Algebra and Discrete Mathematics
Volume **28** (2019). Number 1, pp. 107–122 RESEARCH ARTICLE

On the existence of degree-magic labellings of the n-fold self-union of complete bipartite graphs

Phaisatcha Inpoonjai and Thiradet Jiarasuksakun

Communicated by D. Simson

Abstract. Magic rectangles are a classical generalization of the well-known magic squares, and they are related to graphs. A graph G is called degree-magic if there exists a labelling of the edges by integers $1, 2, \ldots, |E(G)|$ such that the sum of the labels of the edges incident with any vertex v is equal to $(1 + |E(G)|)\deg(v)/2$. Degree-magic graphs extend supermagic regular graphs. In this paper, we present a general proof of the necessary and sufficient conditions for the existence of degree-magic labellings of the n-fold self-union of complete bipartite graphs. We apply this existence to construct supermagic regular graphs and to identify the sufficient condition for even n-tuple magic rectangles to exist.

1. Introduction

We consider simple graphs without isolated vertices. If G is a graph, then $V(G)$ and $E(G)$ stand for the vertex set and the edge set of G, respectively. Cardinalities of these sets are called the *order* and *size* of G.

Let a graph G and a mapping f from $E(G)$ into positive integers be given. The *index mapping* of f is the mapping f^* from $V(G)$ into positive

The authors would like to thank the anonymous referee for careful reading and the helpful comments improving this paper.

2010 MSC: Primary 05C78; Secondary 05B15.

Key words and phrases: regular graphs, bipartite graphs, tripartite graphs, supermagic graphs, degree-magic graphs, balanced degree-magic graphs, magic rectangles.

integers defined by

$$f^*(v) = \sum_{e \in E(G)} \eta(v,e) f(e) \quad \text{for every} \quad v \in V(G),$$

where $\eta(v,e)$ is equal to 1 when e is an edge incident with vertex v, and 0 otherwise. An injective mapping f from $E(G)$ into positive integers is called a *magic labelling* of G for an *index* λ if its index mapping f^* satisfies

$$f^*(v) = \lambda \quad \text{for all} \quad v \in V(G).$$

A magic labelling f of a graph G is called a *supermagic labelling* if the set $\{f(e) : e \in E(G)\}$ consists of consecutive positive integers. We say that a graph G is *supermagic* (*magic*) whenever a supermagic (magic) labelling of G exists.

A bijective mapping f from $E(G)$ into $\{1, 2, \ldots, |E(G)|\}$ is called a *degree-magic labelling* (or *d-magic labelling*) of a graph G if its index mapping f^* satisfies

$$f^*(v) = \frac{1 + |E(G)|}{2} \deg(v) \quad \text{for all} \quad v \in V(G).$$

A d-magic labelling f of a graph G is called *balanced* if for all $v \in V(G)$, the following equation is satisfied

$$\begin{aligned} |\{e \in E(G) &: \eta(v,e) = 1, f(e) \leqslant \lfloor |E(G)|/2 \rfloor\}| \\ &= |\{e \in E(G) : \eta(v,e) = 1, f(e) > \lfloor |E(G)|/2 \rfloor\}|. \end{aligned}$$

We say that a graph G is *degree-magic* (*balanced degree-magic*) or only *d-magic* when a d-magic (balanced d-magic) labelling of G exists.

The concept of magic graphs was introduced by Sedláček [1]. Later, supermagic graphs were introduced by Stewart [2]. There are now many papers published on magic and supermagic graphs; see Gallian [3] for more comprehensive references. The concept of degree-magic graphs was then introduced by Bezegová and Ivančo [4] as an extension of supermagic regular graphs. They established the basic properties of degree-magic graphs and characterized degree-magic and balanced degree-magic complete bipartite graphs in [4]. They also characterized degree-magic complete tripartite graphs in [5]. Some of these concepts are investigated in [6-8].

Magic rectangles are a natural generalization of the magic squares which have widely intrigued mathematicians and the general public. A magic (p,q)-rectangle R is a $p \times q$ array in which the first pq positive integers are placed such that the sum over each row of R is constant and the sum over each column of R is another (different if $p \neq q$) constant. Harmuth [9, 10] studied magic rectangles over a century ago and proved that

Theorem 1. ([9, 10]) *For $p, q > 1$, there is a magic (p, q)-rectangle R if and only if $p \equiv q \pmod 2$ and $(p, q) \neq (2, 2)$.*

In 1990, Sun [11] studied the existence of magic rectangles. Later, Bier and Rogers [12] studied on balanced magic rectangles, and Bier and Kleinschmidt [13] studied about centrally symmetric and magic rectangles. Then Hagedorn [14] presented a simplified modern proof of the necessary and sufficient conditions for a magic rectangle to exist. The concept of magic rectangles was generalized to n-dimensions and several existence theorems were proven by Hagedorn [15].

We will hereinafter use the following auxiliary results from these studies.

