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Lie algebras of derivations with large abelian ideals

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Abstract. Let $\mathbb{K}$ be a field of characteristic zero, $A = \mathbb{K}[x_1, \ldots, x_n]$ the polynomial ring and $R = \mathbb{K}(x_1, \ldots, x_n)$ the field of rational functions. The Lie algebra $\widetilde{W}_n(\mathbb{K}) := \mathrm{Der}_{\mathbb{K}} R$ of all $\mathbb{K}$-derivation on R is a vector space (of dimension n) over R and every its subalgebra L has rank $\mathrm{rk}_R L = \dim_R RL$. We study subalgebras L of rank m over R of the Lie algebra $\widetilde{W}_n(\mathbb{K})$ with an abelian ideal $I \subset L$ of the same rank m over R. Let F be the field of constants of L in R. It is proved that there exist a basis $D_1, \ldots, D_m$ of FI over F, elements $a_1, \ldots, a_k \in R$ such that $D_i(a_j) = \delta_{ij}$, $i = 1, \ldots, m$, $j = 1, \ldots, k$, and every element $D \in FL$ is of the form $D = \sum_{i=1}^{m} f_i(a_1, \ldots, a_k)D_i$ for some $f_i \in F[t_1, \ldots t_k]$, $\deg f_i \leqslant 1$. As a consequence it is proved that L is isomorphic to a subalgebra (of a very special type) of the general affine Lie algebra $\mathrm{aff}_m(F)$.

Introduction

Let $\mathbb{K}$ be a field of characteristic zero, $A = \mathbb{K}[x_1, \ldots, x_n]$ the polynomial ring and $R = \mathbb{K}(x_1, \ldots, x_n)$ the field of rational functions in n variables. The Lie algebra $\widetilde{W}_n(\mathbb{K}) := \mathrm{Der}_{\mathbb{K}} R$ of all $\mathbb{K}$-derivation on R is of great interest because in case $\mathbb{K} = \mathbb{R}$, the field of real numbers, elements of $\widetilde{W}_n(\mathbb{K})$ (which are of the form

$$D = f_1 \frac{\partial}{\partial x_1} + \cdots + f_n \frac{\partial}{\partial x_n}, f_i \in R)$$

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can be considered as vector fields on the manifold $\mathbb{K}^n$ with rational coefficients $f_1, \ldots, f_n \in R$. Note that in case $\mathbb{K} = C$ or $\mathbb{K} = R$ the Lie algebras $\widetilde{W}_1(\mathbb{K})$ and $\widetilde{W}_2(\mathbb{K})$ were studied by S. Lie [7], A. González-López, N. Kamran and P. J. Olver [2] and others from the viewpoint of structure of finite-dimensional subalgebras.

Since $\widetilde{W}_n(\mathbb{K})$ is a vector space of dimension n over R one can define the rank $\mathrm{rk}_R L$ over R for any subalgebra $L \subseteq \widetilde{W}_n(\mathbb{K})$ by the rule: $\mathrm{rk}_R L := \dim_R RL$. We study subalgebras $L \subseteq \widetilde{W}_n(\mathbb{K})$ of rank m over R which have an abelian ideal I of the same rank m over R. A natural basis over F, the field of constants for L in R, for such Lie algebras is built. Note that analogous results in cases $n = 2$ and $n = 3$ were obtained in [3] and in [1], in case $m = n$ such a basis can be built using results of [6]. As a corollary one can prove that the Lie algebra FL over the field F can be isomorphically embedded into the general affine Lie algebra $\mathrm{aff}_m(F)$. This result can be used to study solvable finite dimensional subalgebras $L \subseteq \widetilde{W}_n(\mathbb{K})$ because such Lie algebras (over an algebraically closed field of characteristic zero) have a series of ideals

$$0 \subset L_1 \subset L_2 \subset \ldots \subset L_m = L \quad \text{with } \mathrm{rk}_R L_s = s, \ s = 1, \ldots, m.$$

