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On some non-periodic groups whose cyclic subgroups are GNA-subgroups

A. A. Pypka

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To Professor L. A. Kurdachenko on the occasion of his 70th birthday

Abstract. In this paper we obtain the description of non-periodic locally generalized radical groups whose cyclic subgroups are GNA-subgroups.

Introduction

Let G be a group. Recall that a subgroup H of G is called *abnormal* in G if $g \in \langle H, H^g \rangle$ for every element $g \in G$. Recall also that a subgroup H of G is *self-normalizing* in G if $N_G(H) = H$. It is well known that every abnormal subgroup of G is self-normalizing in G. Clearly abnormal and self-normalizing subgroups are antipodes of normal subgroups. On the one hand, a subgroup H of G is both normal and abnormal in G iff $H = G$. On the other hand, if H is a normal subgroup of G, then $N_G(H) = G$. This remarks shows that the properties of normal subgroups and abnormal (respectively, self-normalizing) subgroups are diametrically opposite.

In the same time, there are subgroups that combine the concepts of normality and abnormality. Recall that a subgroup H of a group G is called *pronormal* in G if for every element $g \in G$ the subgroups H and H^g are conjugate in $\langle H, H^g \rangle$. Thus, every normal and abnormal subgroup of G is

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pronormal in G. Note that the normalizer $N_G(H)$ of pronormal subgroup H is abnormal in G (see, for example, [1]), and hence self-normalizing in G.

In the paper [6] the authors introduced the following generalization of normal and abnormal subgroups.

Definition 1. A subgroup H of a group G is called a *GNA-subgroup* of G if for every element $g \in G$ either $H^g = H$ or $N_K(N_K(H)) = N_K(H)$, where $K = \langle H, g \rangle$.

Clearly every pronormal subgroup is a *GNA*-subgroup. Moreover, example from [6] shows that there are *GNA*-subgroups, which are not pronormal.

In the paper [6], the authors obtained the description of locally finite groups whose all subgroups are *GNA*-subgroups. Later, in the paper [5], it has been obtained the description of locally finite groups whose cyclic subgroups are *GNA*-subgroups.

In this article, we continue to study the influence of *GNA*-subgroups on the group structure. More precisely, we investigate the structure of some non-periodic groups whose cyclic subgroups are *GNA*-subgroups.

First, we recall some definitions. A *locally nilpotent radical* of a group G is a subgroup generated by all normal locally nilpotent subgroups of G. We will denote this subgroup by $\mathrm{Lnr}(G)$. We recall also that a *locally finite radical* of a group G is a subgroup generated by all normal locally finite subgroups of G. We will denote this subgroup by $\mathrm{Lfr}(G)$.

A group G is called *radical* if G has an ascending series whose factors are locally nilpotent. A group G is called *generalized radical* if G has an ascending series whose factors are locally nilpotent or locally finite. Hence a generalized radical group G either has an ascendant locally nilpotent subgroup or an ascendant locally finite subgroup. In the first case, the locally nilpotent radical $\mathrm{Lnr}(G)$ of G is non-identity. In the second case, it is not hard to see that G contains a non-identity normal locally finite subgroup. Clearly, in every group G the subgroup $\mathrm{Lfr}(G)$ is the largest normal locally finite subgroup. Thus, every generalized radical group has an ascending series of normal subgroups with locally nilpotent or locally finite factors.

Observe also that a periodic generalized radical group is locally finite, and hence periodic locally generalized radical group is also locally finite.

The main result of this paper is the following

Theorem 1. *Let G be a non-periodic locally generalized radical group. Suppose that R is a locally nilpotent radical of G. If every cyclic subgroup* *of G is a GNA-subgroup, then either G is abelian or $G = R\langle b\rangle$, where R is abelian, $b^2 \in R$ and $a^b = a^{-1}$ for each element $a \in R$. Moreover, in the second case, the following conditions hold:*

(i) *if $b^2 = 1$, then the Sylow 2-subgroup D of R is elementary abelian;*

(ii) *if $b^2 \neq 1$, then either D is elementary abelian or $D = E \times \langle v\rangle$, where E is elementary abelian and $\langle b, v\rangle$ is a quaternion group.*

Conversely, if a group G satisfies the above conditions, then every cyclic subgroup of G is a GNA-subgroup.