Theorem 2. ([4]) *Let G be a regular graph. Then G is supermagic if and only if it is d-magic.*

Theorem 3. ([4]) *Let G be a d-magic graph of even size. Then every vertex of G has an even degree and every component of G has an even size.*

Theorem 4. ([4]) *Let G be a balanced d-magic graph. Then G has an even number of edges and every vertex has an even degree.*

Theorem 5. ([4]) *Let G be a d-magic graph having a half-factor. Then $2G$ is a balanced d-magic graph.*

Theorem 6. ([4]) *Let H_1 and H_2 be edge-disjoint subgraphs of a graph G which form its decomposition. If H_1 is d-magic and H_2 is balanced d-magic, then G is a d-magic graph. Moreover, if H_1 and H_2 are both balanced d-magic, then G is a balanced d-magic graph.*

Proposition 1. ([4]) *For $p, q > 1$, the complete bipartite graph $K_{p,q}$ is d-magic if and only if $p \equiv q \pmod 2$ and $(p, q) \neq (2, 2)$.*

Theorem 7. ([4]) *The complete bipartite graph $K_{p,q}$ is balanced d-magic if and only if the following statements hold:*

(i) $p \equiv q \equiv 0 \pmod 2$;

(ii) if $p \equiv q \equiv 2 \pmod 4$, then $\min\{p, q\} \geqslant 6$.

2. The n-fold self-union of complete bipartite graphs

For any integer $n \geqslant 1$, the *n-fold self-union* of a graph G, denoted by nG, is the union of n disjoint copies of G. For integers $p, q > 1$, we consider the n-fold self-union $nK_{p,q}$ of complete bipartite graphs. Let $nK_{p,q}$ be a d-magic graph. Since $\deg(v)$ is p or q and $f^*(v) = (npq+1)\deg(v)/2$ for any $v \in V(nK_{p,q})$, we then have

Proposition 2. *Let $nK_{p,q}$ be a d-magic graph. Then the following conditions hold:*

(i) if n is odd, then $p \equiv q \pmod 2$;

(ii) if n is even, then $p \equiv q \equiv 0 \pmod 2$.

Theorem 8. *Let $nK_{p,q}$ be a balanced d-magic graph. Then the following conditions hold:*

(i) p and q are both even;

(ii) if n is odd and $p \equiv q \equiv 2 \pmod 4$, then $\min\{p, q\} \geqslant 6$.

Proof. For any integer $n \geqslant 1$, suppose that $nK_{p,q}$ is balanced d-magic. By Theorem 4, p and q are both even because $nK_{p,q}$ has vertices of degrees p and q.

For any $t \in \{1, 2, \ldots, n\}$, let $K^t_{2,2s}$ be the $t^{\underline{th}}$ copy of a graph $K_{2,2s}$ and let $e^t(v^t)$ be its edge (vertex) corresponding to $e \in E(K_{2,2s})(v \in V(K_{2,2s}))$. Let f be a balanced d-magic labelling of a graph $nK_{2,2s}$ and let $\{u^t, v^t\}$ be a partite set of $K^t_{2,2s}$ with two vertices. Put

$$A^t := \{e^t \in E(K^t_{2,2s}) : \eta(u^t, e^t) = 1, f(e^t) \leqslant 2ns\}$$

and

$$B^t := \{e^t \in E(K^t_{2,2s}) : \eta(w^t, e^t) = 1, f(e^t) \leqslant 2ns\}.$$

Clearly, $A^t \cap B^t = \varnothing$ and $|A^t| = |B^t| = s$ because f is balanced d-magic. We can see that any edge of $K^t_{2,2s}$ is incident to either u^t or w^t and the set of labels of edges incident to a vertex of degree two is $\{r, 4ns - r + 1\}$ for some $r \in \{1, 2, \ldots, 2ns\}$. As a result, we have

$$\begin{aligned}\frac{4ns+1}{2}2s = \frac{|E(nK_{2,2s})|+1}{2}\deg(u^t) &= f^*(u^t) \\ &= \sum_{e^t \in A^t} f(e^t) + \sum_{e^t \in B^t} (4ns - f(e^t) + 1) \\ &= (4ns+1)s + \sum_{e^t \in A^t} f(e^t) - \sum_{e^t \in B^t} f(e^t).\end{aligned}$$

This means that $\sum_{e^t\in A^t} f(e^t) = \sum_{e^t\in B^t} f(e^t)$. Consequently,

$$(2ns+1)ns = \sum_{i=1}^{2ns} i = \sum_{t=1}^{n}(\sum_{e^t\in A^t} f(e^t) + \sum_{e^t\in B^t} f(e^t))$$
$$= \sum_{t=1}^{n}(2\sum_{e^t\in A^t} f(e^t)) = 2\sum_{t=1}^{n}(\sum_{e^t\in A^t} f(e^t)) \equiv 0 \pmod 2.$$

Since n is odd, s is even. This proves that condition (ii) holds. □

Proposition 2 allows the set of d-magic graphs $nK_{p,q}$ to be divided into sets of odd and even d-magic graphs. Inspection quickly shows that for $(p,q) = (2,2)$, a d-magic graph $nK_{2,2}$ does not exist. In the next results, we prove the existence of d-magic graphs $nK_{p,q}$ for other even integers $(p,q) \neq (2,2)$.

