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Adjoint functors, preradicals and closure operators in module categories

A. I. Kashu

Abstract. In this article preradicals and closure operators are studied in an adjoint situation, defined by two covariant functors between the module categories R-Mod and S-Mod. The mappings which determine the relationship between the classes of preradicals and the classes of closure operators of these categories are investigated. The goal of research is to elucidate the concordance (compatibility) of these mappings. For that some combinations of them, consisting of four mappings, are considered and the commutativity of corresponding diagrams (squares) is studied. The obtained results show the connection between considered mappings in adjoint situation.

1. Preliminary notions and facts

The present work is devoted to the study of preradicals and closure operators in module categories. The behaviour of these constructions is investigated in an adjoint situation, i.e. in the case of two adjoint covariant functors between the module categories. There exists a series of mappings in this case, which realize the relationship between preradicals and closure operators of considered categories. The principal attention is given to investigation of compatibility of these mappings, which is expressed as commutativity of suitable diagrams.

Firstly we will clarify the situation in what we intend to work. Let ${}_RU_S$ be an arbitrary (R, S)-bimodule and consider the following two covariant

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functors defined by this bimodule,

$$R\text{-Mod} \underset{T=U\otimes_S \text{-}}{\overset{H=\mathrm{Hom}_R(U,\text{-})}{\rightleftarrows}} S\text{-Mod},$$

where R-Mod (S-Mod) is the category of left R-modules (S-modules) and T is left adjoint to H (notation: $T \dashv H$) ([1, 2]). By the definition this means that for every pair of modules $X \in R$-Mod and $Y \in S$-Mod there exists a natural isomorphism $\mathrm{Hom}_R\big(T(Y), X\big) \cong \mathrm{Hom}_S\big(Y, H(X)\big)$. This adjoint situation is completely defined by two associated natural transformations,

$$\Phi : TH \to \mathbb{1}_{R-\mathrm{Mod}}, \qquad \Psi : \mathbb{1}_{S-\mathrm{Mod}} \to HT,$$

which satisfy the following relations:

$$H(\Phi_X) \cdot \Psi_{H(X)} = 1_{H(X)}, \quad \Phi_{T(Y)} \cdot T(\Psi_Y) = 1_{T(Y)},$$

where $X \in R$-Mod and $Y \in S$-Mod. We remark that any adjoint situation with covariant functors between two module categories has this form (up to an isomorphism).

Recall that a *preradical* r of R-Mod is a subfunctor of the identical functor of R-Mod, i.e. r is a function which associates to every module $M \in R$-Mod a submodule $r(M) \subseteq M$ such that $f\big(r(M)\big) \subseteq r(M')$ for every R-morphism $f : M \to M'$ ([3, 4]). Denote by $\mathbb{PR}(R)$ the class of all preradicals of R-Mod. An order relation in $\mathbb{PR}(R)$ is defined as follows: $r \leqslant s \Leftrightarrow r(M) \subseteq s(M)$ for every $M \in R$-Mod.

Remind also that a *closure operator* of R-Mod is a function C, which associates to every pair $N \subseteq M$, where $N \in \mathbb{L}(M)$, a submodule of M denoted by $C_M(N)$, such that the following conditions are satisfied:

(c_1) $N \subseteq C_M(N)$ (*extension*);

(c_2) If $N_1, N_2 \in \mathbb{L}(M)$ and $N_1 \subseteq N_2$, then $C_M(N_1) \subseteq C_M(N_2)$ (*monotony*);

(c_3) For every R-morphism $f : M \to M'$ and $N \in \mathbb{L}(M)$ the relation $f\big(C_M(N)\big) \subseteq C_{M'}\big(f(N)\big)$ is true (*continuity*),

where $M \in R$-Mod and $\mathbb{L}(M)$ is the lattice of submodules of M ([5–7]). Let $\mathbb{CO}(R)$ be the class of all closure operators of R-Mod with the order relation: $C \leqslant D \Leftrightarrow C_M(N) \subseteq D_M(N)$ for every pair $N \subseteq M$ of R-Mod.

The relationship between preradicals and closure operators of R-Mod is expressed by three mappings ([5–7]), which we denote and define as follows.

a) $\varphi^R : \mathbb{CO}(R) \to \mathbb{PR}(R)$. For every closure operator $C \in \mathbb{CO}(R)$ the corresponding preradical $\varphi^R(C) = r_C$ is defined by the rule:

$$r_C(M) \stackrel{\text{def}}{=\!=} C_M(0) \tag{1.1}$$

for every $M \in R$-Mod.

b) $\psi_1^R : \mathbb{PR}(R) \to \mathbb{CO}(R)$. For every preradical $r \in \mathbb{PR}(R)$ and every pair $M \subseteq X$ of R-Mod we define: $\psi_1^R(r) = C^r$, where

$$C_X^r(M)/M \stackrel{\text{def}}{=\!=} r(X/M). \tag{1.2}$$

c) $\psi_2^R : \mathbb{PR}(R) \to \mathbb{CO}(R)$. For $r \in \mathbb{PR}(R)$ and $M \subseteq X$ of R-Mod we define: $\psi_2^R(r) = C_r$, where

$$(C_r)_X(M) \stackrel{\text{def}}{=\!=} r(X) + M. \tag{1.3}$$

Then C^r is the greatest among the closure operators $C \in \mathbb{CO}(R)$ with the property $\varphi^R(C) = r$, while C_r is the least between such operators. If we define in $\mathbb{CO}(R)$ the equivalence relation

$$C \sim D \quad \Longleftrightarrow \quad r_C = r_D,$$

then $\mathbb{PR}(R) \cong \mathbb{CO}(R)/\sim$, where to every preradical $r \in \mathbb{PR}(R)$ corresponds the equivalence class $[C_r, C^r]$, i.e. the interval between C_r and C^r.

2. Mappings between preradicals and closure operators in adjoint situation

In this section we consider the adjoint situation $T \dashv H$ indicated above and recall the definitions of the mappings between the preradicals of categories R-Mod and S-Mod ([8–10]), as well as the definitions of the mappings between the closure operators of these categories ([5, 11]).

