname{Im} \alpha \simeq \mathbb{Z}_2(G/T)$. So,

$$\begin{aligned} H^1(G, W; \mathcal{F}_TG) &\overset{\text{Theorem 1}}{\simeq} H^1(G, \{G_1, G_2\}; \mathcal{F}_TG) \\ &:= H^0(G; \operatorname{Hom}_{\mathbb{Z}_2}(\Delta, \mathcal{F}_TG)) \simeq H^0(G; \operatorname{Hom}_{\mathbb{Z}_2}(\mathbb{Z}_2(G/T), \mathcal{F}_TG)) \\ &\overset{\text{Shapiro}}{\simeq} H^0(T; \mathcal{F}_TG) \overset{\text{Remark 4.3}}{\simeq} H^0(T; \operatorname{Ind}_T^G \mathcal{P}(T)) \\ &\simeq H^0(T; \bigoplus_{g\in G/T} g\mathcal{P}(T)) \simeq \bigoplus_{g\in G/T} H^0(T; g\mathcal{P}(T)) \\ &\overset{\text{Remark 4.4}}{\simeq} \bigoplus_{g\in G/T} H^0(T; \mathcal{P}(T)). \end{aligned}$$

Since $H^0(T; \mathcal{P}(T)) \overset{\text{Example 3}}{\simeq} \mathbb{Z}_2$ and $[G : T] = \infty$, we have

$$\dim H^1(G, W; \mathcal{F}_TG) = \infty.$$

Furthermore, it follows from $(*)$ that

$$\dim \ker \operatorname{res}^G_W = \infty.$$

Therefore, $E(G, W, \mathcal{F}_T G) = \infty$.

If $G = G_1 *_{T,\theta}$, similarly to the previous case, we obtain $H^0(G; \mathcal{F}_T G) = 0 = H^0(G_1; \mathcal{F}_T G)$ and $\dim H^1(G, W; \mathcal{F}_T G) = \dim \ker \operatorname{res}^G_W$. Besides, it follows from the short exact sequence given in Proposition 1.2 that $\Delta = \ker \varepsilon = \operatorname{Im} \alpha \simeq \mathbb{Z}_2(G/T)$. Thus, as before, we conclude that

$$\dim H^1(G, W; \mathcal{F}_T G) = H^1(G, \{G_1\}; \mathcal{F}_T G) = \infty,$$

consequently,

$$\dim \ker \operatorname{res}^G_W = \infty \text{ and } E(G, W, \mathcal{F}_T G) = \infty. \qquad \square$$

Remark 5. It is important to know on the existence of a subgroup T of a group G satisfying the conditions of the previous theorem, that is, T is a finitely generated and normal subgroup of G with $[G_1 : T] = \infty = [G_2 : T]$. We note that Scott and Wall in [17] showed the following fact: If $G = G_1 * G_2$ where G_1 and G_2 are non-trivial and H is a finitely generated, normal subgroup of G, then H is trivial or has finite index in G. Then the authors questioned whether there is an analogous result when $G = G_1 *_T G_2$ or $G = G_1 *_{T,\theta}$. Clearly, if this were true, our result would not make sense, since in this case, considering $H = T$ with T a finitely generated and normal subgroup of G, we should have T trivial or $[G : T]$ finite. We are interested in the case where T is infinite, so non-trivial, and $[G : T] = [G : G_1][G_1 : T] = \infty$. The following example was suggested to us by G. P. Scott.

Example 4. Consider any group T and G_1 and G_2 two finitely generated groups for which T is a normal subgroup. For example, $G_1 = A \times T$ and $G_2 = B \times T$, where A and B are any finitely generated infinite groups. A and B can be taken, for example, cyclic groups. Then $[G_1 : T] = \infty = [G_2 : T]$ and we can see that T is normal in $G := G_1 *_T G_2$, by using the Normal Form Theorem, given in [8].

Based on the previous example, we present an application of the Theorem 3.

Example 5. Given T a finitely generated group, not necessarily finite, consider $G = G_1 *_T G_2$ as in the previous example, $W = G/G_1 \,\dot\cup\, G/G_2$ and a map $\phi : G \times W \to W$, such that, if $w_1 = gG_1 \in W$ then $g' \cdot w_1 = g'gG_1$

and if $w_2 = gG_2 \in W$, then $g' \cdot w_2 = g'gG_2$. Observe that ϕ is a G-action in W. Now, if $w_1 = 1 \cdot G_1$ and $w_2 = 1 \cdot G_2$, one has

$$G(w_1) = \{g \cdot 1 \cdot G_1,\ g \in G\} = \{g \cdot G_1,\ g \in G\} = G/G_1,$$
$$G(w_2) = \{g \cdot 1 \cdot G_2,\ g \in G\} = \{g \cdot G_2,\ g \in G\} = G/G_2.$$

Furthermore,

$$G_{w_1} = \{g \in G,\ g \cdot w_1 = w_1\} = \{g \in G,\ g \cdot G_1 = G_1\} = G_1.$$

Similarly, $G_{w_2} = G_2$. Thus, there exists a G-set W, such that $\mathbb{E} = \{w_1, w_2\}$ with $G_{w_1} = G_1$ and $G_{w_2} = G_2$. Therefore, by Theorem 3, $E(G, W, \mathcal{F}_T G) = \infty$.

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