Now let us consider a concept of a half-factor of a graph G defined by Bezegová and Ivančo [4]. A spanning subgraph H of a graph G is called a *half-factor* of G whenever $\deg_H(v) = \deg_G(v)/2$ for every vertex $v \in V(G)$. Note that a spanning subgraph of G with the edge set $E(G)\setminus E(H)$ is also a half-factor of G. Similarly, if f is a balanced d-magic labelling of G, then the spanning subgraphs with the edge sets $\{e \in E(G) : f(e) \leqslant \lfloor |E(G)|/2 \rfloor\}$ and $\{e \in E(G) : f(e) > \lfloor |E(G)|/2 \rfloor\}$ are half factors of G.

Theorem 9. *For even integers $p,q > 1$. If the complete bipartite graph $K_{p,q}$ is d-magic, then the following conditions hold:*

(i) $nK_{p,q}$ is a d-magic graph for all odd integers $n \geqslant 3$;

(ii) $nK_{p,q}$ is a balanced d-magic graph for all even integers $n \geqslant 2$.

Proof. For any integer $n \geqslant 2$ and $t \in \{1,2,\dots,n\}$, let $K^t_{p,q}$ be the $t^{\underline{th}}$ copy of a graph $K_{p,q}$ and let $e^t(v^t)$ be its edge (vertex) corresponding to $e \in E(K_{p,q})(v \in V(K_{p,q}))$. Since $K_{p,q}$ is d-magic, there is a d-magic labelling g of $K_{p,q}$. Since p and q are both even, there is a half-factor H of $K_{p,q}$. First suppose that $nK_{p,q} = K^1_{p,q} \cup K^2_{p,q} \cup \cdots \cup K^n_{p,q}$ and we consider a mapping f from $E(nK_{p,q})$ into positive integers given by

$$f(e^t) = \begin{cases} g(e) + (t-1)pq & \text{if} \quad e \in E(H), \\ g(e) + (n-t)pq & \text{if} \quad e \in E(K_{p,q}) \setminus E(H). \end{cases}$$

We can then check that f is a bijection from $E(nK_{p,q})$ onto $\{1, 2, \ldots, npq\}$. For any vertex $v^t \in V(K^t_{p,q})$, we have

$$\begin{aligned}
f^*(v^t) = &\sum_{e^t \in E(K^t_{p,q})} \eta(v^t, e^t) f(e^t) = \sum_{e \in E(H)} \eta(v, e)(g(e) + (t-1)pq) \\
&+ \sum_{e \in E(K_{p,q}) \setminus E(H)} \eta(v, e)(g(e) + (n-t)pq) \\
&= \sum_{e \in E(H)} \eta(v, e) g(e) + \sum_{e \in E(K_{p,q}) \setminus E(H)} \eta(v, e) g(e) + (t-1)pq \deg_H(v) \\
&+ (n-t)pq \deg_{K_{p,q} - E(H)}(v) \\
&= \sum_{e \in E(H)} \eta(v, e) g(e) + \sum_{e \in E(K_{p,q}) \setminus E(H)} \eta(v, e) g(e) + \frac{(t-1)pq}{2} \deg_{K_{p,q}}(v) \\
&+ \frac{(n-t)pq}{2} \deg_{K_{p,q}}(v) \\
&= g^*(v) + \frac{(n-1)pq}{2} \deg_{K_{p,q}}(v) = \frac{pq+1}{2} \deg_{K_{p,q}}(v) \\
&+ \frac{(n-1)pq}{2} \deg_{K_{p,q}}(v) \\
&= \frac{npq+1}{2} \deg_{K_{p,q}}(v) = \frac{npq+1}{2} \deg_{nK_{p,q}}(v^t).
\end{aligned}$$

Hence f is d-magic and $nK_{p,q}$ is a d-magic graph for all integers $n \geqslant 2$.

If n is even, then for any $v^t \in V(K^t_{p,q})$ and $t \leqslant n/2$, we have

$$\begin{aligned}
&|\{e^t \in E(K^t_{p,q}) : \eta(v^t, e^t) = 1, f(e^t) \leqslant npq/2\}| \\
&\quad = \deg_H(v) = \frac{\deg_{K_{p,q}}(v)}{2} = \deg_{K_{p,q} - E(H)}(v) \\
&\quad = |\{e^t \in E(K^t_{p,q}) : \eta(v^t, e^t) = 1, f(e^t) > npq/2\}|,
\end{aligned}$$

and for any $v^t \in V(K^t_{p,q})$ and $t > n/2$, we have

$$\begin{aligned}
&|\{e^t \in E(K^t_{p,q}) : \eta(v^t, e^t) = 1, f(e^t) \leqslant npq/2\}| \\
&\quad = \deg_{K_{p,q} - E(H)}(v) = \frac{\deg_{K_{p,q}}(v)}{2} = \deg_H(v) \\
&\quad = |\{e^t \in E(K^t_{p,q}) : \eta(v^t, e^t) = 1, f(e^t) > npq/2\}|.
\end{aligned}$$

Thus f is balanced d-magic and $nK_{p,q}$ is a balanced d-magic graph for all even integers $n \geqslant 2$. □

Combining Proposition 1 and Theorem 9, we obtain the following result.

Proposition 3. *Let p and q be even positive integers with $(p,q) \neq (2,2)$. Then the following conditions hold:*

(i) $nK_{p,q}$ is a d-magic graph for all odd integers $n \geqslant 1$;

(ii) $nK_{p,q}$ is a balanced d-magic graph for all even integers $n \geqslant 2$.