1. Preliminary results

Lemma 1. *Let G be a group whose cyclic subgroups are GNA-subgroups.*

(i) *If H is a subgroup of G, then every cyclic subgroup of H is a GNA-subgroup.*

(ii) *If H is a normal subgroup of G, then every cyclic subgroup of G/H is a GNA-subgroup.*

Proof. It follows from the definition of GNA-subgroups. □

In the paper [3], B.H. Neumann proved the following classical result: *if the factor-group $G/\zeta(G)$ is finite, then the derived subgroup $[G, G]$ is also finite.* As a corollary, we can come to the following generalization: *if the factor-group $G/\zeta(G)$ is locally finite, then the derived subgroup $[G, G]$ is also locally finite.*

Lemma 2. *Let G be a generalized radical group. If every cyclic subgroup of G is a GNA-subgroup, then G is soluble of class at most 3.*

Proof. Suppose that the locally finite radical $\mathrm{Lfr}(G) = F$ of G is non-identity. Then $[F, F]$ is abelian [5, Corollary 14]. It follows that in any case the locally nilpotent radical $\mathrm{Lnr}(G) = R$ of G is non-identity. We will prove that G is a radical group. Suppose the contrary. Then G includes the normal subgroups T and S such that $R \leqslant T \leqslant S$, T is radical, S/T is locally finite and $\mathrm{Lnr}(S/T) = \langle 1\rangle$. By [5, Corollary 4], R is a Dedekind group. Corollary 1 from [5] shows that every subgroup of R is G-invariant. Then $S/C_S(R)$ is abelian (see, for example [7, Theorem 1.5.1]). We observe that $C_S(R) \cap T \leqslant R$ (see [4, Lemma 4]). Suppose first that R is periodic. Then

$$C_S(R)/(C_S(R) \cap R) = C_S(R)/(C_S(R) \cap T) \cong C_S(R)T/T \leqslant S/T.$$

In particular, $C_S(R)/(C_S(R) \cap R)$ is locally finite. Since R is periodic and locally nilpotent, $C_S(R)$ is locally finite. Being locally finite, $C_S(R)$ is metabelian by [5, Corollary 14]. Since S/T does not include non-identity normal abelian subgroups, $C_S(R) \leqslant T$. We have now

$$S/T \cong (S/C_S(R))/(T/C_S(R)).$$

We have remarked above that the factor-group $S/C_S(R)$ is abelian, and therefore S/T is abelian. Contradiction.

Suppose now that R is not periodic. Corollary 4 from [5] shows that R is abelian. Let V be the periodic part of R and put $C = C_S(R)$. By proved above, $C/R \cong C/(C \cap R)$ is locally finite. Also, the inclusion $R \leqslant \zeta(C)$ implies that $[C, C]$ is a locally finite subgroup. Using [5, Corollary 14], we obtain that C is soluble. It follows that $C_S(R) \leqslant T$, and using the arguments from above, we again obtain a contradiction. This contradiction shows that G is a radical group.

Then $C_G(R) \leqslant R$ [4, Lemma 4]. By [5, Corollary 4], R is a Dedekind group, in particular, R is metabelian. Corollary 1 from [5] shows that every subgroup of R is G-invariant. Then $G/C_G(R)$ is abelian (see, e.g., Theorem 1.5.1 in [7]). The inclusion $C_G(R) \leqslant R$ implies that G/R is abelian, so that G is soluble and $\mathrm{scl}(G) \leqslant 3$. □

Corollary 1. *Let G be a locally generalized radical group. If every cyclic subgroup of G is a GNA-subgroup, then G is soluble of class at most 3.*

Lemma 3. *Let G be a group and A be a normal abelian subgroup of G. Suppose that $G = A\langle b\rangle$ where $b^2 \in A$ and $a^b = a^{-1}$ for each element $a \in A$. If the subgroup $\langle b\rangle$ is a GNA-subgroup, then*

(i) *if $b^2 = 1$, then the Sylow 2-subgroup D of A is elementary abelian;*

(ii) *if $b^2 \neq 1$, then either D is elementary abelian or $D = E \times \langle v\rangle$ where E is elementary abelian and $\langle b, v\rangle$ is a quaternion group.*

Proof. Suppose that $a \in C_A(b)$, then $a^b = a$. On the other hand, by our conditions, $a^b = a^{-1}$, that is $a^{-1} = a$ and $1 = a^2$. Thus $C_A(b)$ is an elementary abelian 2-subgroup. If $c = b^2 \neq 1$, then $c \in C_A(b)$, and by proved above, $1 = c^2 = b^4$. Conversely, if $|a| = 2$, then $a \in C_A(b)$.