We use standard notation. The ground field $\mathbb{K}$ is arbitrary of characteristic zero. Recall that the general affine Lie algebra $\mathrm{aff}_m(\mathbb{K})$ is the semidirect product $\mathrm{aff}_m(\mathbb{K}) = \mathrm{gl}_m(\mathbb{K}) \leftthreetimes V_m$, where V_m is a vector space over $\mathbb{K}$ of dimension m with a zero multiplication and the general linear Lie algebra $\mathrm{gl}_m(\mathbb{K})$ acts on V_m in the natural way. If $L \subseteq \widetilde{W}_n(\mathbb{K})$ is a subalgebra, then the field of constants for L in R is the subfield of the field R of the form $F(L) = \{r \in R \mid D(r) = 0 \text{ for all } D \in L\}$.

1. Preliminary results

The next two lemmas contain some technical results about derivations (see for example, [5] or [3]).

Lemma 1. *Let $D_1, D_2 \in \widetilde{W}_n(\mathbb{K})$ and $a, b \in R$. Then:*
1) $[aD_1, bD_2] = ab[D_1, D_2] + aD_1(b)D_2 - bD_2(a)D_1$,
2) *if* $[D_1, D_2] = 0$, *then* $[aD_1, bD_2] = aD_1(b)D_2 - bD_2(a)D_1$,
3) *if* $a, b \in \mathrm{Ker}\, D_1 n \,\mathrm{Ker}\, D_2$, *then* $[aD_1, bD_2] = ab[D_1, D_2]$.

Let L be a subalgebra of $\widetilde{W}_n(\mathbb{K})$ and $F = F(L)$ its field of constants. Then the set FL of all linear combinations of elements aD, where $a \in F$, $D \in L$ is a Lie algebra over the field F.

Lemma 2. *If L is an abelian, nilpotent or solvable subalgebra of $\widetilde{W}_n(\mathbb{K})$, then so is FL respectively.*

Lemma 3. *Let L be a subalgebra of rank $m \geqslant 1$ over R of the Lie algebra $\widetilde{W}_n(\mathbb{K})$ and let L contain a proper abelian ideal I of the same rank m over R. If an inner derivation $\operatorname{ad} T$ for some $T \in L$ is of rank k on the F-space FI (as a linear operator), then there exist a basis $T_1, \ldots, T_m$ of FI over F and elements $a_1, \ldots, a_k \in R$ such that $T_i(a_j) = \delta_{ij}$, $i = 1, \ldots, m$, $j = 1, \ldots, k$. Besides, T can be written in the form*

$$T = f_1(a_1, \ldots, a_k)T_1 + \ldots + f_m(a_1, \ldots, a_k)T_m,$$

for some $f_i \in F[t_1, \ldots, t_k]$, $\deg f_i \leqslant 1$, $i = 1, \ldots, m$.

Proof. Choose any basis $D_1, \ldots, D_m$ of the vector space FI over F. Since by [3] (Lemma 3) $\operatorname{rk}_R I = \dim_F FI$ it holds $T = a_1D_1 + \ldots + a_mD_m$ for some elements $a_i \in R$. Without loss of generality one can assume that $[D_1, T], \ldots, [D_k, T]$ form a basis of the vector space $T(FI) = [T, FI]$ (recall that the linear operator $\operatorname{ad} T$ is of rank k on FI by the conditions of the lemma). Any element $[D_s, T]$, $k+1 \leqslant s \leqslant m$, is a linear combination of $[D_1, T], \ldots, [D_k, T]$ over F, so we can choose D_s in such a way that $[D_s, T] = 0$. The latter means that in this basis the matrix $B = (D_i(a_j))$ is of the form

$$B = \begin{pmatrix} D_1(a_1) & \ldots & D_1(a_m) \\ \cdot & & \\ \cdot & & \\ D_k(a_1) & \ldots & D_k(a_m) \\ 0 & \ldots & 0 \\ \cdot & \ldots & \cdot \\ 0 & \ldots & 0 \end{pmatrix}$$