Corollary 1. *Let p and q be even positive integers with $(p,q) \neq (2,2)$. If $p = q$, then $nK_{p,q}$ is a supermagic graph.*

Proof. Follows from Theorem 2 and Proposition 3. □

Example 1. Consider the complete bipartite graph $K_{2,6}$. One can confirm that $K_{2,6}$ is d-magic, but it is not a balanced d-magic graph (see Figure 1), and the labels on edges u_iv_j of $K_{2,6}$, where $1 \leqslant i \leqslant 2$ and $1 \leqslant j \leqslant 6$, are shown in Table 1.

Then we can construct a balanced d-magic graph $4K_{2,6}$ (see Figure 2) with the labels on edges $u_i^tv_j^t$ of $4K_{2,6}$, where $1 \leqslant i \leqslant 2$, $1 \leqslant j \leqslant 6$ and $1 \leqslant t \leqslant 4$, in Table 2.

Theorem 10. *Let p and q be integers with $p, q > 1$. If the complete bipartite graph $K_{p,q}$ is balanced d-magic, then $nK_{p,q}$ is a balanced d-magic graph for all integers $n \geqslant 2$.*

Proof. For any integer $n \geqslant 2$ and $t \in \{1, 2, \dots, n\}$, let $K_{p,q}^t$ be the $t^{\underline{th}}$ copy of a graph $K_{p,q}$ and let $e^t(v^t)$ be its edge (vertex) corresponding to $e \in E(K_{p,q})(v \in V(K_{p,q}))$. Since $K_{p,q}$ is balanced d-magic, there is a balanced d-magic labelling g of $K_{p,q}$. Let $H_1^{'} := \{e \in E(K_{p,q}) : g(e) \leqslant \lfloor pq/2 \rfloor\}$ and $H_2^{'} := \{e \in E(K_{p,q}) : g(e) > \lfloor pq/2 \rfloor\}$. Thus, the spanning subgraphs H_1 and H_2 of $K_{p,q}$ with the edge sets $H_1^{'}$ and $H_2^{'}$ are half-factors of $K_{p,q}$, respectively. Suppose that $nK_{p,q} = K_{p,q}^1 \cup K_{p,q}^2 \cup \dots \cup K_{p,q}^n$ and we consider a mapping f from $E(nK_{p,q})$ into positive integers given by

$$f(e^t) = \begin{cases} g(e) + (t-1)pq & \text{if} \quad e \in H_1^{'}, \\ g(e) + (n-t)pq & \text{if} \quad e \in H_2^{'}. \end{cases}$$

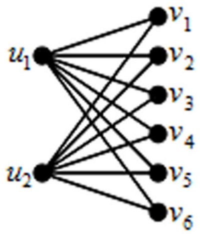

FIGURE 1. A d-magic complete bipartite graph $K_{2,6}$ with 8 vertices and 12 edges.

Vertices	v_1	v_2	v_3	v_4	v_5	v_6
u_1	1	11	3	9	8	7
u_2	12	2	10	4	5	6

TABLE 1. The labels on edges of d-magic complete bipartite graph $K_{2,6}$.

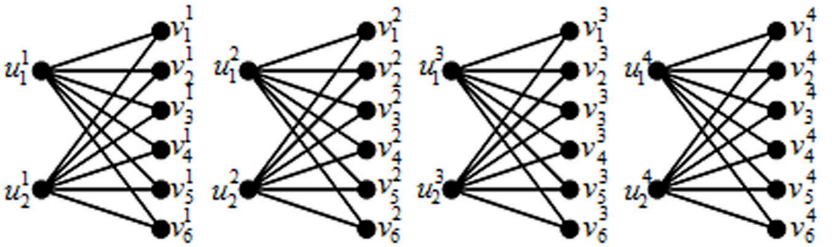

FIGURE 2. A balanced d-magic graph $4K_{2,6}$ with 32 vertices and 48 edges.

Vertices	v_1^1	v_2^1	v_3^1	v_4^1	v_5^1	v_6^1
u_1^1	37	47	39	9	8	7
u_2^1	12	2	10	40	41	42
Vertices	v_1^2	v_2^2	v_3^2	v_4^2	v_5^2	v_6^2
u_1^2	25	35	27	21	20	19
u_2^2	24	14	22	28	29	30
Vertices	v_1^3	v_2^3	v_3^3	v_4^3	v_5^3	v_6^3
u_1^3	13	23	15	33	32	31
u_2^3	36	26	34	16	17	18
Vertices	v_1^4	v_2^4	v_3^4	v_4^4	v_5^4	v_6^4
u_1^4	1	11	3	45	44	43
u_2^4	48	38	46	4	5	6

TABLE 2. The labels on edges of balanced d-magic graph $4K_{2,6}$.