Note that if $a \in \langle b\rangle$, then $\langle b\rangle^a = \langle b\rangle$. Let a be an arbitrary element of A. Then $b^{-1}a^{-1}ba = aa = a^2$, and $b^a = a^{-1}ba = ba^2$. Furthermore, $b^{-1}ab = a^{-1}$ and $ab = ba^{-1}$. Then we have

$$(ba)(ba) = b(ab)a = b(ba^{-1})a = b^2.$$

Since this is valid for arbitrary element a, we obtain $(ba^2)^2 = b^2$.

Since $\langle b\rangle$ is a GNA-subgroup, we have two possibilities: either $\langle b\rangle^a = \langle b\rangle$ or $N_K(\langle b\rangle) = N_K(N_K(\langle b\rangle))$, where $K = \langle\langle b\rangle, a\rangle = \langle b, a\rangle$, $a \in A$. In the first case, we obtain that a subgroup

$$\langle b\rangle = \langle b\rangle^a = \langle ba^2\rangle = \langle b, a^2\rangle$$

is a 2-subgroup, in particular, a^2 (and hence a) is a 2-element. In the second case, we again obtain that a subgroup

$$\langle b\rangle = N_K(\langle b\rangle) = N_K(N_K(\langle b\rangle))$$

is a 2-subgroup.

Suppose first $|b| = 2$. Then $\langle b\rangle \cap A = \langle 1\rangle$. Assume that A has an element u of order 4. By proved above $u^{-1}bu = bu^2$. Since $|u^2| = 2$, $u^2 \in C_A(b)$. It follows that $\langle b, u^2\rangle$ is abelian. On the one hand, $\langle b\rangle \neq \langle b\rangle^u$. On the other hand $N_K(\langle b\rangle) = \langle b, u^2\rangle \neq \langle b\rangle$, $K = \langle\langle b\rangle, u\rangle = \langle b, u\rangle$. So that $N_K(\langle b\rangle) \neq N_K(N_K(\langle b\rangle))$, and we obtain a contradiction. This contradiction shows that a Sylow 2-subgroup of A is elementary abelian.

Suppose now that $c = b^2 \neq 1$. Let D be a Sylow 2-subgroup of A. Since the subgroup $\langle c\rangle$ is normal in G, its image in the factor-group $G/\langle c\rangle$ is a GNA-subgroup. As proved above, $D/\langle c\rangle$ is an elementary abelian 2-subgroup. Then either D is elementary abelian or D has an element v of order 4 such that $v^2 = c = b^2$. Consider the last situation. Since v has a maximal order among all the elements of D, $D = E \times \langle v\rangle$. Since $\langle v\rangle$ is $\langle b\rangle$-invariant, we have

$$|\langle b\rangle\langle v\rangle| = (|\langle b\rangle||\langle v\rangle|)/|\langle b\rangle \cap \langle v\rangle| = 8.$$

Furthermore, as proved above, $v^{-1}bv = bv^2 = bb^2 = b^3$. Hence $\langle b, v\rangle$ is a product of two normal cyclic subgroups of order 4. It follows that $\langle b, v\rangle$ is a quaternion group. □

Corollary 2. *Let G be a group and A be a normal abelian non-periodic subgroup of G. Suppose that $G = A\langle b\rangle$ where $b^2 \in A$, and $a^b = a^{-1}$ for each element $a \in A$. Then G has a subgroup, which is not a GNA-subgroup.*

Proof. Indeed, let h be an element of A of infinite order. Put $H = \langle h^4\rangle$. Then H is normal in G, the element hH has order 4, and $\langle hH\rangle \cap \langle bH\rangle = H$. Lemma 3 shows that the subgroup $\langle b, h^4\rangle$ can not be a GNA-subgroup. □

Lemma 4. *Let G be a non-periodic finitely generated soluble group. Suppose that R is a locally nilpotent radical of G. If every cyclic subgroup of G is a GNA-subgroup, then either G is abelian or $G = R\langle b\rangle$ where R is abelian, $b^2 \in R$, and $a^b = a^{-1}$ for each element $a \in R$.*