and the first k rows $R_1, \ldots, R_k$ of B are linearly independent over the field F. Since the matrix B is of rank k over F we can choose k columns $C_{i_1}, \ldots, C_{i_k}$ of B which are linearly independent over F. It is easy to see that there exists a linear combination $\gamma_{11}R_1 + \cdots + \gamma_{k1}R_k$ of the first k rows $R_1, \ldots, R_k$ of the matrix B such that

$$\gamma_{11}R_1 + \cdots + \gamma_{k1}R_k = (*, \ldots \underbrace{1}_{i_1}, *, \ldots \underbrace{0}_{i_2}, * \ldots, \underbrace{0}_{i_k}, \ldots *),$$

where the right side is the row with 1 on i_1st place, 0 on the i_2nd place, …, 0 on the i_kth place. Denote $D_1' = \gamma_{11}D_1 + \cdots + \gamma_{k1}D_k$. Then

$$[D_1', T] = r_1D_1 + \cdots + 1 \cdot D_{i_1} + \cdots + 0 \cdot D_{i_2} + \cdots + 0 \cdot D_{i_k} + \cdots + r_mD_m$$

for some $r_i \in R$, $i \notin \{i_1, \dots, i_k\}$. The latter means that

$$D'_1(a_{i_1}) = 1, \quad D'_1(a_{i_2}) = 0, \quad \dots, \quad D'_1(a_{i_k}) = 0.$$

Analogously one can build $D'_2, \dots, D'_k$ with properties $D'_j(a_{i_s}) = \delta_{js}$, $s = 1, \dots, k$. So we now have a basis $D'_1, \dots, D'_k, D_{k+1}, \dots, D_m$ of the vector space FI over F. Denote for convenience $T_1 = D'_1, \dots T_k = D'_k, T_{k+1} = D_{k+1}, \dots, T_m = D_m$. Then we have $T_j(a_{i_s}) = \delta_{js}$, $j, s = 1, \dots, k$. Besides, by the choice of the initial basis of the vector space FI it holds $T_{k+1}(a_{i_s}) = 0, \dots, T_m(a_{i_s}) = 0$, $s = 1, \dots, k$.

Further any column $C_j, j \notin \{i_1, \dots, i_s\}$ is a linear combination of the columns $C_{i_1}, \dots, C_{i_k}$ of the matrix B, so we can write down $C_j = \beta_{1j} C_{i_1} + \dots + \beta_{kj} C_{i_k}$ for some $\beta_{ij} \in F$. Then

$$D_t(a_j - \sum_{s=1}^{k} \beta_{sj} a_{i_s}) = 0, \qquad t = 1, \dots, m$$

and therefore $a_j = \sum_{s=1}^{k} \beta_{sj} a_{i_s} + \delta_j, \delta_j \in F$. The latter means, that all the coefficients a_j are of the form

$$a_j = f_j(a_{i_1}, \dots, a_{i_k}), \qquad f_j \in F[t_1, \dots, t_k], \quad \deg f_j \leqslant 1.$$

After renumbering the elements $a_{i_1}, \dots, a_{i_k}$ we get the proof of the last part of the lemma. The proof is complete. □

Lemma 4. *Let L be a subalgebra of rank m over R of the Lie algebra $\widetilde{W}_n(\mathbb{K})$ with an abelian ideal I of the same rank m over R and $D \in FL$. If there exist a basis $D_1, \dots, D_m$ of FI over F and elements $a_1, \dots, a_k \in R$ with $D_i(a_j) = \delta_{ij}$, $i = 1, \dots, m$, $j = 1, \dots, k$, then there exists an element $\overline{D} \in \widetilde{W}_n(\mathbb{K})$ such that $[D - \overline{D}, D_i] = 0$, $i = 1, \dots, k$.*

Proof. Since $D_1, \dots, D_m$ is a basis of L over R (see Lemma 3 in [3]) the element D can be written in the form

$$D = s_1 D_1 + \dots + s_m D_m \quad \text{for some } s_i \in R.$$

Then

$$[D_i, D] = D_i(s_1) D_1 + \dots + D_i(s_m) D_m$$

and therefore $D_i(s_j) \in F$ because $[D_i, D] \in FI$.