As above, the bimodule ${}_RU_S$ defines the adjoint functors

$$R\text{-Mod} \quad \underset{T = U \otimes_S -}{\overset{H = \mathrm{Hom}_R(U, -)}{\rightleftarrows}} \quad S\text{-Mod}$$

with the associated transformations $\Phi : TH \to \mathbb{1}_{R-\mathrm{Mod}}$ and $\Psi : \mathbb{1}_{S-\mathrm{Mod}} \to HT$. In this situation two mappings can be defined

$$\mathbb{PR}(R) \quad \underset{(-)^*}{\overset{(-)^*}{\rightleftarrows}} \quad \mathbb{PR}(S)$$

between the *classes of preradicals* of studied categories.

a) The mapping $r \rightsquigarrow r^*$ from $\mathbb{PR}(R)$ to $\mathbb{PR}(S)$ is defined as follows. Let $r \in \mathbb{PR}(R)$ and $Y \in S$-Mod. Applying T and r, we obtain in R-Mod the sequence

$$0 \to r\big(T(Y)\big) \xrightarrow{\subseteq} T(Y) \xrightarrow[\text{nat}]{\pi^r_{T(Y)}} T(Y)/r\big(T(Y)\big) \to 0,$$

where $\pi^r_{T(Y)}$ is the natural epimorphism. Applying H and using Ψ, we have in R-Mod the composition of morphisms

$$Y \xrightarrow{\Psi_Y} HT(Y) \xrightarrow{H(\pi^r_{T(Y)})} H[T(Y)/r\big(T(Y)\big)].$$

Preradical r^* is defined by the rule

$$r^*(Y) \overset{\text{def}}{=\!=} \operatorname{Ker}[H(\pi^r_{T(Y)}) \cdot \Psi_Y]. \tag{2.1}$$

b) Now we define the inverse mapping $s \rightsquigarrow s^*$ from $\mathbb{PR}(S)$ to $\mathbb{PR}(R)$. Let $s \in \mathbb{PR}(S)$ and $X \in R$-Mod. By H and s we have in S-Mod the inclusion $i^s_{H(X)} : s\big(H(X)\big) \xrightarrow{\subseteq} H(X)$. Applying T and using Φ, we obtain in R-Mod the morphisms:

$$T\big(s(H(X))\big) \xrightarrow{T(i^s_{H(X)})} TH(X) \xrightarrow{\Phi_X} X.$$

Preradical $s^* \in \mathbb{PR}(R)$ is defined by the rule

$$s^*(X) \overset{\text{def}}{=\!=} \operatorname{Im}[\Phi_X \cdot T(i^s_{H(X)})]. \tag{2.2}$$

In the works [8–10] a series of properties of these mappings is shown.

In continuation we will define two mappings between the *classes of closure operators* of the studied categories in adjoint situation ([5, 11]):

$$\mathbb{CO}(R) \underset{(-)^*}{\overset{(-)^*}{\rightleftarrows}} \mathbb{CO}(S).$$

c) We begin with the mapping $C \rightsquigarrow C^*$ from $\mathbb{CO}(R)$ to $\mathbb{CO}(S)$. Let $C \in \mathbb{CO}(R)$ and $n : N \xrightarrow{\subseteq} Y$ be an arbitrary inclusion of S-Mod. Apply T and consider in R-Mod the decomposition of the morphism $T(n)$ by the operator C:

$$\begin{tikzcd} T(N) \arrow[r, "\overline{T(n)}"] \arrow[rrr, bend left, "T(n)"] & \operatorname{Im} T(n) \arrow[r, "j^n_C", "\subseteq"'] & C_{T(Y)}\big(\operatorname{Im} T(n)\big) \arrow[r, "i^n_C", "\subseteq"'] & T(Y), \end{tikzcd}$$

where $\overline{T(n)}$ is the restriction of $T(n)$ to its image, and j^n_C, i^n_C are the inclusions. Consider the natural epimorphism

$$\pi^n_C:\ T(Y) \xrightarrow{\text{nat}} T(Y)/C_{T(Y)}\big(\operatorname{Im} T(n)\big).$$

By H and Ψ we obtain in S-Mod the composition of morphisms

$$Y \xrightarrow{\Psi_Y} HT(Y) \xrightarrow{H(\pi^n_C)} H\big[T(Y)/C_{T(Y)}\big(\operatorname{Im} T(n)\big)\big].$$

Operator C^* is defined by the rule:

$$C^*_Y(N) \overset{\text{def}}{=\!=} \operatorname{Ker}\,[H(\pi^n_C)\cdot \Psi_Y]. \tag{2.3}$$

d) Finally, we show the inverse mapping $D \rightsquigarrow D^*$ from $\mathbb{CO}(S)$ to $\mathbb{CO}(R)$. Let $D \in \mathbb{CO}(S)$ and $m: M \xrightarrow{\subseteq} X$ be an inclusion of R-Mod. Apply H and consider the decomposition of $H(m)$ by D:

$$\begin{tikzcd} H(M) \arrow[r, "\overline{H(m)}"] \arrow[rrr, bend left, "H(m)"] & \operatorname{Im} H(m) \arrow[r, "j^m_D", "\subseteq"'] & D_{H(X)}\big(\operatorname{Im} H(m)\big) \arrow[r, "i^m_D", "\subseteq"'] & H(X), \end{tikzcd}$$

where $\overline{H(m)}$ is the restriction of $H(m)$ and j^m_D, i^m_D are the inclusions. Returning in R-Mod by T and using Φ, we obtain the composition of morphisms,

$$T[D_{H(X)}\big(\operatorname{Im} H(m)\big)] \xrightarrow{T(i^m_D)} TH(X) \xrightarrow{\Phi_X} X.$$

The operator D^* is defined by the rule:

$$D^*_X(M) \overset{\text{def}}{=\!=} \operatorname{Im}[\Phi_X \cdot T(i^m_D)] + M. \tag{2.4}$$

Totalizing the exposed above definitions of the mappings and considering them together, we obtain the general situation for the pair $T \dashv H$ of adjoint functors,

$$\begin{tikzcd} \mathbb{PR}(R) \arrow[rr, shift left, "(-)^*"] \arrow[d, bend right, "\psi^R_1"'] \arrow[d, bend left, "\psi^R_2"] & & \mathbb{PR}(S) \arrow[ll, shift left, "(-)^*"] \arrow[d, bend right, "\psi^S_1"'] \arrow[d, bend left, "\psi^S_2"] \\ \mathbb{CO}(R) \arrow[u, "\varphi^R"] \arrow[rr, shift left, "(-)^*"] & & \mathbb{CO}(S) \arrow[u, "\varphi^S"] \arrow[ll, shift left, "(-)^*"] \end{tikzcd} .$$

The goal of the following investigations consists in the elucidation of the relations between these ten mappings, in the search of concordance and compatibility of them. For that we analyze separately diverse combinations by four mappings (six cases) and we study the commutativity of respective squares.

In continuation we will consider three pairs of mappings:

$$\text{I)}\ \ (\varphi^R, \varphi^S), \qquad \text{II)}\ \ (\psi_1^R, \psi_1^S), \qquad \text{III)}\ \ (\psi_2^R, \psi_2^S).$$

3. Squares containing the first pair of mappings

We start with two combinations of considered mappings in which participate φ^R and φ^S.

a) Firstly we analyze the square which consists in the following mappings:

$$\begin{tikzcd} \mathbb{PR}(R) \arrow[r, "(-)^*"] & \mathbb{PR}(S) \\ \mathbb{CO}(R) \arrow[u, "\varphi^R"'] \arrow[r, "(-)^*"] & \mathbb{CO}(S) \arrow[u, "\varphi^S"'] \end{tikzcd} .$$

Theorem 3.1. *For every closure operator $C \in \mathbb{CO}(R)$ the relation*

$$r_C^* = r_{C^*}$$

is true (in this sense we can say that the previous square is commutative).