We can then check that f is a bijection from $E(nK_{p,q})$ onto $\{1, 2, \ldots npq\}$. For any vertex $v^t \in V(K^t_{p,q})$, we have

$$\begin{aligned}
f^*(v^t) &= \sum_{e^t \in E(K^t_{p,q})} \eta(v^t, e^t) f(e^t) \\
&= \sum_{e \in H'_1} \eta(v, e)(g(e) + (t-1)pq) + \sum_{e \in H'_2} \eta(v, e)(g(e) + (n-t)pq) \\
&= \sum_{e \in H'_1} \eta(v, e) g(e) + \sum_{e \in H'_2} \eta(v, e) g(e) + (t-1)pq \deg_{H_1}(v) \\
&\quad + (n-t)pq \deg_{H_2}(v) \\
&= \sum_{e \in H'_1} \eta(v, e) g(e) + \sum_{e \in H'_2} \eta(v, e) g(e) + \frac{(t-1)pq}{2} \deg_{K_{p,q}}(v) \\
&\quad + \frac{(n-t)pq}{2} \deg_{K_{p,q}}(v) \\
&= g^*(v) + \frac{(n-1)pq}{2} \deg_{K_{p,q}}(v) = \frac{pq+1}{2} \deg_{K_{p,q}}(v) \\
&\quad + \frac{(n-1)pq}{2} \deg_{K_{p,q}}(v) \\
&= \frac{npq+1}{2} \deg_{K_{p,q}}(v) = \frac{npq+1}{2} \deg_{nK_{p,q}}(v^t).
\end{aligned}$$

Hence f is a d-magic labelling. For $v^t \in V(K^t_{p,q})$ and $t \leqslant (n+1)/2$, we have

$$\begin{aligned}
&|\{e^t \in E(K^t_{p,q}) : \eta(v^t, e^t) = 1, f(e^t) \leqslant \lfloor npq/2 \rfloor\}| \\
&\qquad = \deg_{H_1}(v) = \frac{\deg_{K_{p,q}}(v)}{2} = \deg_{H_2}(v) \\
&\qquad = |\{e^t \in E(K^t_{p,q}) : \eta(v^t, e^t) = 1, f(e^t) > \lfloor npq/2 \rfloor\}|,
\end{aligned}$$

and for any $v^t \in V(K^t_{p,q})$ and $t > (n+1)/2$, we have

$$\begin{aligned}
&|\{e^t \in E(K^t_{p,q}) : \eta(v^t, e^t) = 1, f(e^t) \leqslant \lfloor npq/2 \rfloor\}| \\
&\qquad = \deg_{H_2}(v) = \frac{\deg_{K_{p,q}}(v)}{2} = \deg_{H_1}(v) \\
&\qquad = |\{e^t \in E(K^t_{p,q}) : \eta(v^t, e^t) = 1, f(e^t) > \lfloor npq/2 \rfloor\}|.
\end{aligned}$$

Hence f is balanced d-magic. Therefore, $nK_{p,q}$ is a balanced d-magic graph for all integers $n \geqslant 2$. □

According to Theorems 7 and 10, we obtain the following result.

Proposition 4. *Let p and q be even positive integers and the following condition holds:*

$$\textit{if } p \equiv q \equiv 2 \pmod 4, \textit{ then } \min\{p, q\} \geqslant 6.$$

Then $nK_{p,q}$ is a balanced d-magic graph for all integers $n \geqslant 1$.

3. A construction of supermagic regular graphs

In this section we construct supermagic regular graphs by applying the existence of the n-fold self-union of complete bipartite graphs. Herein, we consider the ξ-multiplication of a graph introduced by Bezegová and Ivančo [4] to prove the next result. Let G be a graph and ξ be a mapping from $V(G)$ into the positive integers. The ξ-*multiplication* of G, denoted by G^ξ, is a graph whose vertices are all ordered pairs (v, i), where $v \in V(G)$ and $1 \leqslant i \leqslant \xi(v)$, and two vertices $(u, i), (v, j)$ are joined by an edge in G^ξ if and only if u, v are adjacent in G. Note that G^ξ is isomorphic to lexicographic product $G[D_n]$ of G and a totally disconnected graph D_n, when $\xi(v) = n$ for all $v \in V(G)$.

Proposition 5. *For any integer $n \geqslant 1$ and $t \in \{1, 2, \ldots, n\}$. Let G^t be the $t^{\underline{th}}$ copy of a graph G and let v^t be its vertex corresponding to $v \in V(G)$. Let ξ be a mapping from $V(nG)$ into even positive integers such that the following conditions hold:*

(i) $\xi(v^t) = \xi(v^s)$ for all $t, s \in \{1, 2, \ldots, n\}$;

(ii) for any adjacent vertices $u^t, v^t \in V(G^t)$, if $\xi(u^t) \equiv \xi(v^t) \equiv 2 \pmod 4$, then $\min\{\xi(u^t), \xi(v^t)\} \geqslant 6$.

Then the ξ-multiplication $(nG)^\xi$ is a balanced d-magic graph.

