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Generalized 2-absorbing and strongly generalized 2-absorbing second submodules

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Abstract. Let R be a commutative ring with identity. A proper submodule N of an R-module M is said to be a *2-absorbing submodule* of M if whenever $abm \in N$ for some $a, b \in R$ and $m \in M$, then $am \in N$ or $bm \in N$ or $ab \in (N :_R M)$. In [3], the authors introduced two dual notion of 2-absorbing submodules (that is, *2-absorbing and strongly 2-absorbing second submodules*) of M and investigated some properties of these classes of modules. In this paper, we will introduce the concepts of generalized 2-absorbing and strongly generalized 2-absorbing second submodules of modules over a commutative ring and obtain some related results.

1. Introduction

Throughout this paper, R will denote a commutative ring with identity and $\mathbb{Z}$ will denote the ring of integers.

Let M be an R-module. A proper submodule P of M is said to be *prime* if for any $r \in R$ and $m \in M$ with $rm \in P$, we have $m \in P$ or $r \in (P :_R M)$ [14]. A non-zero submodule S of M is said to be *second* if for each $a \in R$, the homomorphism $S \xrightarrow{a} S$ is either surjective or zero [17]. In this case $\mathrm{Ann}_R(S)$ is a prime ideal of R. A proper submodule N of M is said to be *completely irreducible* if $N = \bigcap_{i\in I} N_i$, where $\{N_i\}_{i\in I}$

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is a family of submodules of M, implies that $N = N_i$ for some $i \in I$. It is easy to see that every submodule of M is an intersection of completely irreducible submodules of M [15].

Badawi gave a generalization of prime ideals in [9] and said such ideals 2-absorbing ideals. A proper ideal I of R is a *2-absorbing ideal* of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. He proved that I is a 2-absorbing ideal of R if and only if whenever I_1, I_2, and I_3 are ideals of R with $I_1I_2I_3 \subseteq I$, then $I_1I_2 \subseteq I$ or $I_1I_3 \subseteq I$ or $I_2I_3 \subseteq I$. In [10], the authors introduced the concept of 2-absorbing primary ideal which is a generalization of primary ideal. A proper ideal I of R is called a *2-absorbing primary ideal* of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$.

The authors in [13] and [16], extended the concept of 2-absorbing ideals to the concept of 2-absorbing submodules. A proper submodule N of M is called a *2-absorbing submodule* of M if whenever $abm \in N$ for some $a, b \in R$ and $m \in M$, then $am \in N$ or $bm \in N$ or $ab \in (N :_R M)$.

In [3], the authors introduced two dual notion of 2-absorbing submodules (that is, *2-absorbing and strongly 2-absorbing second submodules*) of M and investigated some properties of these classes of modules. A non-zero submodule N of M is said to be a *2-absorbing second submodule of* M if whenever $a, b \in R$, L is a completely irreducible submodule of M, and $abN \subseteq L$, then $aN \subseteq L$ or $bN \subseteq L$ or $ab \in \mathrm{Ann}_R(N)$. A non-zero submodule N of M is said to be a *strongly 2-absorbing second submodule of* M if whenever $a, b \in R$, K is a submodule of M, and $abN \subseteq K$, then $aN \subseteq K$ or $bN \subseteq K$ or $ab \in \mathrm{Ann}_R(N)$.

The purpose of this paper is to introduce the concepts of generalized and strongly generalized 2-absorbing second submodules of an R-module M as generalizations of 2-absorbing and strongly 2-absorbing second submodules of M respectively, and provide some information concerning these new classes of modules.

2. Main results

Definition 2.1. We say that a non-zero submodule N of an R-module M is a *generalized 2-absorbing second submodule* or *G2-absorbing second submodule* of M if whenever $a, b \in R$, L is a completely irreducible submodule of M and $abN \subseteq L$, then $a \in \sqrt{(L :_R N)}$ or $b \in \sqrt{(L :_R N)}$ or $ab \in \mathrm{Ann}_R(N)$. By a *generalized 2-absorbing second module*, we mean a module which is a generalized 2-absorbing second submodule of itself.