Proof. 1) We begin with the route $C \overset{\varphi^R}{\rightsquigarrow} r_C \overset{(-)^*}{\rightsquigarrow} r_C^*$ for $C \in \mathbb{CO}(R)$. The rule (1.1) shows that $r_C(X) \overset{\text{def}}{=\!=} C_X(0)$ for every $X \in R$-Mod. The following step $r_C \overset{(-)^*}{\rightsquigarrow} r_C^*$ uses the rule (2.1). Namely, for every $Y \in S$-Mod we have:

$$(r_C^*)\,(Y) \overset{\text{def}}{=\!=} \operatorname{Ker}\,[H(\pi_{T(Y)}^{r_C}) \cdot \Psi_Y], \tag{3.1}$$

where $\pi_{T(Y)}^{r_C} :\ T(Y) \to T(Y)/r_C\big(T(Y)\big)$ is the natural epimorphism which leads to the composition of morphisms

$$Y \xrightarrow{\Psi_Y} HT(Y) \xrightarrow{H(\pi_{T(Y)}^{r_C})} H\big[T(Y)/r_C\big(T(Y)\big)\big].$$

2) For the same operator $C \in \mathbb{CO}(R)$ now we follow the way: $C \overset{(-)^*}{\rightsquigarrow} C^* \overset{\varphi^S}{\rightsquigarrow} r_{C^*}$. The transition $C \overset{(-)^*}{\rightsquigarrow} C^*$ is realized by the rule (2.3), i.e. for every inclusion $n : N \xrightarrow{\subseteq} Y$ of S-Mod we have $C_Y^*(N) = \operatorname{Ker}\,[H(\pi_C^n)\cdot\Psi_Y]$, where $\pi_C^n :\ T(Y) \to T(Y)/C_{T(Y)}\big(\operatorname{Im} T(n)\big)$ is the natural epimorphism.

On the following step $C^* \overset{\varphi^{\mathrm{S}}}{\rightsquigarrow} r_{C^*}$ we use the definition (1.1): $r_{C^*}(Y) \overset{\text{def}}{=\!=} C^*_Y(0)$ for every $Y \in S$-Mod. Now we come back to the construction of C^* and assume $N = 0$. Then the situation is simplified, since $n = 0$, $T(n) = 0$, $\operatorname{Im} T(n) = 0$, $\pi^n_C = \pi^0_C : \ T(Y) \to T(Y)/C_{T(Y)}(0)$, therefore,

$$r_{C^*}(Y) = C^*_Y(0) \overset{\text{def}}{=\!=} \operatorname{Ker}[H(\pi^0_C) \cdot \Psi_Y]. \tag{3.2}$$

3) Now we compare the expressions (3.1) and (3.2) for $r^*_C(Y)$ and $r_{C^*}(Y)$. It is obvious that the relation $r_C\big(T(Y)\big) \overset{\text{def}}{=\!=} C_{T(Y)}(0)$ implies the coincidence of epimorphisms $\pi^{r_C}_{T(Y)}$ and π^0_C, so by (3.1) and (3.2) we have $r^*_C(Y) = r_{C^*}(Y)$ for every $Y \in S$-Mod, i.e. $r^*_C = r_{C^*}$. $\square$

b) Further we consider the second combination of mappings which contains φ^R and φ^S, namely the square

$$\begin{tikzcd}
\mathbb{PR}(R) & & \mathbb{PR}(S) \arrow[ll, "(-)^*"'] \\
\mathbb{CO}(R) \arrow[u, "\varphi^R"'] & & \mathbb{CO}(S), \arrow[ll, "(-)^*"'] \arrow[u, "\varphi^S"']
\end{tikzcd}$$

analyzing the concordance of these mappings.

Theorem 3.2. *For every closure operator $D \in \mathbb{CO}(S)$ the relation*

$$r_{D^*} = r^*_D$$

is true, i.e. the previous square is commutative.

Proof. 1) We begin with the route $D \overset{(-)^*}{\rightsquigarrow} D^* \overset{\varphi^{\mathrm{R}}}{\rightsquigarrow} r_{D^*}$, where $D \in \mathbb{CO}(S)$. The transition $D \overset{(-)^*}{\rightsquigarrow} D^*$ is defined by the rule (2.4), i.e. for every inclusion $m : M \xrightarrow{\subseteq} X$ of R-Mod we have: $D^*_X(M) = \operatorname{Im}[\Phi_X \cdot T(i^m_D)] + M$, where $i^m_D : D_{H(X)}\big(\operatorname{Im} H(m)\big) \xrightarrow{\subseteq} H(X)$ is the inclusion, which leads to the composition

$$T[D_{H(X)}\big(\operatorname{Im} H(m)\big)] \xrightarrow{T(i^m_D)} TH(X) \xrightarrow{\Phi_X} X.$$

The following step $D^* \overset{\varphi^{\mathrm{R}}}{\rightsquigarrow} r_{D^*}$ is defined by the rule (1.1), i.e. $r_{D^*}(X) \overset{\text{def}}{=\!=} D^*_X(0)$ for every $X \in R$-Mod. To specify the module $D^*_X(0)$ we assume $M = 0$ in the above construction of $D^*_X(M)$. Then $m = 0$, $H(m) = 0$, $\overline{H(m)} = 0$, $TH(M) = 0$, $T\big(\operatorname{Im} H(m)\big) = 0$, therefore $D_{H(X)}\big(\operatorname{Im} H(m)\big) = D_{H(X)}(0)$,

$i_D^m = i_D^0 : D_{H(X)}(0) \xrightarrow{\subseteq} H(X)$ and $T(i_D^0) : T\big(D_{H(X)}(0)\big) \to TH(X)$. In such a way we obtain

$$r_{D^*}(X) = D_X^*(0) = \mathrm{Im}[\Phi_X \cdot T(i_D^0)]. \tag{3.3}$$

2) Further we follow the way $D \overset{\varphi^{\mathrm{S}}}{\rightsquigarrow} r_D \overset{(-)^*}{\rightsquigarrow} r_D^*$ for $D \in \mathbb{CO}(S)$. By the rule (1.1) we have $r_D(Y) \overset{\text{def}}{=\!=} D_Y(0)$ for every $Y \in S$-Mod. The second step $r_D \overset{(-)^*}{\rightsquigarrow} r_D^*$ is defined by the rule (2.2), i.e. for every $X \in R$-Mod we have

$$r_D^*(X) = \mathrm{Im}[\Phi_X \cdot T(i_{H(X)}^{r_D})], \tag{3.4}$$

where the inclusion $i_{H(X)}^{r_D} : r_D\big(H(X)\big) \xrightarrow{\subseteq} H(X)$ implies the composition $T\big[r_D\big(H(X)\big)\big] \xrightarrow{T(i_{H(X)}^{r_D})} TH(X) \xrightarrow{\Phi_X} X$.