Proof. For any integer $n \geqslant 1$ and $t \in \{1, 2, \ldots, n\}$. Let edge $e^t = u^t v^t \in E(G^t)$ and let $(G^t_{e^t})^\xi$ be a subgraph of $(G^t)^\xi$ induced by $\{(u^t, i) : 1 \leqslant i \leqslant \xi(u^t)\} \cup \{(v^t, j) : 1 \leqslant j \leqslant \xi(v^t)\}$. Evidently, $(G^t_{e^t})^\xi$ is isomorphic to a complete bipartite graph $K_{\xi(u^t),\xi(v^t)}$. According to Theorem 7, $K_{\xi(u^t),\xi(v^t)}$ is a balanced d-magic graph. By condition (i), we obtain that $K_{\xi(u^t),\xi(v^t)}$ is isomorphic to $K_{\xi(u^s),\xi(v^s)}$ for all $t, s \in \{1, 2, .., n\}$. Thus, by Proposition 4, $\bigcup_{t=1}^n K_{\xi(u^t),\xi(v^t)}$ is a balanced d-magic graph. Since $\bigcup_{t=1}^n (G^t_{e^t})^\xi$ is isomorphic to $\bigcup_{t=1}^n K_{\xi(u^t),\xi(v^t)}$, $\bigcup_{t=1}^n (G^t_{e^t})^\xi$ is a balanced d-magic graph. The ξ-multiplication $(nG)^\xi$ is decomposed into edge-disjoint subgraphs $\bigcup_{t=1}^n (G^t_{e^t})^\xi$ for all $e^t \in E(G^t)$. Therefore, by Theorem 6, $(nG)^\xi$ is a balanced d-magic graph. □

Note that the subgraph of $(nG)^\xi$ induced by $\bigcup_{t=1}^{n}\{(v^t,1) : v^t \in V(G^t)\}$ is isomorphic to nG. Thus, by Proposition 5, for any graph G there is a balanced d-magic graph which contains an induced subgraph isomorphic to nG for all integers $n \geqslant 1$.

We end this section with a similar result for supermagic regular graphs.

Theorem 11. *For any graph G there is a supermagic regular graph which contains an induced subgraph isomorphic to nG for all integers $n \geqslant 1$.*

Proof. Let G_1 be a graph obtained from G by attaching a pendant edge at each vertex of G. For any integer $n \geqslant 1$ and $t \in \{1, 2, \ldots, n\}$. Let $G^t(G_1^t)$ be the $t^{\underline{th}}$ copy of a graph $G(G_1)$. Put $m := |V(G^t)|$ and denote the vertices of G_1^t by $u_1^t, u_2^t, \ldots, u_m^t, w_1^t, w_2^t, \ldots, w_m^t$ in such a way that $V(G^t) = \{u_1^t, \ldots, u_m^t\}$ and $u_i^t w_i^t$, for all $i \in \{1, \ldots, m\}$, is an attached edge of G_1^t. Consider a mapping ξ from $V(nG_1)$ into positive integers given by

$$\xi(u_i^t) = 4 \quad \text{and} \quad \xi(w_i^t) = 4(1 + \Delta - \deg_{G^t}(u_i^t))$$

for all $i \in \{1, \ldots, m\}$ and $t \in \{1, 2, \ldots, n\}$, where Δ is the maximum degree of G^t. Let $H_1 := (nG_1)^\xi$. By Proposition 5, H_1 is a balanced d-magic graph.

We set

$$U := \bigcup_{t=1}^{n} \bigcup_{i=1}^{m} \bigcup_{j=1}^{\xi(u_i^t)} \{(u_i^t, j)\} \quad \text{and} \quad W := \bigcup_{t=1}^{n} \bigcup_{i=1}^{m} \bigcup_{j=1}^{\xi(w_i^t)} \{(w_i^t, j)\}.$$

It is clear that $U \cap W = \varnothing$ and $U \cup W = V(H_1)$. The set W is an independent set of H_1 and $|W| = 4nh$, where $h = \sum_{i=1}^{m}(1 + \Delta - \deg_{G^t}(u_i^t))$ for any $t \in \{1, 2, \ldots, n\}$. Consider h, we have

$$h = m + \sum_{i=1}^{m}(\Delta - \deg_{G^t}(u_i^t)) \geqslant m > \Delta.$$

Moreover, $h = (1 + \Delta)m - \sum_{i=1}^{m} \deg_{G^t}(u_i^t) = (1 + \Delta)m - 2|E(G^t)|$. Thus, if Δ is odd, then h is even. Since $h > \Delta$ and both of h, Δ are not odd, there is a Δ-regular graph R order h. According to Proposition 5, $(nR)[D_4]$ is balanced d-magic, 4Δ-regular graph of order $4nh$. Therefore, there is a balanced d-magic graph H_2, 4Δ-regular graph, such that $V(H_2) = W$.

Let H denote the graph such that the graphs H_1 and H_2 form its decomposition. As H_1 and H_2 are balanced d-magic, by Theorem 6, the graph H is balanced d-magic. Clearly, any vertex of U has degree $4(1+\Delta)$ in H_1. Similarly, the degree of any vertex belonging to W is 4 in H_1 and 4Δ in H_2. So, H is a regular graph of degree $4(1+\Delta)$. According to Theorem 2, the graph H is supermagic. Therefore, H is a desired graph because its subgraph induced by $\bigcup_{t=1}^{n}\bigcup_{i=1}^{m}\{(u_i^t,1)\}$ is isomorphic to nG. □

Example 2. Considering a path P_2, we can construct a supermagic regular graph H which contains an induced subgraph isomorphic to $3P_2$ (see Figure 3), and the labels on edges $u_i^t v_j^t$, $v_i^t x_j^t$, $x_i^t y_j^t$ and $u_i^t y_j^t$ of H, where $1 \leqslant i \leqslant 4, 1 \leqslant j \leqslant 4$ and $1 \leqslant t \leqslant 3$, are shown in Table 3.