Example 2.2. Clearly every 2-absorbing second submodule is a $G2$-absorbing second submodule. But the converse is not true in general as we will see in the Example 2.8.

We recall that an R-module M is said to be a *cocyclic module* if $Soc_R(M)$ is a large and simple submodule of M [18]. (Here $Soc_R(M)$ denotes the sum of all minimal submodules of M.). A submodule L of M is a completely irreducible submodule of M if and only if M/L is a cocyclic R-module [15].

Proposition 2.3. Let N be a $G2$-absorbing second submodule of an R-module M. Then we have the following.

(a) If L is a completely irreducible submodule of M such that $N \not\subseteq L$, then $(L :_R N)$ is a 2-absorbing primary ideal of R.

(b) If M is a cocyclic module, then $\mathrm{Ann}_R(N)$ is a 2-absorbing primary ideal of R.

(c) If $\mathrm{Ann}_R(N)$ is a primary ideal of R, then $(L :_R N)$ is a primary ideal of R for all completely irreducible submodule L of M such that $N \not\subseteq L$.

Proof. (a) Since $N \not\subseteq L$, we have $(L :_R N) \neq R$. Let $a, b, c \in R$ and $abc \in (L :_R N)$. Then $abN \subseteq (L :_M c)$. Thus $a^tN \subseteq (L :_M c)$ for some positive integer t or $b^sN \subseteq (L :_M c)$ for some positive integer s or $abN = 0$ because by [7, 2.1], $(L :_M c)$ is a completely irreducible submodule of M. Therefore, $ac \in \sqrt{(L :_R N)}$ or $bc \in \sqrt{(L :_R N)}$ or $ab \in (L :_R N)$.

(b) Since M is cocyclic, the zero submodule of M is a completely irreducible submodule of M. Thus the result follows from part (a).

(c) Let $a, b \in R$, L be a completely irreducible submodule of M such that $N \not\subseteq L$, and $ab \in (L :_R N)$. Then $a^tN \subseteq L$ for some positive integer t or $b^sN \subseteq L$ for some positive integer s or $abN = 0$. If $abN = 0$, then by assumption, $a \in \sqrt{\mathrm{Ann}_R(N)}$ or $bN = 0$. Thus in any cases we get that, $a \in \sqrt{(L :_R N)}$ or $b \in \sqrt{(L :_R N)}$. □

Lemma 2.4. Let I be an ideal of R and N be a $G2$-absorbing second submodule of M. If $a \in R$, L is a completely irreducible submodule of M, and $IaN \subseteq L$, then $a \in \sqrt{(L :_R N)}$ or $I \subseteq \sqrt{(L :_R N)}$ or $Ia \subseteq \mathrm{Ann}_R(N)$.

Proof. Let $a \notin \sqrt{(L :_R N)}$ and $Ia \not\subseteq \mathrm{Ann}_R(N)$. Then there exists $b \in I$ such that $abN \neq 0$. Now as N is a $G2$-absorbing second submodule of M, $baN \subseteq L$ implies that $b \in \sqrt{(L :_R N)}$. We show that $I \subseteq \sqrt{(L :_R N)}$. To see this, let c be an arbitrary element of I. Then $(b + c)aN \subseteq L$. Hence, either $b+c \in \sqrt{(L :_R N)}$ or $(b+c)a \in \mathrm{Ann}_R(N)$. If $b+c \in \sqrt{(L :_R N)}$, then since $b \in \sqrt{(L :_R N)}$ we have $c \in \sqrt{(L :_R N)}$. If $(b+c)a \in \mathrm{Ann}_R(N)$, then $ca \notin \mathrm{Ann}_R(N)$, but $caN \subseteq L$. Thus $c \in \sqrt{(L :_R N)}$. Hence, we conclude that $I \subseteq \sqrt{(L :_R N)}$. □

Theorem 2.5. *Let I and J be two ideals of R and N be a $G2$-absorbing second submodule of M. If L is a completely irreducible submodule of M and $IJN \subseteq L$, then $I \subseteq \sqrt{(L :_R N)}$ or $J \subseteq \sqrt{(L :_R N)}$ or $IJ \subseteq \mathrm{Ann}_R(N)$.*