3) Now we compare the expressions (3.3) and (3.4). Since by the definition $r_D\big(H(X)\big) \overset{\text{def}}{=\!=} D_{H(X)}(0)$, it is clear that the inclusions i_D^0 and $i_{H(X)}^{r_D}$ coincide, so by the indicated above relations it follows that $r_{D^*}(X) = r_D^*(X)$ for every $X \in R$-Mod, i.e. $r_{D^*} = r_D^*$. □

4. Squares containing the second pair of mappings

In continuation we analyze two combinations of the studied mappings in which participate ψ_1^R and ψ_1^S.

a) Now we consider the square

$$\begin{array}{ccc} \mathbb{PR}(R) & \xrightarrow{\quad (-)^* \quad} & \mathbb{PR}(S) \\ \Big\downarrow{\scriptstyle \psi_1^R} & & \Big\downarrow{\scriptstyle \psi_1^S} \\ \mathbb{CO}(R) & \xrightarrow{\quad (-)^* \quad} & \mathbb{CO}(S)\,. \end{array}$$

Theorem 4.1. *For every preradical $r \in \mathbb{PR}(R)$ the relation*

$$C^{r^*} = (C^r)^*$$

is true, i.e. the previous diagram is commutative.

Proof. 1) Let $r \in \mathbb{PR}(R)$. Firstly we follow the route: $r \overset{(-)^*}{\rightsquigarrow} r^* \overset{\psi_1^S}{\rightsquigarrow} C^{r^*}$. The translation $r \overset{(-)^*}{\rightsquigarrow} r^*$ is defined by (2.1), i.e. for every $Y \in S$-Mod we have: $r^*(Y) \stackrel{\text{def}}{=\!=} \operatorname{Ker}[H(\pi^r_{T(Y)}) \cdot \Psi_Y]$, where $\pi^r_{T(Y)} : T(Y) \to T(Y)/r\big(T(Y)\big)$ is the natural epimorphism, which defines the composition

$$Y \xrightarrow{\Psi_Y} HT(Y) \xrightarrow{H(\pi^r_{T(Y)})} H\big[T(Y)/r\big(T(Y)\big)\big].$$

The following step $r^* \overset{\psi_1^S}{\rightsquigarrow} C^{r^*}$ is defined by (1.2), i.e. for every inclusion $n : N \xrightarrow{\subseteq} Y$ of S-Mod we have: $C^{r^*}_Y(N)/N \stackrel{\text{def}}{=\!=} r^*(Y/N)$. To precise the expression of $r^*(Y/N)$, in the above construction of r^* we substitute Y by Y/N. Then we obtain the natural epimorphism

$$\pi^r_{T(Y/N)} \;:\; T(Y/N) \xrightarrow{\text{nat}} T(Y/N)/r\big(T(Y/N)\big)$$

and the composition

$$Y/N \xrightarrow{\Psi_{Y/N}} HT(Y/N) \xrightarrow{H\left(\pi^r_{T(Y/N)}\right)} H\big[T(Y/N)/r\big(T(Y/N)\big)\big].$$

By the definition we have $r^*(Y/N) = \operatorname{Ker}[H(\pi^r_{T(Y/N)}) \cdot \Psi_{Y/N}]$, therefore $C^{r^*}_Y = \operatorname{Ker}[H(\pi^r_{T(Y/N)}) \cdot \Psi_{Y/N}]$. Denoting by $\pi_N : Y \to Y/N$ the natural epimorphism, now it is easy to see that

$$C^{r^*}_Y(N) = \operatorname{Ker}[H(\pi^r_{T(Y/N)}) \cdot \Psi_{Y/N} \cdot \pi_N]. \tag{4.1}$$

2) Further, for $r \in \mathbb{PR}(R)$ we consider the way: $r \overset{\psi_1^{\mathrm{R}}}{\rightsquigarrow} C^r \overset{(-)^*}{\rightsquigarrow} (C^r)^*$. The first step $r \overset{\psi_1^{\mathrm{R}}}{\rightsquigarrow} C^r$ is defined by (1.2), i.e. for every pair $M \subseteq X$ of R-Mod we have $C^r_X(M)/M \stackrel{\text{def}}{=\!=} r(X/M)$. The transition $C^r \overset{(-)^*}{\rightsquigarrow} (C^r)^*$ is determined by (2.3). This means that for every inclusion $n : N \xrightarrow{\subseteq} Y$ of S-Mod we consider in R-Mod the decomposition of $T(n)$ by the operator C^r:

$$\begin{tikzcd}
T(N) \arrow[r, "\overline{T(n)}"] \arrow[rrr, bend left, "T(n)"] & \operatorname{Im} T(n) \arrow[r, "j^n_{C^r}", "\subseteq"'] & C^r_{T(Y)}\big(\operatorname{Im} T(n)\big) \arrow[r, "i^n_{C^r}", "\subseteq"'] & T(Y).
\end{tikzcd}$$

By the natural epimorphism $\pi^n_{C^r} \;:\; T(Y) \xrightarrow{\text{nat}} T(Y)/C^r_{T(Y)}\big(\operatorname{Im} T(n)\big)$ we obtain in S-Mod the composition

$$Y \xrightarrow{\Psi_Y} HT(Y) \xrightarrow{H(\pi^n_{C^r})} H\big[T(Y)/C^r_{T(Y)}\big(\operatorname{Im} T(n)\big)\big].$$

Using (2.3) we have

$$(C^r)^*_Y(N) \stackrel{\text{def}}{=\!=} \operatorname{Ker}[H(\pi^n_{C^r}) \cdot \psi_Y]. \tag{4.2}$$