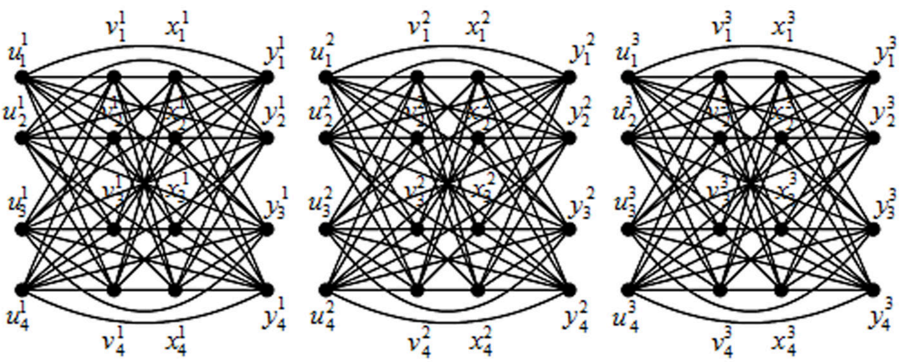

FIGURE 3. A supermagic regular graph H containing an induced subgraph isomorphic to $3P_2$.

4. The n-tuple magic rectangles

In this section we introduce n-tuple magic rectangles and obtain a sufficient condition for even n-tuple magic rectangles to exist.

Definition 1. *An n-tuple magic (p,q)-rectangle $R := (r_{i,j}^1)(r_{i,j}^2)\dots(r_{i,j}^n)$ is a class of n arrays in which each array has p rows and q columns, and the first npq positive integers are placed such that the sum over each row of any array of R is constant and the sum over each column of R is another (different if $p \neq q$) constant.*

Let R be an n-tuple magic (p,q)-rectangle. Since each row sum of any array of R is $q(npq+1)/2$ and each column sum of R is $p(npq+1)/2$ and both are integer, we have

TABLE 3. The labels on edges of supermagic regular graph H.

Vertices	u_1^1	u_2^1	u_3^1	u_4^1	x_1^1	x_2^1	x_3^1	x_4^1
v_1^1	49	54	139	144	73	80	117	116
v_2^1	56	51	142	137	78	75	114	119
v_3^1	141	138	52	55	115	118	76	77
v_4^1	140	143	53	50	120	113	79	74
y_1^1	1	6	187	192	25	30	163	168
y_2^1	8	3	190	185	32	27	166	161
y_3^1	189	186	4	7	165	162	28	31
y_4^1	188	191	5	2	164	167	29	26
Vertices	u_1^2	u_2^2	u_3^2	u_4^2	x_1^2	x_2^2	x_3^2	x_4^2
v_1^2	65	70	123	128	89	96	101	100
v_2^2	72	67	126	121	94	91	98	103
v_3^2	125	122	68	71	99	102	92	93
v_4^2	124	127	69	66	104	97	95	90
y_1^2	17	22	171	176	41	46	147	152
y_2^2	24	19	174	169	48	43	150	145
y_3^2	173	170	20	23	149	146	44	47
y_4^2	172	175	21	18	148	151	45	42
Vertices	u_1^3	u_2^3	u_3^3	u_4^3	x_1^3	x_2^3	x_3^3	x_4^3
v_1^3	129	134	59	64	105	112	85	84
v_2^3	136	131	62	57	110	107	82	87
v_3^3	61	58	132	135	83	86	108	109
v_4^3	60	63	133	130	88	81	111	106
y_1^3	177	182	11	16	153	158	35	40
y_2^3	184	179	14	9	160	155	38	33
y_3^3	13	10	180	183	37	34	156	159
y_4^3	12	15	181	178	36	39	157	154

Proposition 6. *If R is an n-tuple magic (p, q)-rectangle, then the following conditions hold:*

(i) if n is odd, then $p \equiv q \pmod 2$;

(ii) if n is even, then $p \equiv q \equiv 0 \pmod 2$.

Proposition 6 allows the set of n-tuple magic rectangles to be divided into sets of odd and even rectangles. We quickly see that an n-tuple magic $(2, 2)$-rectangle does not exist, because the row sums and column sums of any array are different.

Theorem 12. *For any integer $n \geqslant 1$ and even integers $p, q > 1$, let $K^t_{p,q}$ be the $t^{\underline{th}}$ copy of $K_{p,q}$ for all $t \in \{1, 2, \ldots, n\}$. A mapping f from $E(nK_{p,q})$ into positive integers given by*

$$f(u^t_i v^t_j) = r^t_{i,j} \quad \text{for every} \quad u^t_i v^t_j \in E(K^t_{p,q}),$$

is a d-magic labelling of $nK_{p,q}$ if and only if $R := (r^1_{i,j})(r^2_{i,j}) \ldots (r^n_{i,j})$ is an n-tuple magic (p, q)-rectangle.