Proof. Let $I \not\subseteq \sqrt{(L :_R N)}$ and $J \not\subseteq \sqrt{(L :_R N)}$. We show that $IJ \subseteq \mathrm{Ann}_R(N)$. Assume that $c \in I$ and $d \in J$. By assumption there exists $a \in I$ such that $a \notin \sqrt{(L :_R N)}$ but $aJN \subseteq L$. Now Lemma 2.4 shows that $aJ \subseteq \mathrm{Ann}_R(N)$ and so $(I \setminus \sqrt{(L :_R N)})J \subseteq \mathrm{Ann}_R(N)$. Similarly there exists $b \in (J \setminus \sqrt{(L :_R N)})$ such that $Ib \subseteq \mathrm{Ann}_R(N)$ and also $I(J \setminus \sqrt{(L :_R N)}) \subseteq \mathrm{Ann}_R(N)$. Thus we have $ab \in \mathrm{Ann}_R(N)$, $ad \in \mathrm{Ann}_R(N)$ and $cb \in \mathrm{Ann}_R(N)$. As $a + c \in I$ and $b + d \in J$, we have $(a+c)(b+d)N \subseteq L$. Therefore, $a+c \in \sqrt{(L :_R N)}$ or $b+d \in \sqrt{(L :_R N)}$ or $(a+c)(b+d) \in \mathrm{Ann}_R(N)$. If $a+c \in \sqrt{(L :_R N)}$, then $c \notin \sqrt{(L :_R N)}$. Hence $c \in I \setminus \sqrt{(L :_R N)}$ which implies that $cd \in \mathrm{Ann}_R(N)$. Similarly if $b + d \in \sqrt{(L :_R N)}$, we can deduce that $cd \in \mathrm{Ann}_R(N)$. Finally if $(a + c)(b + d) \in \mathrm{Ann}_R(N)$, then $ab + ad + cb + cd \in \mathrm{Ann}_R(N)$ so that $cd \in \mathrm{Ann}_R(N)$. Therefore, $IJ \subseteq \mathrm{Ann}_R(N)$. □

Theorem 2.6. *Let N be a non-zero submodule of an R-module M. The following statements are equivalent:*

(a) *If $abN \subseteq K$ for some $a, b \in R$ and a submodule K of M, then $a \in \sqrt{(K :_R N)}$ or $b \in \sqrt{(K :_R N)}$ or $ab \in \mathrm{Ann}_R(N)$.*

(b) *If $IJN \subseteq K$ for some ideals I and J of R and submodule K of M, then $I \subseteq \sqrt{(K :_R N)}$ or $J \subseteq \sqrt{(K :_R N)}$ or $IJ \subseteq \mathrm{Ann}_R(N)$.*

Proof. $(a) \Rightarrow (b)$ The proof is similar to the proof of Theorem 2.5.

$(b) \Rightarrow (a)$ This is clear. □

Definition 2.7. We say that a non-zero submodule N of an R-module M is a *strongly generalized 2-absorbing second submodule* or *strongly $G2$-absorbing second submodule* of M if satisfies the equivalent conditions of Theorem 2.6. By a *strongly generalized 2-absorbing second module*, we mean a module which is a strongly generalized 2-absorbing second submodule of itself.

Example 2.8. Clearly every strongly 2-absorbing second submodule is a strongly $G2$-absorbing second submodule. But the converse is not true in general. For example, for any prime integer p, let $M = \mathbb{Z}_{p^\infty}$ and $N = \langle 1/p^3 + \mathbb{Z}\rangle$. Then N is a strongly $G2$-absorbing second submodule which is not a strongly 2-absorbing second submodule of M.

This is clear that every strongly $G2$-absorbing second submodule is a $G2$-absorbing second submodule. It is natural to ask the following question:

Question 2.9. Let M be an R-module. Is every $G2$-absorbing second submodule of M a strongly $G2$-absorbing second submodule of M?