3) Now we verify the relation between $C^{r^*}_Y(N)$ and $(C^r)^*_Y(N)$. For that we consider in S-Mod the diagram

$$\begin{tikzcd}
 & C^{r^*}_Y(N) \arrow[rr, dashed, "\pi'_N", "\mathrm{nat}"'] \arrow[d, "\subseteq"] & & r^*(Y/N) = C^{r^*}_Y(N)/N \arrow[d, "\subseteq"] \\
N \arrow[r, "n", "\subseteq"'] \arrow[d, "\Psi_N"] & Y \arrow[rr, "\pi_N", "\mathrm{nat}"'] \arrow[d, "\Psi_Y"] \arrow[dd, bend right] & & Y/N \arrow[d, "\Psi_{Y/N}"] \arrow[dd, bend right] \\
HT(N) \arrow[r, "HT(n)"] & HT(Y) \arrow[rr, "HT(\pi_N)"] \arrow[d, "H(\pi^n_{C^r})"] & & HT(Y/N) \arrow[d, "H(\pi^r_{T(Y/N)})"] \\
 & H[T(Y)/C^r_{T(Y)}\big(\operatorname{Im} T(n)\big)] \arrow[rr, dashed, "\cong"] & & H[T(Y/N)/r\big(T(Y/N)\big)],
\end{tikzcd}$$

where π'_N is the natural epimorphism and $r^*(Y/N) \overset{\mathrm{def}}{=\!=} C^{r^*}_Y(N)/N$. We search the relation between the modules of the last line. For that we look for the connection between the modules $T(Y)/C^r_{T(Y)}\big(\operatorname{Im} T(n)\big)$ and $T(Y/N)/r\big(T(Y/N)\big)$. Since T is right exact, it transforms the short exact sequence $0 \longrightarrow N \underset{\subseteq}{\xrightarrow{n}} Y \underset{\mathrm{nat}}{\xrightarrow{\pi_N}} Y/N \to 0$ in an exact sequence $T(N) \xrightarrow{T(n)} T(Y) \xrightarrow{T(\pi_N)} T(Y/N) \to 0$, therefore we have the exact sequence $0 \to \operatorname{Im} T(n) \xrightarrow{\subseteq} T(Y) \xrightarrow{T(\pi_N)} T(Y/N) \to 0$. Then it is clear that $T(Y/N) \cong T(Y)/\operatorname{Im} T(n)$, which implies the isomorphism

$$T(Y/N)\,/\,r\big(T(Y/N)\big) \cong [T(Y)/\operatorname{Im} T(n)]\,/\,r[T(Y)/\operatorname{Im} T(n)].$$

Using (1.2) for C^r, we have $r[T(Y)/\operatorname{Im} T(n)] = C^r_{T(Y)}\big(\operatorname{Im} T(n)\big)/\operatorname{Im} T(n)$, and substituting this module in the previous relation we obtain

$$\begin{aligned} T(Y/N)/r\big(T(Y/N)\big) &\cong [T(Y)/\operatorname{Im} T(n)]/\,[C^r_{T(Y)}\big(\operatorname{Im} T(n)\big)/\operatorname{Im} T(n)] \\ &\cong T(Y)/C^r_{T(Y)}\big(\operatorname{Im} T(n)\big). \end{aligned}$$

Applying H now we have in S-Mod the isomorphism

$$H[T(Y)/C^r_{T(Y)}\big(\operatorname{Im} T(n)\big)] \cong H[T(Y/N)/r\big(T(Y/N)\big)],$$

which closes the previous diagram. Therefore,

$$\operatorname{Ker}[H(\pi^n_{C^r}) \cdot \Psi_Y] = \operatorname{Ker}[H(\pi^r_{T(Y/N)}) \cdot \Psi_{Y/N} \cdot \pi_N]$$

and by (4.1) and (4.2) this means that $(C^r)^*_Y(N) = C^{r^*}_Y(N)$ for every inclusion $N \subseteq Y$ of S-Mod. Thus $(C^r)^* = C^{r^*}$. $\square$

b) In continuation we consider the square

$$\begin{tikzcd} \mathbb{PR}(R) \arrow[d, "\psi_1^R"] & \mathbb{PR}(S) \arrow[l, "(-)^*"'] \arrow[d, "\psi_1^S"] \\ \mathbb{CO}(R) & \mathbb{CO}(S) \arrow[l, "(-)^*"'] \end{tikzcd}$$

and verify the concordance of his mappings.

Theorem 4.2. *For every preradical $s \in \mathbb{PR}(S)$ the relation*

$$(C^s)^* \leqslant C^{s^*}$$

is true. If the module ${}_RU$ is projective, then $(C^s)^* = C^{s^*}$, i.e. the studied square is commutative.

Proof. 1) Let $s \in \mathbb{PR}(S)$. We begin with the route: $s \overset{(-)^*}{\rightsquigarrow} s^* \overset{\psi_1^{\mathrm{R}}}{\rightsquigarrow} C^{s^*}$. The transition $s \overset{(-)^*}{\rightsquigarrow} s^*$ is defined by (2.2), i.e. for every $X \in R$-Mod we have the inclusion $i^s_{H(X)} : s\big(H(X)\big) \xrightarrow{\subseteq} H(X)$, which leads to the composition $T\big[s\big(H(X)\big)\big] \xrightarrow{T(i^s_{H(X)})} TH(X) \xrightarrow{\Phi_X} X$. By the rule (2.2) we have $s^*(X) \overset{\text{def}}{=\!=} \mathrm{Im}[\Phi_X \cdot T(i^s_{H(X)})]$.

The following step $s^* \overset{\psi_1^{\mathrm{R}}}{\rightsquigarrow} C^{s^*}$ is realized by the rule (1.2), i.e. for every inclusion $M \subseteq X$ of R-Mod we have: $C^{s^*}_X(M)/M = s^*(X/M)$. In the above construction of s^* we substitute the module X by X/M, obtaining the inclusion $i^s_{H(X/M)} :\ s\big(H(X/M)\big) \xrightarrow{\subseteq} H(X/M)$ and the composition

$$T\big[s\big(H(X/M)\big)\big] \xrightarrow{T(i^s_{H(X/M)})} TH(X/M) \xrightarrow{\Phi_{X/M}} X/M.$$

By the definition (1.2) in this case we have

$$s^*(X/M) \overset{\text{def}}{=\!=} \mathrm{Im}[\Phi_{X/M} \cdot T(i^s_{H(X/M)})],$$

therefore,

$$C^{s^*}_X(M)/M = \mathrm{Im}[\Phi_{X/M} \cdot T(i^s_{H(X/M)})]. \tag{4.3}$$

2) For $s \in \mathbb{PR}(S)$ now we consider the transitions: $s \overset{\psi_1^{\mathrm{S}}}{\rightsquigarrow} C^s \overset{(-)^*}{\rightsquigarrow} (C^s)^*$. The operator C^s is obtained by (1.2), i.e. $C^s_Y(N)/N \overset{\text{def}}{=\!=} s(Y/N)$ for every inclusion $N \subseteq Y$ of S-Mod.

Further, for the step $C^s \overset{(-)^*}{\rightsquigarrow} (C^s)^*$ we use the rule (2.4). Namely, for every inclusion $m : M \xrightarrow{\subseteq} X$ of R-Mod we consider the inclusion $i^m_{C^s} : C^s_{H(X)}\big(\operatorname{Im} H(m)\big) \xrightarrow{\subseteq} H(X)$ of S-Mod and the composition $T\big[C^s_{H(X)}\big(\operatorname{Im} H(m)\big)\big] \xrightarrow{T(i^m_{C^s})} TH(X) \xrightarrow{\Phi_X} X$ in R-Mod.