Proof. By [5, Corollary 4], R is a Dedekind group. Corollary 1 from [5] shows that every subgroup of R is G-invariant. Then $G/C_G(R)$ is abelian (see, for example [7, Theorem 1.5.1]). The inclusion $C_G(R) \leqslant R$ [4, Lemma 4] implies that G/R is abelian. Being abelian and finitely generated G/R is finitely presented. It follows that R has the elements $x_1, \dots, x_k$ such that $R = \langle x_1 \rangle^G \dots \langle x_k \rangle^G$ (see, for example, [2, p. 421]). Since every subgroup of R is G-invariant, $\langle x_j \rangle^G = \langle x_j \rangle$, $1 \leqslant j \leqslant k$. It follows that R is finitely generated. If we suppose that R is periodic, then R is finite. The inclusion $C_G(R) \leqslant R$ [4, Lemma 4] implies that G/R is also finite, and hence G is finite. This contradiction proves that R is non-periodic.

Then Corollary 2 and 3 from [5] shows that R is abelian. Suppose that the center $\zeta(G)$ contains every element of R of infinite order. Clearly, R is generated by elements of infinite order, so that $R \leqslant \zeta(G)$. Then the fact that G/R is abelian implies that G is nilpotent. Using again Corollary 2 and 3 from [5] we obtain that G is abelian. Therefore, we consider the case when a subgroup R contains an element of infinite order, which is not central. Since R is abelian and finitely generated,

$$R = \langle u_1 \rangle \times \dots \times \langle u_n \rangle \times \langle v_1 \rangle \times \dots \times \langle v_t \rangle,$$

where the elements $u_1, \dots, u_n$ have infinite orders and $v_1, \dots, v_t$ have finite orders. Suppose that $u_j \in \zeta(G)$ for all j, $1 \leqslant j \leqslant n$. Since $\zeta(G)$ does not include R, there exists an index m such that $v_m \notin \zeta(G)$. Then there exists an element g such that $v_m^g = v_m^r \neq v_m$ where r is a certain positive integer. Consider the element $u_1 v_m$. We have

$$(u_1 v_m)^g = u_1^g v_m^g = u_1 v_m^r \neq u_1 v_m.$$

We remark that $u_1 v_m$ has infinite order. By [5, Corollary 1], a subgroup $\langle u_1 v_m \rangle$ is G-invariant. Then the fact that $g \notin C_G(u_1 v_m)$ implies $(u_1 v_m)^g = (u_1 v_m)^{-1} = u_1^{-1} v_m^{-1}$. On the other hand, we have $(u_1 v_m)^g = u_1 v_m^r$, which implies that $u_1 = u_1^{-1}$. Contradiction. So, there exists an index j such that $u_j \notin \zeta(G)$. Without loss of generality we can suppose that $j = 1$. Let b be an element of G such that $G = \langle b \rangle C_G(\langle u_1 \rangle)$. Then $u_1^b = u_1^{-1}$, and $b^2 \in C_G(\langle u_1 \rangle)$. Suppose now that there exists an index s, $1 < s \leqslant n$, such that $[b, u_s] = 1$. Then

$$(u_1 u_s)^b = u_1^b u_s^b = u_1^{-1} u_s \neq u_1 u_s.$$

On the other hand, an infinite cyclic subgroup $\langle u_1 u_s \rangle$ is G-invariant by [5, Corollary 1]. Then it follows that

$$(u_1 u_s)^b = (u_1 u_s)^{-1} = u_1^{-1} u_s^{-1}.$$

Hence $u_s = u_s^{-1}$, and we obtain a contradiction. This contradiction shows that $u_j^b = u_j^{-1}$ for all j, $1 \leqslant j \leqslant n$. Using the same arguments we can prove that $v_j^b = v_j^{-1}$ for all j, $1 \leqslant j \leqslant t$. It follows that $a^b = a^{-1}$ for all elements $a \in R$.

With the help of similar arguments we can prove that

$$C_G(\langle u_1 \rangle) = C_G(R) = R.$$

Hence $G = R\langle b \rangle$ and $b^2 \in R$. □

Corollary 3. *Let G be a non-periodic locally generalized radical group. Suppose that R is a locally nilpotent radical of G. If every cyclic subgroup of G is a GNA-subgroup, then either G is abelian or $G = R\langle b \rangle$ where R is abelian, $b^2 \in R$, and $a^b = a^{-1}$ for each element $a \in R$.*

Proof. By Corollary 1, G is soluble. Suppose that G is not abelian. Then G includes a non-periodic finitely generated non-abelian subgroup K. By Lemma 4, $K = \mathrm{Lnr}(K)\langle b \rangle$, where $\mathrm{Lnr}(K)$ is abelian, $b^2 \in \mathrm{Lnr}(K)$, $b^4 = 1$, and $a^b = a^{-1}$ for each element $a \in \mathrm{Lnr}(K)$.