Denote $\alpha_{ij} = D_i(s_j)$, $i, j = 1, \dots, m$ and consider elements $f_j = \sum_{s=1}^{k} \alpha_{ij} a_i$, $j = 1, \dots, m$. Then $D_i(f_j) = \alpha_{ij}, i, j = 1, \dots, m$ and therefore $D_i(s_j - f_j) = 0$, $i = 1, \dots, k$, $j = 1, \dots, m$. The latter means that $[D_i, D - \overline{D}] = 0$, $i = 1, \dots, k$. □

2. The main result

Theorem 1. *Let L be a subalgebra of rank m over R of the Lie algebra $\widetilde{W}_n(\mathbb{K})$ with a proper abelian ideal $I \subset L$ of the same rank m over R and F be the field of constants for L. Then there exist a basis $D_1, \ldots, D_m$ of the ideal FI over F and elements $a_1, \ldots, a_k \in R, k \geqslant 1$ such that $D_i(a_j) = \delta_{ij}$, $i = 1, \ldots, m$, $j = 1, \ldots, k$. Every element $D \in FL$ can be written in the form $D = \sum_{i=1}^{m} f_i(a_1, \ldots, a_k)D_i$ for some linear polynomials $f_i \in F[t_1, \ldots, t_k]$.*

Proof. Take any element $D \in L \setminus I$. Then the inner derivation $\operatorname{ad} D$ on FL is nonzero and by Lemma 3 there exist a basis $D_1, \ldots, D_m$ of the vector space FI over F and elements $a_1, \ldots, a_{k_1} \in R$ such that $D_i(a_j) = \delta_{ij}$, $i = 1, \ldots, m$, $j = 1, \ldots, k_1$ (here k_1 is the rank of the linear operator $\operatorname{ad} D$ on FI). By the same Lemma 3 the element D can be written in the form

$$D = f_1(a_1, \ldots, a_{k_1})D_1 + \cdots + f_m(a_1, \ldots, a_{k_1})D_m \tag{1}$$

for some linear polynomials $f_i \in F[t_1, \ldots, t_{k_1}]$. If every element of the Lie algebra FL can be expressed in such a form, then we put $k = k_1$ and the proof is complete. Let $T \in FL$ be any element that is not of form (1). Then by Lemma 4 there exists an element

$$\overline{T} = \sum_{i=1}^{m} g_i D_i,$$

where $g_i = g_i(a_1, \ldots, a_{k_1})$, $g_i \in F[t_1, \ldots, t_{k_1}]$ and $\deg g_i \leqslant 1$, such that

$$[D_i, T - \overline{T}] = 0, \qquad i = 1, \ldots, k_1. \tag{2}$$

Without loss of generality one can assume that $[D_i, T] = 0$, $i = 1, \ldots, k_1$. The element T can be written in the form

$$T = s_1D_1 + \cdots + s_mD_m, \qquad s_i \in R, \quad i = 1, \ldots, m.$$

Then the matrix $B = (D_i(s_j))$ is of the form

$$B = \begin{pmatrix} 0 & \ldots & 0 \\ . & \ldots & . \\ 0 & \ldots & 0 \\ D_{k_1+1}(s_1) & \ldots & D_{k_1+1}(s_m) \\ . & & \\ . & & \\ D_m(s_1) & \ldots & D_m(s_m) \end{pmatrix}$$

because $[D_i, T] = \sum_{j=1}^m D_i(s_j)D_j = 0$, $i = 1, \ldots, k_1$, and therefore

$$D_i(s_j) = 0, \qquad i = 1, \ldots, k_1, \quad j = 1, \ldots, m.$$

The matrix B is nonzero because the derivation $\operatorname{ad} T$ is a nonzero linear operator on the vector F-space FI. Using Lemma 3 one can find $D'_{k_1+1}, \ldots, D'_m \in FI$ and $a_{k_1+1}, \ldots, a_{k_1+k_2} \in R$ such that

$$D'_i(a_j) = \delta_{ij}, \qquad j = k_1 + 1, \ldots, k_1 + k_2, \quad i = 1, \ldots, m.$$