By (2.4) we have

$$(C^s)^*_X(M) \overset{\text{def}}{=\!=} \operatorname{Im}[\Phi_X \cdot T(i^m_{C^s})] + M. \tag{4.4}$$

We mention also that by the definition of C^s for the pair $\operatorname{Im} H(m) \subseteq H(X)$ we have

$$C^s_{H(X)}\big(\operatorname{Im} H(m)\big)/\operatorname{Im} H(m) \overset{\text{def}}{=\!=} s\big(H(X)/\operatorname{Im} H(m)\big).$$

3) It remains to compare the obtained expressions for $C^{s^*}_X(M)$ and $(C^s)^*_X(M)$. To this end we consider in S-Mod the commutative diagram

$$\begin{tikzcd}
H(M) \arrow[r, "H(m)"] \arrow[d, "\overline{H(m)}"] & H(X) \arrow[rrr, "H(\pi_X)"] \arrow[d, equal] & & & H(X/M) \arrow[d, equal] \\
\operatorname{Im} H(m) \arrow[r, "\subseteq"] \arrow[d, "\cap" ', "j^m_{C^s}"] & H(X) \arrow[r, "\pi", "\text{nat}" '] & H(X)/\operatorname{Im} H(m) \arrow[r, "\cong"] & \operatorname{Im} H(\pi_X) \arrow[r, "\subseteq"] & H(X/M) \\
C^s_{H(X)}\big(\operatorname{Im} H(m)\big) \arrow[r, dashed, "\pi'", "\text{nat}" '] \arrow[ur, "\subseteq", "i^m_{C^s}" '] \arrow[rrrr, dashed, bend right, "f" '] & A \arrow[r, equal, "\text{def}"] \arrow[ur, "\subseteq"] & s\big(H(X)/\operatorname{Im} H(m)\big) \arrow[u, "\cup" '] \arrow[rr, dashed, "\varphi'"] & & s\big(H(X/M)\big), \arrow[u, "\cup" ']
\end{tikzcd}$$

where $A = C^s_{H(X)}\big(\operatorname{Im} H(m)\big)/\operatorname{Im} H(m)$, the first line is the image of the short exact sequence $0 \longrightarrow M \xrightarrow[\subseteq]{m} X \xrightarrow[\text{nat}]{\pi_X} X/M \longrightarrow 0$, and π, π' are natural epimorphisms. Denoting $\varphi : H(X)/\operatorname{Im} H(m) \xrightarrow{\cong} \operatorname{Im} H(\pi_X) \xrightarrow{\subseteq} H(X/M)$, we have its restriction φ' (by definition of preradical) and $f = \varphi' \cdot \pi'$.

Applying T to this diagram and using Φ, we obtain in R-Mod the following commutative diagram

$$\begin{tikzcd}
(C^s)^*_X(M) \arrow[r, dashed, "\subseteq"] \arrow[d, "\cap"] & C^{s^*}_X(M) \arrow[r, dashed, "\pi'_X", "\text{nat}"'] & s^*(X/M) = C^{s^*}_X(M)/M \arrow[d, "\cap"] \\
X \arrow[rr, "\pi_X", "\text{nat}"'] & & X/M \\
TH(X) \arrow[u, "\Phi_X"'] \arrow[rr, "TH(\pi_X)"] & & TH(X/M) \arrow[u, "\Phi_{X/M}"'] \\
T[C^s_{H(X)}\big(\operatorname{Im} H(m)\big)] \arrow[u, "T(i^m_{C^s})"'] \arrow[uu, bend left] \arrow[rr, dashed, "T(f)"] & & T[s\big(H(X/M)\big)] \arrow[u, "T\left(i^s_{H(X/M)}\right)"'] \arrow[uu, bend left]
\end{tikzcd}$$

By the mentioned above relations we have $(C^s)^*_X(M) = \operatorname{Im}[\Phi_X \cdot T(i^m_{C^s})] + M$ and $s^*(X/M) \overset{\text{def}}{=\!=} C^{s^*}_X(M)/M = \operatorname{Im}[\Phi_{X/M} \cdot T(i^s_{H(X/M)})]$. The commutativity of this diagram implies that

$$\pi_X \cdot \Phi_X \cdot T(i^m_{C^s}) = \Phi_{X/M} \cdot T(i^s_{H(X/M)}) \cdot T(f).$$

Therefore,

$$\begin{aligned}
\operatorname{Im}[\pi_X \cdot \Phi_X \cdot T(i^m_{C^s})] &= \operatorname{Im}[\Phi_{X/M} \cdot T(i^s_{H(X/M)}) \cdot T(f)] \\
&\subseteq \operatorname{Im}[\Phi_{X/M} \cdot T(i^s_{H(X/M)})] \overset{\text{def}}{=\!=} s^*(X/M) = C^{s^*}_X(M)/M.
\end{aligned}$$

Since $\operatorname{Im}[\pi_X \cdot \Phi_X \cdot T(i^m_{C^s})] = \pi_X\big(\operatorname{Im}[\Phi_X \cdot T(i^m_{C^s})]\big) = (\operatorname{Im}[\Phi_X \cdot T(i^m_{C^s}) + M])/M \overset{\text{def}}{=\!=} [(C^s)^*_X(M)]/M$, from the previous relation it follows that $[(C^s)^*_X(M)]/M \subseteq C^{s^*}_X(M)/M$. Therefore $(C^s)^*_X(M) \subseteq C^{s^*}_X(M)$ for every $M \subseteq X$, which means that $(C^s)^* \leqslant C^{s^*}$, proving the first statement of the theorem.

4) Now we will prove the second statement, assuming that ${}_RU$ is a *projective* module. Since $H = \operatorname{Hom}_R(U, \text{-})$, this means that H is an exact functor, i.e. every epimorphism π of R-Mod is transformed into an epimorphism $H(\pi)$ of S-Mod.

Following the above proof, we observe that since π_X is an epimorphism, in this case $H(\pi_X)$ is an epimorphism, i.e. $\operatorname{Im} H(\pi_X) = H(X/M)$. Then φ is an isomorphism, therefore φ' is an isomorphism. But then $f = \varphi' \cdot \pi'$ is an epimorphism, hence $T(f)$ is an epimorphism (the functor T is right exact). Therefore from the last diagram is clear that

$$\operatorname{Im}[\Phi_{X/M} \cdot T(i^s_{H(X/M)}) \cdot T(f)] = \operatorname{Im}[\Phi_{X/M} \cdot T(i^s_{H(X/M)})].$$

From the proof of the first part now it is obvious that in this case instead of inclusion we obtain the equality $(C^s)^*_X(M) = C^{s^*}_X(M)$, so $(C^s)^* = C^{s^*}$. $\square$

5. Squares containing the third pair of mappings

In this section we consider the last two cases and we study the combinations of the mappings which contain ψ_2^R and ψ_2^S.

a) We will examine the following square:

$$\begin{tikzcd} \mathbb{PR}(R) \arrow[r, "(-)^*"] \arrow[d, "\psi_2^R"] & \mathbb{PR}(S) \arrow[d, "\psi_2^S"] \\ \mathbb{CO}(R) \arrow[r, "(-)^*"] & \mathbb{CO}(S) \end{tikzcd} ,$$

verifying the compatibility of his mappings.