Theorem 2.10. *Let N be a non-zero submodule of an Artinian R-module M. The following statements are equivalent:*

(a) *If $abN \subseteq L_1 \cap L_2$ for some $a, b \in R$ and completely irreducible submodules L_1, L_2 of M, then we have $a \in \sqrt{(L_1 \cap L_2 :_R N)}$ or $b \in \sqrt{(L_1 \cap L_2 :_R N)}$ or $ab \in \mathrm{Ann}_R(N)$.*

(b) *N is a strongly $G2$-absorbing second submodule.*

Proof. $(a) \Rightarrow (b)$. Assume that $abN \subseteq K$ for some $a, b \in R$, a submodule K of M, and $ab \not\subseteq \mathrm{Ann}_R(N)$. Since M is Artinian, there exist completely irreducible submodules $L_1, L_2, \dots, L_n$ of M such that $K = \cap_{i=1}^n L_i$. Then for each L_i $(1 \leqslant i \leqslant n)$ either $a \in \sqrt{(L_i :_R N)}$ or $b \in \sqrt{(L_i :_R N)}$. If $a \in \sqrt{(L_i :_R N)}$ for each $1 \leqslant i \leqslant n$, then

$$a \in \cap_{i=1}^n \sqrt{(L_i :_R N)} = \sqrt{\cap_{i=1}^n (L_i :_R N)}$$
$$= \sqrt{(\cap_{i=1}^n L_i :_R N)} = \sqrt{(K :_R N)}.$$

Similarly, if $b \in \sqrt{(L_i :_R N)})$ for each $1 \leqslant i \leqslant n$, then we get that $b \in \sqrt{(K :_R N)}$. Now suppose that there exist $1 \leqslant i, j \leqslant n$ such that $a \notin \sqrt{(L_i :_R N)}$ and $b \notin \sqrt{(L_j :_R N)}$. Then $a \in \sqrt{(L_j :_R N)}$ and $b \in \sqrt{(L_i :_R N)}$. Since $abN \subseteq L_i \cap L_j$, we have either $a \in \sqrt{(L_i \cap L_j :_R N)}$ or $b \in \sqrt{(L_i \cap L_j :_R N)}$. If $a \in \sqrt{(L_i \cap L_j :_R N)}$, then $a \in \sqrt{(L_i :_R N)}$ which is a contradiction. Similarly from $b \in \sqrt{(L_i \cap L_j :_R N)}$ we get a contradiction.

$(b) \Rightarrow (a)$. This is clear. □

Proposition 2.11. Let M be an R-module. If either N is a secondary submodule of M or N is a finite sum of p-secondary submodules of M, then N is strongly $G2$-absorbing second submodule.

Proof. The first assertion is clear. Now the second assertion follows from [11, 3.1.4]. □

Lemma 2.12. Let M be an R-module, $N \subseteq K$ be two submodules of M, and K be a strongly $G2$-absorbing second submodule of M. Then K/N is a strongly $G2$-absorbing second submodule of M/N.

Proof. This is straightforward. □

Proposition 2.13. Let N be a strongly $G2$-absorbing second submodule of an R-module M. Then we have the following.

(a) $\mathrm{Ann}_R(N)$ is a 2-absorbing primary ideal of R.

(b) If K is a submodule of M such that $N \not\subseteq K$, then $(K :_R N)$ is a 2-absorbing primary ideal of R.

Proof. (a) Let $a, b, c \in R$ and $abc \in \mathrm{Ann}_R(N)$. Then $abN \subseteq abN$ implies that $a^t N \subseteq abN$ for some positive integer t or $b^s N \subseteq abN$ for some positive integer s or $abN = 0$. If $abN = 0$, then we are done. If $a^t N \subseteq abN$, then $(ca)^t N \subseteq ca^t N \subseteq cabN = 0$. Thus $ca \in \sqrt{\mathrm{Ann}_R(N)}$. In other case, we do the same.