One can easy to see that $D'_{k_1+1}, \ldots, D'_{k_1+k_2}$ are linear combinations of the derivations $D_{k_1}, \ldots, D_m$. Returning to the old notation we can write $D_{k_1+1} = D'_{k_1+1}, \ldots, D_m = D'_m$. Then $D_i(a_j) = \delta_{ij}$, $i = 1, \ldots, m$, $j = 1, \ldots, k_1 + k_2$. By Lemma 3 the element T can be written in the form

$$T = \sum_{i=1}^m f_i(a_1, \ldots, a_{k_1+k_2})D_i, \tag{3}$$

where $f_i \in F[t_1, \ldots, t_{k_1+k_2}]$, $\deg f_i \leqslant 1$. If every element of FL is of the form (3), then all is done. If not, then we can repeat the above considerations and build elements

$$D_{k_1+k_2+1}, \ldots, D_{k_1+k_2+k_3} \in FL, \qquad a_{k_1+k_2+1}, \ldots, a_{k_1+k_2+k_3} \in R$$

with properties $D_i(a_j) = \delta_{ij}$, $i = 1, \ldots, m$, $j = 1, \ldots, k_1 + k_2 + k_3$. This process eventually stops and we get the needed basis $D_1, \ldots, D_m$ of the ideal FL, some elements $a_1, \ldots, a_k \in R$ with the property $D_i(a_j) = \delta_{ij}$ and possibility to write any element of FL in the form

$$D = \sum_{i=1}^m f_i(a_1, \ldots, a_k)D_i,$$

where $f_i \in F[t_1, \ldots, t_k]$, $\deg f_i \leqslant 1$. The proof is complete. □

Corollary 1. *Let L be a subalgebra of rank m over R of the Lie algebra $\widetilde{W}_n(\mathbb{K})$. If L contains an abelian ideal I of the same rank m over R, then FL is isomorphic to a subalgebra of the general affine Lie algebra* $\operatorname{aff}_m(F)$.

Proof. By Theorem 1 every element $D \in FL$ can be written in the form

$$D = f_1(a_1, \ldots, a_k)D_1 + \cdots + f_m(a_1, \ldots, a_k)D_m, \qquad f_i \in F[t_1, \ldots, t_k],$$

with $\deg f_i \leqslant 1$, $D_i(a_j) = \delta_{ij}$, $i = 1, \ldots, m$, $j = 1, \ldots, k$. The linear polynomial f_i can be written in the form $f_i = \overline{f}_i + c_i$, where $c_i \in F$, $\overline{f}_i$ is a homogeneous polynomial of degree 1, i.e. a linear form $\overline{f}_i = \sum_{j=1}^{k} a_{ij} x_j$. One can establish a correspondence φ between the Lie algebra FL and a subalgebra of the Lie algebra $\mathrm{gl_m}(F)$ by the rule: if $D \in FL$ is of the form

$$D = \sum_{i=1}^{m} f_i D_i = \sum_{i=1}^{m} (\sum_{j=1}^{k} a_{ij} x_j + c_i) D_i,$$

then $\varphi(D) = A + \overline{c}$, where

$$A = (a_{ij}) \in \mathrm{gl_m}(F) \quad \text{and} \quad \overline{c} = (c_1, \ldots, c_m) \in V_m.$$

One can easily verify that this correspondence is an injective homomorphism from the Lie algebra FL into the general affine Lie algebra $\mathrm{aff}_m(F)$. Therefore FL is isomorphic to a subalgebra of the general affine Lie algebra $\mathrm{aff}_m(F)$. □

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