Theorem 5.1. *For every preradical $r \in \mathbb{PR}(R)$ the relation*

$$C_{r^*} \leqslant C_r^*$$

is true.

Proof. 1) Let $r \in \mathbb{PR}(R)$ and consider the way: $r \overset{(-)^*}{\rightsquigarrow} r^* \overset{\psi_2^S}{\rightsquigarrow} C_{r^*}$. For $Y \in S$-Mod applying T and r we obtain in R-Mod the sequence

$$0 \longrightarrow r\big(T(Y)\big) \xrightarrow{\subseteq} T(Y) \xrightarrow[\text{nat}]{\pi^r_{T(Y)}} T(Y)/r\big(T(Y)\big) \longrightarrow 0.$$

Using H and Ψ, we have in S-Mod the composition

$$Y \xrightarrow{\Psi_Y} HT(Y) \xrightarrow{H(\pi^r_{T(Y)})} H[T(Y)/r\big(T(Y)\big)],$$

and by the rule (2.1) we obtain: $r^*(Y) \overset{\text{def}}{=\!=} \operatorname{Ker}[H(\pi^r_{T(Y)}) \cdot \Psi_Y]$.

The following step $r^* \overset{\psi_2^S}{\rightsquigarrow} C_{r^*}$ is defined by (1.3), i.e. for every inclusion $n : N \xrightarrow{\subseteq} Y$ of S-Mod we have

$$(C_{r^*})_Y(N) \overset{\text{def}}{=\!=} r^*(Y) + N = \operatorname{Ker}[H(\pi^r_{T(Y)}) \cdot \Psi_Y] + N. \qquad (5.1)$$

2) For $r \in \mathbb{PR}(R)$ now we follow the route: $r \overset{\psi_2^R}{\rightsquigarrow} C_r \overset{(-)^*}{\rightsquigarrow} C_r^*$. The operator C_r is defined by the rule (1.3): $(C_r)_X(M) \overset{\text{def}}{=\!=} r(X) + M$ for every

inclusion $M \subseteq X$ of R-Mod. Further, the transition $C_r \overset{(-)^*}{\rightsquigarrow} C_r^*$ is defined by (2.3). Namely, for an inclusion $n : N \xrightarrow{\subseteq} Y$ of S-Mod, applying T and C_r, we obtain in R-Mod the situation

$$0 \to (C_r)_{T(Y)}(\operatorname{Im} T(n)) \xrightarrow[\subseteq]{i^n_{C_r}} T(Y) \xrightarrow[\text{nat}]{\pi^n_{C_r}} T(Y)/(C_r)_{T(Y)}(\operatorname{Im} T(n)) \to 0,$$

and by (1.3) we have $(C_r)_{T(Y)}\big(\operatorname{Im} T(n)\big) \overset{\text{def}}{=\!=} r\big(T(Y)\big) + \operatorname{Im} T(n)$.

Returning back in S-Mod by H, we obtain the composition

$$Y \xrightarrow{\Psi_Y} HT(Y) \xrightarrow{H(\pi^n_{C_r})} H[T(Y)/(C_r)_{T(Y)}\big(\operatorname{Im} T(n)\big)].$$

By the rule (2.3) we have

$$(C_r^*)_Y(N) \overset{\text{def}}{=\!=} \operatorname{Ker}[H(\pi^n_{C_r}) \cdot \Psi_Y]. \tag{5.2}$$

3) To compare the modules $(C_{r^*})_Y(N)$ and $(C_r^*)_Y(N)$, indicated in (5.1) and (5.2), we consider in R-Mod the situation

$$\begin{tikzcd}
{} & r\big(T(Y)\big) \arrow[dr, "i", "\subseteq"'] \arrow[dd, dashed, "j", "\cap\vert"'] & {} & {} & T(Y)/r\big(T(Y)\big) \arrow[dd, dashed, "\pi"] \\
T(N) \arrow[rr, "T(n)"] & {} & T(Y) \arrow[urr, "\pi^r_{T(Y)}", "\text{nat}"'] \arrow[drr, "\pi^n_{C_r}", "\text{nat}"'] & {} & {} \\
{} & r\big(T(Y)\big) + \operatorname{Im} T(n) \arrow[ur, "i^n_{C_r}", "\subseteq"'] & {} & {} & T(Y)/\big(r(T(Y)) + \operatorname{Im} T(n)\big),
\end{tikzcd}$$

where $r(T(Y)) + \operatorname{Im} T(n) = (C_r)_{T(Y)}\big(\operatorname{Im} T(n)\big)$. The inclusion $j : r\big(T(Y)\big) \xrightarrow{\subseteq} r\big(T(Y)\big) + \operatorname{Im} T(n)$ implies the epimorphism π such that the above diagram is commutative. Applying H we obtain in R-Mod the commutative diagram

$$\begin{tikzcd}
H\big(r\big(T(Y)\big)\big) \arrow[drr, "H(i)"] \arrow[dd, dashed, "H(j)"] & {} & {} & H[T(Y)/r\big(T(Y)\big)] \arrow[dd, dashed, "H(\pi)"] \\
{} & Y \arrow[r, "\Psi_Y"] & H(T(Y)) \arrow[ur, "H(\pi^r_{T(Y)})"] \arrow[dr, "H(\pi^n_{C_r})"] & {} \\
H[r\big(T(Y)\big) + \operatorname{Im} T(n)] \arrow[urr, "H(i^n_{C_r})"] & {} & {} & H[T(Y)/\big(r(T(Y)) + \operatorname{Im} T(n)\big)].
\end{tikzcd}$$

The commutativity of the right triangle implies $\operatorname{Ker}[H(\pi) \cdot H(\pi^r_{T(Y)})] = \operatorname{Ker} H(\pi^n_{C_r})$. Therefore $\operatorname{Ker} H(\pi^r_{T(Y)}) \subseteq \operatorname{Ker} H(\pi^n_{C_r})$, so the relation

$\mathrm{Ker}\,[H(\pi^r_{T(Y)}) \cdot \Psi_Y] \subseteq \mathrm{Ker}\,[H(\pi^n_{C_r}) \cdot \Psi_Y] \overset{\text{def}}{=\!=} (C^*_r)_Y(N)$ is true. Since $N \subseteq (C^*_r)_Y(N)$, it is obvious that $\mathrm{Ker}\,[H(\pi^r_{T(Y)}) \cdot \Psi_Y + N] \subseteq (C^*_r)_Y(N)$. Now by (5.1) we have $(C_{r^*})_Y(N) \subseteq (C^*_r)_Y(N)$ for every $N \subseteq Y$, i.e. $C_{r^*} \leqslant C^*_r$. □

b) The last possible square consisting of the studied mappings is the following:

$$\begin{tikzcd}
\mathbb{PR}(R) \arrow[d, "\psi_2^R"] & & \mathbb{PR}(S) \arrow[ll, "(-)^*"'] \arrow[d, "\psi_2^S"] \\
\mathbb{CO}(R) & & \mathbb{CO}(S) . \arrow[ll, "(-)^*"']
\end{tikzcd}$$

As usual, we examine the relation between these mappings.