(b) Let $a, b, c \in R$ and $abc \in (K :_R N)$. Then $a^t cN \subseteq K$ for some positive integer t or $b^s cN \subseteq K$ for some positive integer s or $abN = 0$. If $a^t cN \subseteq K$ or $b^s cN \subseteq K$, then $(ac)^t N \subseteq K$ or $(bc)^s N \subseteq K$ and so we are done. If $abN = 0$, then the result follows from part (a). □

An R-module M is said to be a *comultiplication module* if for every submodule N of M there exists an ideal I of R such that $N = (0 :_M I)$, equivalently, for each submodule N of M, we have $N = (0 :_M \mathrm{Ann}_R(N))$ [5].

Corollary 2.14. Let M be a comultiplication R-module. If N is a strongly $G2$-absorbing second submodule of M such that $\sqrt{\mathrm{Ann}_R(N)} = \mathrm{Ann}_R(N)$, then N is a strongly 2-absorbing second submodule of M.

Proof. By Proposition 2.13 (a), $\mathrm{Ann}_R(N)$ is a 2-absorbing primary ideal of R. Thus $\sqrt{\mathrm{Ann}_R(N)} = \mathrm{Ann}_R(N)$ is a 2-absorbing ideal of R by [10, 2.2.]. Now the result follows from [3, 3.10]. □

A submodule N of an R-module M is said to be *coidempotent* if $N = (0 :_M \mathrm{Ann}_R(N)^2)$. Also, M is said to be *fully coidempotent* if every submodule of M is coidempotent [1]. Clearly, every fully coidempotent R-module is a comultiplication R-module.

Theorem 2.15. *Let R be a Noetherian ring and N be a submodule of a fully coidempotent R-module M. Then we have the following.*

(a) *If $\mathrm{Ann}_R(N)$ is a 2-absorbing primary ideal of R, then N is a strongly $G2$-absorbing second submodule of M.*

(b) *If M is a cocyclic module and N is a $G2$-absorbing second submodule of M, then N is a strongly $G2$-absorbing second submodule of M.*

Proof. (a) Let $a, b \in R$, K be a submodule of M, and $abN \subseteq K$. Then we have $\mathrm{Ann}_R(K)abN = 0$. Now since R is Noetherian, $(\mathrm{Ann}_R(K)a)^t N = 0$ for some positive integer t or $(\mathrm{Ann}_R(K)b)^s N = 0$ for some positive integer s or $abN = 0$ by [10, 2.18]. If $abN = 0$, we are done. If $(\mathrm{Ann}_R(K)a)^t N = 0$ or $(\mathrm{Ann}_R(K)b)^s N = 0$, then $(\mathrm{Ann}_R(K))^t \subseteq \mathrm{Ann}_R(a^t N)$ or $(\mathrm{Ann}_R(K))^s \subseteq \mathrm{Ann}_R(b^s N)$. Hence, $a^t N \subseteq K$ or $b^s N \subseteq K$ since M is a fully coidempotent R-module. Therefore, N is a strongly $G2$-absorbing second submodule of M.

(b) By Proposition 2.3, $\mathrm{Ann}_R(N)$ is a 2-absorbing primary ideal of R. Thus the result follows from part (a). □

The following example shows that Theorem 2.15 (a) is not satisfied in general.

Example 2.16. By [5, 3.9], the $\mathbb{Z}$-module $\mathbb{Z}$ is not a comultiplication $\mathbb{Z}$-module and so it is not a fully coidempotent $\mathbb{Z}$-module. The submodule $N = p\mathbb{Z}$ of $\mathbb{Z}$, where p is a prime number, is not strongly $G2$-absorbing second submodule. But $\mathrm{Ann}_{\mathbb{Z}}(p\mathbb{Z}) = 0$ is a 2-absorbing primary ideal of R.

For a submodule N of an R-module M the *second radical* (or second socle) of N is defined as the sum of all second submodules of M contained in N and it is denoted by $\mathrm{sec}(N)$ (or $soc(N)$). In case N does not contain any second submodule, the second radical of N is defined to be (0) (see [12] and [2]).

Proposition 2.17. Let M be a finitely generated comultiplication R-module. If N is a strongly $G2$-absorbing second submodule of M, then $\mathrm{sec}(N)$ is a strongly 2-absorbing second submodule of M.