Choose in G a local family $\mathfrak{L}$ of finitely generated subgroups containing K, and let $L \in \mathfrak{L}$. Using again Lemma 4 we obtain that $L = \mathrm{Lnr}(L)\langle b_1 \rangle$, where $\mathrm{Lnr}(L)$ is abelian, $b_1^2 \in \mathrm{Lnr}(L)$, $b_1^4 = 1$, and $a^{b_1} = a^{-1}$ for each element $a \in \mathrm{Lnr}(L)$. Since K is not locally nilpotent, $\mathrm{Lnr}(L) \cap K \neq K$. On the other hand,

$$|K : \mathrm{Lnr}(L) \cap K| \leqslant |L : \mathrm{Lnr}(L)| = 2,$$

so that $\mathrm{Lnr}(K) = \mathrm{Lnr}(L) \cap K$. In particular, $b \notin \mathrm{Lnr}(L)$. It follows that $b = b_1 u$ for some element $u \in \mathrm{Lnr}(L)$. As in the proof of Lemma 3, we can show that $b^2 = (b_1 u)^2 = b_1^2$. So, instead of b_1 we can put b. In other words, if L is an arbitrary subgroup of the family $\mathfrak{L}$, then $L = \mathrm{Lnr}(L)\langle b \rangle$, where $\mathrm{Lnr}(L)$ is abelian, $b^2 \in \mathrm{Lnr}(L)$, $b^4 = 1$, and $a^b = a^{-1}$ for each element $a \in \mathrm{Lnr}(L)$. Since $\mathfrak{L}$ is a local family, $G = \mathrm{Lnr}(G)\langle b \rangle$, where $\mathrm{Lnr}(G)$ is abelian, $b^2 \in \mathrm{Lnr}(G)$, $b^4 = 1$ and $a^b = a^{-1}$ for each element $a \in \mathrm{Lnr}(G)$. □

2. Proof of the main result, Theorem 1

The necessity follows from Lemma 3 and Corollary 3.

Conversely, let a group G satisfies the theorem conditions and let x be an arbitrary element of G. If $x \in R$, then $\langle x \rangle$ is normal in G, in particular, $\langle x \rangle$ is a *GNA*-subgroup. Suppose that $x \notin R$. Then $x = bu$ for some element $u \in R$. In this case, $G = R\langle x\rangle$. As in the proof of Lemma 3, we can show that $x^2 = (bu)^2 = b^2$. Since R is abelian, $a^x = a^b = a^{-1}$ for each element $a \in R$.

Let g be an arbitrary element of G, then $g = x^k a$ for some element $a \in R$. It follows that $g^{-1}xg = a^{-1}xa$. We have $x^{-1}a^{-1}xa = aa = a^2$, and $a^{-1}xa = xa^2$. Furthermore, $x^{-1}ax = a^{-1}$, and $ax = xa^{-1}$. Then we have $(xa)(xa) = x(ax)a = x(xa^{-1})a = x^2$.

Consider $\langle x\rangle^a$. We have $\langle x\rangle^a = \langle xa^2\rangle = \langle x, a^2\rangle$. In particular, it shows that $\langle x\rangle^a$ is a 2-subgroup. In turn, it follows that a^2 is a 2-element, so that a is also a 2-element. Then $a = vc$ where $c^2 = 1$. A subgroup $\langle b, v\rangle$ is a quaternion group, so that $\langle b\rangle$ is $\langle v\rangle$-invariant. It follows that $\langle x\rangle$ is $\langle v\rangle$-invariant. Since $c^2 = 1$, $[c, x] = 1$. It follows that $\langle x\rangle^a = \langle x\rangle$, which shows that $\langle x\rangle$ is a *GNA*-subgroup.

The following result follows directly from Theorem 1 and Corollary 2.

Corollary 4. *Let G be a non-periodic locally generalized radical group. Then every subgroup of G is a GNA-subgroup if and only if G is abelian.*

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Contact information

Aleksandr A. Pypka Department of Geometry and Algebra, Faculty of Mechanics and Mathematics, Oles Honchar Dnipro National University, Gagarin ave., 72, Dnipro, 49010, Ukraine
E-Mail(s): pypka@ua.fm

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