Proof. Let $U^t = \{u^t_1, u^t_2, \ldots, u^t_p\}$ and $V^t = \{v^t_1, v^t_2, \ldots, v^t_q\}$ be partite sets of $K^t_{p,q}$. Suppose that R is an n-tuple magic (p, q)-rectangle. It is easy to see that the map $f : E(nK_{p,q}) \to \{1, 2, \ldots, npq\}$ is bijective. For any $u^t_i \in U^t$, we have

$$f^*(u^t_i) = \sum_{j=1}^{q} f(u^t_i v^t_j) = \sum_{j=1}^{q} r^t_{i,j} = \frac{q(npq + 1)}{2} = \frac{npq + 1}{2} \deg(u^t_i),$$

and for any $v^t_j \in V^t$, we have

$$f^*(v^t_j) = \sum_{i=1}^{p} f(u^t_i v^t_j) = \sum_{i=1}^{p} r^t_{i,j} = \frac{p(npq + 1)}{2} = \frac{npq + 1}{2} \deg(v^t_j).$$

i.e., f is a d-magic labelling of $nK_{p,q}$.

Now suppose that f is a d-magic labelling of $nK_{p,q}$. For all $1 \leqslant i \neq s \leqslant p$, we have

$$\sum_{j=1}^{q} r^t_{i,j} = \sum_{j=1}^{q} f(u^t_i v^t_j) = f^*(u^t_i) = f^*(u^t_s) = \sum_{j=1}^{q} f(u^t_s v^t_j) = \sum_{j=1}^{q} r^t_{s,j}. \quad (1)$$

For all $1 \leqslant j \neq z \leqslant q$, we have

$$\sum_{i=1}^{p} r^t_{i,j} = \sum_{i=1}^{p} f(u^t_i v^t_j) = f^*(v^t_j) = f^*(v^t_z) = \sum_{i=1}^{p} f(u^t_i v^t_z) = \sum_{i=1}^{p} r^t_{i,z}. \quad (2)$$

By (1), we have

$$\sum_{j=1}^{q} r_{i,j}^{t} = \sum_{j=1}^{q} r_{s,j}^{t} = \frac{q(npq+1)}{2}.$$

By (2), we have

$$\sum_{i=1}^{p} r_{i,j}^{t} = \sum_{i=1}^{p} r_{i,z}^{t} = \frac{p(npq+1)}{2}.$$

Therefore, R is an n-tuple magic (p, q)-rectangle. □

According to Proposition 3 and Theorem 12, we obtain the following result.

Proposition 7. *Let p and q be even positive integers with $(p, q) \neq (2, 2)$. Then an n-tuple magic (p, q)-rectangle exists.*

References

[1] J. Sedláček, *Theory of graphs and its applications*, in: Problem 27, Proc. Symp. Smolenice, Praha, 1963, pp. 163-164.

[2] B.M. Stewart, *Magic graphs*, Canad. J. Math., Vol. **18**, 1966, pp. 1031-1059.

[3] J.A. Gallian, *A dynamic survey of graph labeling*, Electron. J. Combin., Vol. **16**, 2009, #DS6.

[4] L'. Bezegová, J. Ivančo, *An extension of regular supermagic graphs*, Discrete Math., Vol. **310**, 2010, pp. 3571-3578.

[5] L'. Bezegová, J. Ivančo, *A characterization of complete tripartite degree-magic graphs*, Discuss. Math. Graph Theory, Vol. **32**, 2012, pp. 243-253.

[6] L'. Bezegová, J. Ivančo, *On conservative and supermagic graphs*, Discrete Math., Vol. **311**, 2011, pp. 2428-2436.

[7] L'. Bezegová, J. Ivančo, *Number of edges in degree-magic graphs*, Discrete Math., Vol. **313**, 2013, pp. 1349-1357.

[8] L'. Bezegová, *Balanced degree-magic complements of bipartite graphs*, Discrete Math., Vol. **313**, 2013, pp. 1918-1923.

[9] T. Harmuth, *Über magische Quadrate und ähniche Zahlenfiguren*, Arch. Math. Phys., Vol. **66**, 1881, pp. 286-313.

[10] T. Harmuth, *Über magische Rechtecke mit ungeraden Seitenzahlen*, Arch. Math. Phys., Vol. **66**, 1881, pp. 413-447.

[11] R. Sun, *Existence of magic rectangles*, Nei Mongol Daxue Xuebao Ziran Kexue, Vol. **21**, No. **1**, 1990, pp. 10-16.

[12] T. Bier, G. Rogers, *Balanced Magic Rectangles*, European J. Combin., Vol. **14**, 1993, pp. 285-299.

[13] T. Bier, A. Kleinschmidt, *Centrally symmetric and magic rectangles*, Discrete Math., Vol. **176**, 1997, pp. 29-42.

[14] T.R. Hagedorn, *Magic rectangles revisited*, Discrete Math., Vol. **207**, 1999, pp. 65-72.

[15] T.R. Hagedorn, *On the existence of magic n-dimensional rectangles*, Discrete Math., Vol. **207**, 1999, pp. 53-63.

Contact information

P. Inpoonjai Faculty of Sciences and Agricultural Technology, Rajamangala University of Technology Lanna Chiangrai, 99, Sai Khao, Phan District, Chiang Rai, 57120, Thailand
E-Mail(s): phaisatcha_in@outlook.com

T. Jiarasuksakun Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi, 126 Pracha Uthit Rd., Bang Mod, Thung Khru, Bangkok 10140, Thailand
E-Mail(s): thiradet.jia@kmutt.ac.th

Received by the editors: 28.12.2016
and in final form 07.03.2017.