Proof. Let N be a strongly $G2$-absorbing second submodule of M. By Proposition 2.13 (a), $\mathrm{Ann}_R(N)$ is a 2-absorbing primary ideal of R. Thus by [10, 2.2], $\sqrt{\mathrm{Ann}_R(N)}$ is a 2-absorbing ideal of R. By [6, 2.12], $\mathrm{Ann}_R(\mathrm{sec}(N)) = \sqrt{\mathrm{Ann}_R(N)}$. Therefore, $\mathrm{Ann}_R(\mathrm{sec}(N))$ is a 2-absorbing ideal of R. Now the result follows from [3, 3.10]. □

A non-zero submodule N of an R-module M is a *strongly 2-absorbing secondary submodule* of M if whenever $a, b \in R$, K is a submodule of M and $abN \subseteq K$, then $a(\mathrm{sec}(N)) \subseteq K$ or $b(\mathrm{sec}(N)) \subseteq K$ or $ab \in \mathrm{Ann}_R(N)$ [4].

Theorem 2.18. *Let M be a comultiplication R-module and N be a strongly $G2$-absorbing second submodule of M. Then N is a strongly 2-absorbing secondary submodule of M.*

Proof. Let $a, b \in R$, K be a submodule of M, and $abN \subseteq K$. Then we have $a^tN \subseteq K$ for some positive integer t or $b^sN \subseteq K$ for some positive integer s or $abN = 0$. If $abN = 0$, then we are done. Suppose that $a^tN \subseteq K$ for some positive integer t. As M is a comultiplication R-module, $K = (0 :_M I)$ for some ideal I of R. Thus $Ia^tN = 0$. This implies that $Ia \subseteq \sqrt{\mathrm{Ann}_R(N)}$. Thus

$$\sec(N) \subseteq (0 :_M \sqrt{\mathrm{Ann}_R(N)}) \subseteq (0 :_M Ia) = (K :_M a).$$

Hence $a(\sec(N)) \subseteq K$, as needed. □

Example 2.19. The submodule $N = p\mathbb{Z}$ of the $\mathbb{Z}$-module $M = \mathbb{Z}$, where p is a prime number, is not a strongly $G2$-absorbing second submodule. But as $\sec(p\mathbb{Z}) = 0$, we have N is a strongly 2-absorbing secondary submodule of M.

Theorem 2.20. *Let $f : M \to \acute{M}$ be a monomorphism of R-modules. Then we have the following.*

(a) *If N is a strongly $G2$-absorbing second submodule of M, then $f(N)$ is a strongly $G2$-absorbing second submodule of $\acute{M}$.*

(b) *If $\acute{N}$ is a strongly $G2$-absorbing second submodule of $\acute{M}$ and $\acute{N} \subseteq f(M)$, then $f^{-1}(\acute{N})$ is a strongly $G2$-absorbing second submodule of M.*

Proof. (a) Since $N \neq 0$ and f is a monomorphism, we have $f(N) \neq 0$. Let $a, b \in R$, $\acute{K}$ be a submodule of $\acute{M}$, and $abf(N) \subseteq \acute{K}$. Then $abN \subseteq f^{-1}(\acute{K})$. As N is strongly $G2$-absorbing second submodule, $a^tN \subseteq f^{-1}(\acute{K})$ for some positive integer t or $b^sN \subseteq f^{-1}(\acute{K})$ for some positive integer s or $abN = 0$. Therefore,

$$a^tf(N) \subseteq f(f^{-1}(\acute{K})) = f(M) \cap \acute{K} \subseteq \acute{K}$$

or

$$b^sf(N) \subseteq f(f^{-1}(\acute{K})) = f(M) \cap \acute{K} \subseteq \acute{K}$$

or $abf(N) = 0$, as needed.