Theorem 5.2. *For every preradical $s \in \mathbb{PR}(S)$ the relation*

$$C_{s^*} \leqslant C^*_s$$

is true.

Proof. 1) Let $s \in \mathbb{PR}(S)$ and consider the way: $s \overset{\psi_2^S}{\rightsquigarrow} C_s \overset{(-)^*}{\rightsquigarrow} C^*_s$. By the rule (1.3) we have the operator C_s such that $(C_s)_Y(N) \overset{\text{def}}{=\!=} s(Y) + N$ for every inclusion $N \subseteq Y$ of S-Mod.

Further, the transition $C_s \overset{(-)^*}{\rightsquigarrow} C^*_s$ from $\mathbb{CO}(S)$ to $\mathbb{CO}(R)$ is defined by the rule (2.4). This means that for every inclusion $m : M \xrightarrow{\subseteq} X$ of R-Mod we consider the decomposition of $H(m)$ by the operator C_s,

$$\begin{tikzcd}
H(M) \arrow[r, "\overline{H(m)}"] \arrow[rrr, bend left, "H(m)"] & \mathrm{Im}\, H(m) \arrow[r, "j^m_{C_s}", "\subseteq"'] & (C_s)_{H(X)}\big(\mathrm{Im}\, H(m)\big) \arrow[r, "i^m_{C_s}", "\subseteq"'] & H(X),
\end{tikzcd}$$

where $(C_s)_{H(X)}\big(\mathrm{Im}\, H(m)\big) \overset{\text{def}}{=\!=} s\big(H(X)\big) + \mathrm{Im}\, H(m)$. Applying T, we obtain in R-Mod the composition of morphisms

$$T[s\big(H(X)\big) + \mathrm{Im}\, H(m)] = T[(C_s)_{H(X)}(\mathrm{Im}\, H(m))] \xrightarrow{T(i^m_{C_s})} TH(X) \xrightarrow{\Phi_X} X.$$

By the definition (2.4) we have

$$(C^*_s)_X(M) \overset{\text{def}}{=\!=} \mathrm{Im}[\Phi_X \cdot T(i^m_{C_s})] + M. \tag{5.3}$$

2) For the same preradical $s \in \mathbb{PR}(S)$ now we consider the transitions $s \overset{(-)^*}{\rightsquigarrow} s^* \overset{\psi_2^{\mathrm{R}}}{\rightsquigarrow} C_{s^*}$. To obtain s^*, let $X \in R$-Mod for which we have the inclusion $i^s_{H(X)} \,:\, s\big(H(X)\big) \xrightarrow{\subseteq} H(X)$. Applying T and using Φ, we obtain in R-Mod the composition of morphisms

$$T[s\big(H(X)\big)] \xrightarrow{T(i^s_{H(X)})} TH(X) \xrightarrow{\Phi_X} X.$$

The preradical s^* is defined by the rule (2.2), i.e.,

$$s^*(X) \overset{\text{def}}{=\!=} \mathrm{Im}[\Phi_X \cdot T(i^s_{H(X)})].$$

For the following transition $s^* \overset{\psi_2^{\mathrm{R}}}{\rightsquigarrow} C_{s^*}$ the rule (1.3) is used: $(C_{s^*})_X(M) \overset{\text{def}}{=\!=} s^*(X) + M$ for every pair $M \subseteq X$ of R-Mod. Taking into account the form of $s^*(X)$ indicated above, now we have

$$(C_{s^*})_X(M) = \mathrm{Im}[\Phi_X \cdot T(i^s_{H(X)})] + M. \tag{5.4}$$

3) Finally, we compare the constructions of parts 1) and 2). In S-Mod we have the inclusions

$$\begin{tikzcd}
s\big(H(X)\big) \arrow[d, "i", "\cap"'] \arrow[dr, "i^s_{H(X)}", "\subseteq"'] & \\
s\big(H(X)\big) + \mathrm{Im}\, H(m) \arrow[r, "i^m_{C_s}", "\subseteq"'] & H(X),
\end{tikzcd}$$

which implies in R-Mod the situation

$$\begin{tikzcd}
T[s\big(H(X)\big)] \arrow[d, "T(i)"] \arrow[dr, "T(i^s_{H(X)})"] & & & (C_{s^*})_X(M) \overset{\text{def}}{=\!=} \mathrm{Im}[\Phi_X \cdot T(i^s_{H(X)})] + M \arrow[dl, "\supseteq"'] \arrow[dd, dashed, "\cap"] \\
& TH(X) \arrow[r, "\Phi_X"] & X & \\
T[s\big(H(X)\big) + \mathrm{Im}\, H(m)] \arrow[ur, "T(i^m_{C_s})"] & & & (C^*_s)_X(M) \overset{\text{def}}{=\!=} \mathrm{Im}[\Phi_X \cdot T(i^m_{C_s})] + M. \arrow[ul, "\supseteq"]
\end{tikzcd}$$

The commutativity of the left triangle implies $\mathrm{Im}\, T(i^s_{H(X)}) \subseteq \mathrm{Im}\, T(i^m_{C_s})$, therefore $\mathrm{Im}[\Phi_X \cdot T(i^s_{H(X)})] \subseteq \mathrm{Im}[\Phi_X \cdot T(i^m_{C_s})]$. By (5.3) and (5.4) this means that $(C_{s^*})_X(M) \subseteq (C^*_s)_X(M)$ for every $M \subseteq X$, i.e. $C_{s^*} \leqslant C^*_s$. $\square$

In conclusion we can affirm that the indicated ten mappings, which realize the connection between preradicals and closure operators in adjoint situation, are well concordant between them. For the six combinations, which constitute the squares of mappings, in three cases the commutativity of respective diagrams is proved (Theorems 3.1, 3.2, 4.1), and in other three cases the inclusion relations are shown (Theorems 4.2, 5.1, 5.2).

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Contact information

A. I. Kashu Institute of Mathematics and Computer, Science "Vladimir Andrunachievici", Academiei str., 5, MD–2028, Kishinev, Moldova
E-Mail(s): alexei.kashu@math.md

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