(b) If $f^{-1}(\acute{N}) = 0$, then $f(M) \cap \acute{N} = f(f^{-1}(\acute{N})) = f(0) = 0$. Thus $\acute{N} = 0$, a contradiction. Therefore, $f^{-1}(\acute{N}) \neq 0$. Now let $a, b \in R$, K be a submodule of M, and $abf^{-1}(\acute{N}) \subseteq K$. Then

$$ab\acute{N} = ab(f(M) \cap \acute{N}) = abf(f^{-1}(\acute{N})) \subseteq f(K).$$

As $\acute{N}$ is strongly $G2$-absorbing second submodule, $a^t\acute{N} \subseteq f(K)$ for some positive integer t or $b^s\acute{N} \subseteq f(K)$ for some positive integer s or $ab\acute{N} = 0$. Therefore, $a^t f^{-1}(\acute{N}) \subseteq f^{-1}(f(K)) = K$ or $b^s f^{-1}(\acute{N}) \subseteq f^{-1}(f(K)) = K$ or $abf^{-1}(\acute{N}) = 0$ as desired. □

Corollary 2.21. Let M be an R-module and $N \subseteq K$ be two submodules of M. Then N is a strongly $G2$-absorbing second submodule of K if and only if N is a strongly $G2$-absorbing second submodule of M.

Proof. This follows from Theorem 2.20 by using the natural monomorphism $K \to M$. □

Let N be a submodule of an R-module M. Then Corollary 2.21 shows that N is a strongly $G2$-absorbing second submodule of M if and only if N is a strongly $G2$-absorbing second module.

Let R_i be a commutative ring with identity and M_i be an R_i-module, for $i = 1, 2$. Let $R = R_1 \times R_2$. Then $M = M_1 \times M_2$ is an R-module and each submodule of M is in the form of $N = N_1 \times N_2$ for some submodules N_1 of M_1 and N_2 of M_2.

Lemma 2.22. Let $R = R_1 \times R_2$ and $M = M_1 \times M_2$. Then M_i is a fully coidempotent R_i-module, for $i = 1, 2$ if and only if M is a fully coidempotent R-module.

Proof. First suppose that M is a fully coidempotent R-module and N_1 is a submodule of an R_1-module M_1. Then $N = N_1 \times 0$ is a submodule of M. Thus $N = (0 :_M \text{Ann}_R(N)^2) = (0 :_{M_1} \text{Ann}_{R_1}(N_1)^2) \times 0$. Hence $N_1 = (0 :_{M_1} \text{Ann}_{R_1}(N_1)^2)$. Therefore, M_1 is a fully coidempotent R_1-module. Similarly, M_2 is a fully coidempotent R_2-module. Conversely, let N be a submodule of M. Then $N = N_1 \times N_2$ for some submodules N_1 of M_1 and N_2 of M_2. By assumption, $N_i = (0 :_{M_i} \text{Ann}_{R_i}(N_i)^2)$ for $i = 1, 2$. So

$$N = (0 :_{M_1} \text{Ann}_{R_1}(N_1)^2) \times (0 :_{M_2} \text{Ann}_{R_2}(N_2)^2) = (0 :_M \text{Ann}_R(N)^2),$$

as requested. □

Theorem 2.23. *Let $R = R_1 \times R_2$ be a Noetherian ring and $M = M_1 \times M_2$, where M_1 is a fully coidempotent R_1-module and M_2 is a fully coidempotent R_2-module. Then we have the following.*

(a) *A non-zero submodule K_1 of M_1 is a strongly $G2$-absorbing second submodule if and only if $N = K_1 \times 0$ is a strongly $G2$-absorbing second submodule of M.*

(b) *A non-zero submodule K_2 of M_2 is a strongly $G2$-absorbing second submodule if and only if $N = 0 \times K_2$ is a strongly $G2$-absorbing second submodule of M.*

(c) *If K_1 is a secondary submodule of M_1 and K_2 is a secondary submodule of M_2, then $N = K_1 \times K_2$ is a strongly $G2$-absorbing second submodule of M.*

Proof. (a) Let K_1 be a strongly $G2$-absorbing second submodule of M_1. Then $\mathrm{Ann}_{R_1}(K_1)$ is a 2-absorbing primary ideal of R_1 by Proposition 2.13. Now since $\mathrm{Ann}_R(N) = \mathrm{Ann}_{R_1}(K_1) \times R_2$, we have $\mathrm{Ann}_R(N)$ is a 2-absorbing primary ideal of R by [10, 2.23]. Thus the result follows from Theorem 2.15 (a). Conversely, let $N = K_1 \times 0$ be a strongly $G2$-absorbing second submodule of M. Then $\mathrm{Ann}_R(N) = \mathrm{Ann}_{R_1}(K_1) \times R_2$ is a primary ideal of R by Proposition 2.13. Thus $\mathrm{Ann}_{R_1}(K_1)$ is a primary ideal of R_1 by [10, 2.23]. Thus by Theorem 2.15 (a), K_1 is a strongly $G2$-absorbing second submodule of M_1.

(b) This is proved similar to the part (a).

(c) Let K_1 be a secondary submodule of M_1 and K_2 be a secondary submodule of M_2. Then $\mathrm{Ann}_{R_1}(K_1)$ and $\mathrm{Ann}_{R_2}(K_2)$ are primary ideals of R_1 and R_2, respectively. Now since $\mathrm{Ann}_R(N) = \mathrm{Ann}_{R_1}(K_1) \times \mathrm{Ann}_{R_2}(K_2)$, we have $\mathrm{Ann}_R(N)$ is a 2-absorbing primary ideal of R by [10, 2.23]. Thus the result follows from Theorem 2.15 (a). □

Theorem 2.24. *Let $R = R_1 \times R_2$ be a Noetherian decomposable ring and $M = M_1 \times M_2$ be a fully coidempotent R-module, where M_1 is an R_1-module and M_2 is an R_2-module. Suppose that $N = N_1 \times N_2$ is a non-zero submodule of M. Then the following conditions are equivalent:*

(a) *N is a strongly $G2$-absorbing second submodule of M;*

(b) *Either $N_1 = 0$ and N_2 is a strongly $G2$-absorbing second submodule of M_2 or $N_2 = 0$ and N_1 is a strongly $G2$-absorbing second submodule of M_1 or N_1, N_2 are secondary submodules of M_1, M_2, respectively.*

Proof. $(a) \Rightarrow (b)$. Let $N = N_1 \times N_2$ be a strongly $G2$-absorbing second submodule of M. Then $\mathrm{Ann}_R(N) = \mathrm{Ann}_{R_1}(N_1) \times \mathrm{Ann}_{R_2}(N_2)$ is a 2-absorbing primary ideal of R by Proposition 2.13. By [10, 2.23], we have $\mathrm{Ann}_{R_1}(N_1) = R_1$ and $\mathrm{Ann}_{R_2}(N_2)$ is a 2-absorbing primary ideal of R_2 or $\mathrm{Ann}_{R_2}(N_2) = R_2$ and $\mathrm{Ann}_{R_1}(N_1)$ is a 2-absorbing primary ideal of R_1 or $\mathrm{Ann}_{R_1}(N_1)$ and $\mathrm{Ann}_{R_2}(N_2)$ are primary ideals of R_1 and R_2, respectively. Suppose that $\mathrm{Ann}_{R_1}(N_1) = R_1$ and $\mathrm{Ann}_{R_2}(N_2)$ is a 2-absorbing primary ideal of R_2. Then $N_1 = 0$ and N_2 is a strongly $G2$-absorbing second submodule of M_2 by Theorem 2.15 (a) and Lemma 2.22. Similarly if

$\mathrm{Ann}_{R_2}(N_2) = R_2$ and $\mathrm{Ann}_{R_1}(N_1)$ is a 2-absorbing primary ideal of R_1. Then $N_2 = 0$ and N_1 is a strongly $G2$-absorbing second submodule of M_1. If the last case hold, then as M_1 (resp. M_2) is a comultiplication R_1-(resp. R_2-) module, N_1 (resp. N_2) is a secondary submodule of M_1 (resp. M_2) by [4, 2.25].

$(b) \Rightarrow (a)$. This can be proved easily by using Theorem 2.23. □

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