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RESEARCH ARTICLE

Additivity of elementary maps on alternative rings

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Abstract. Let $\mathfrak{R}$ and $\mathfrak{R}'$ be alternative rings. In this article we investigate the additivity of surjective elementary maps of $\mathfrak{R} \times \mathfrak{R}'$. As a main theorem, we prove that if $\mathfrak{R}$ contains a non-trivial idempotent satisfying some conditions, these maps are additive.

1. Alternative rings and elementary maps

Let $\mathfrak{R}$ be a ring not necessarily associative or commutative and consider the following convention for its multiplication operation: $xy \cdot z = (xy)z$ and $x \cdot yz = x(yz)$ for $x, y, z \in \mathfrak{R}$, in order to reduce the number of parentheses. We denote the *associator* of $\mathfrak{R}$ by $(x, y, z) = xy \cdot z - x \cdot yz$ for $x, y, z \in \mathfrak{R}$.

Let $X = \{x_i\}_{i \in \mathbb{N}}$ be an arbitrary set of variables. A *non-associative monomial of degree* 1 is any element of X. Given a natural number $n > 1$, a *non-associative monomial of degree* n is an expression of the form $(u)(v)$, where u is a non-associative monomial of some degree i and v is a non-associative monomial of degree $n - i$. A *non-associative polynomial* f over a ring $\mathfrak{R}$ is any formal linear combination of non-associative monomials with coefficients in $\mathfrak{R}$. If f includes no variables except $x_1, x_2, \ldots, x_n$ and $a_1, a_2, \ldots, a_n$ is a set of elements of $\mathfrak{R}$, then $f(a_1, a_2, \ldots, a_n)$ is an element of $\mathfrak{R}$ which results by applying the sequence of operations forming f to $a_1, a_2, \ldots, a_n$ in place of $x_1, x_2, \ldots, x_n$.

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Let $\mathfrak{R}$ and $\mathfrak{R}'$ be two rings and let $M : \mathfrak{R} \to \mathfrak{R}'$ and $M^* : \mathfrak{R}' \to \mathfrak{R}$ be two maps. We call the ordered pair (M, M^*) an *elementary map of* $\mathfrak{R} \times \mathfrak{R}'$ if for all non-associative monomial $f = f(x_1, x_2, x_3)$ of degree 3

$$M\big(f\big(a, M^*(x), b\big)\big) = f\big(M(a), x, M(b)\big),$$
$$M^*\big(f\big(x, M(a), y\big)\big) = f\big(M^*(x), a, M^*(y)\big)$$

for all $a, b \in \mathfrak{R}$ and $x, y \in \mathfrak{R}'$.

We say that the elementary map (M, M^*) of $\mathfrak{R} \times \mathfrak{R}'$ is *additive* (resp., *injective, surjective, bijective*) if both maps M and M^* are additive (resp., injective, surjective, bijective).

A ring $\mathfrak{R}$ is said to be *alternative* if $(x, x, y) = 0 = (y, x, x)$ for all $x, y \in \mathfrak{R}$. It is easily seen that any associative ring is an alternative ring.

An alternative ring $\mathfrak{R}$ is called *k-torsion free* if $k\,x = 0$ implies $x = 0$, for any $x \in \mathfrak{R}$, where $k \in \mathbb{Z}$, $k > 0$, and *prime* if $\mathfrak{A}\mathfrak{B} \neq 0$ for any two nonzero ideals $\mathfrak{A}, \mathfrak{B} \subseteq \mathfrak{R}$.

Let us consider $\mathfrak{R}$ an alternative ring and fix a non-trivial idempotent $e_1 \in \mathfrak{R}$, i.e, $e_1^2 = e_1,\ e_1 \neq 0$ and e_1 is not a unity element. Let $e_2 \colon \mathfrak{R} \to \mathfrak{R}$ and $e_2' \colon \mathfrak{R} \to \mathfrak{R}$ be linear operators given by $e_2(a) = a - e_1 a$ and $e_2'(a) = a - a e_1$. Clearly $e_2^2 = e_2$, $(e_2')^2 = e_2'$ and we note that if $\mathfrak{R}$ has a unity, then we can consider $e_2 = 1 - e_1 \in \mathfrak{R}$. Let us denote $e_2(a)$ by $e_2 a$ and $e_2'(a)$ by $a e_2$. It is easy to see that $e_i a \cdot e_j = e_i \cdot a e_j\ (i, j = 1, 2)$ for all $a \in \mathfrak{R}$. Then $\mathfrak{R}$ has a Peirce decomposition $\mathfrak{R} = \mathfrak{R}_{11} \oplus \mathfrak{R}_{12} \oplus \mathfrak{R}_{21} \oplus \mathfrak{R}_{22}$, where $\mathfrak{R}_{ij} = e_i \mathfrak{R} e_j\ (i, j = 1, 2)$, (see [3]) satisfying the multiplicative relations:

(i) $\mathfrak{R}_{ij}\mathfrak{R}_{jl} \subseteq \mathfrak{R}_{il}\ (i, j, l = 1, 2)$;

(ii) $\mathfrak{R}_{ij}\mathfrak{R}_{ij} \subseteq \mathfrak{R}_{ji}\ (i, j = 1, 2)$;

(iii) $\mathfrak{R}_{ij}\mathfrak{R}_{kl} = 0$, if $j \neq k$ and $(i, j) \neq (k, l),\ (i, j, k, l = 1, 2)$;

(iv) $x_{ij}^2 = 0$, for all $x_{ij} \in \mathfrak{R}_{ij}\ (i, j = 1, 2;\ i \neq j)$.

According to [4], "*The first surprising result on how the multiplicative structure of a ring determines its additive structure is due to Martindale III (1969). In Martindale III (1969, [5] Theorem), he established a condition on a ring R such that every multiplicative bijective map on R is additive.*" Li and Lu [4] also considered this question in the context of associative rings containing a non-trivial idempotent. They proved the following theorem.

Theorem 1. *[4, Li and Lu] Let $\mathfrak{R}$ and $\mathfrak{R}'$ be two associative rings. Suppose that $\mathfrak{R}$ is a 2-torsion free ring containing a family $\{e_\alpha | \alpha \in \Lambda\}$ of idempotents which satisfies:*

(i) *If $x \in \mathfrak{R}$ is such that $x\mathfrak{R} = 0$, then $x = 0$;*

(ii) *If* $x \in \mathfrak{R}$ *is such that* $e_\alpha \mathfrak{R} x = 0$ *for all* $\alpha \in \Lambda$, *then* $x = 0$ *(and hence* $\mathfrak{R}x = 0$ *implies* $x = 0$*);*

(iii) *For each* $\alpha \in \Lambda$ *and* $x \in \mathfrak{R}$, *if* $e_\alpha x e_\alpha \mathfrak{R}(1 - e_\alpha) = 0$ *then* $e_\alpha x e_\alpha = 0$.

Then every surjective elementary map (M, M^*) *of* $\mathfrak{R} \times \mathfrak{R}'$ *is additive.*

The hypotheses in Li and Lu's Theorem [4] allowed the author to make its proof based on calculus using Peirce decomposition notion for associative rings.

The notion of Peirce decomposition for alternative rings is similar to that one for associative rings. However, this similarity is restricted to its written form, not including its theoretical structure since Peirce decomposition for alternative rings is a generalization of that classical one for associative rings. Taking this fact into account, in the present paper we generalize the main Li and Lu's Theorem [4] to the class of alternative rings. For this purpose, we adopt and follow the same structure of the proof proposed by [4], in order to preserve the author's ideas and to highlight the generalization of associative results to the alternative results. Therefore, our lemmas and the main theorem, which seem to be equal in written form to those presented in lemmas and the theorem proposed in Li and Lu [4], are distinguished by a fundamental item: the use of the non-associative multiplications. The symbol "$\cdot$", as defined in the introduction section of this article, is essential to elucidate how the non-associative multiplication should be done, and also the symbol "$\cdot$" is used to simplify the notation. Therefore, the symbol "$\cdot$" is crucial to the logic, characterization and generalization of associative results to alternative results.

2. The main result

Our main result reads as follows.

Theorem 2. *Let* $\mathfrak{R}$ *and* $\mathfrak{R}'$ *be two alternative rings. Suppose that* $\mathfrak{R}$ *is a* 2*-torsion free ring containing a family* $\{e_\alpha | \alpha \in \Lambda\}$ *of idempotents which satisfies:*

(i) *If* $x \in \mathfrak{R}$ *is such that* $x\mathfrak{R} = 0$, *then* $x = 0$;

(ii) *If* $x \in \mathfrak{R}$ *is such that* $e_\alpha \mathfrak{R} \cdot x = 0$ *(or* $e_\alpha \cdot \mathfrak{R}x = 0$*) for all* $\alpha \in \Lambda$, *then* $x = 0$ *(and hence* $\mathfrak{R}x = 0$ *implies* $x = 0$*);*

(iii) *For each* $\alpha \in \Lambda$ *and* $x \in \mathfrak{R}$, *if* $(e_\alpha x e_\alpha) \cdot \mathfrak{R}(1 - e_\alpha) = 0$ *then* $e_\alpha x e_\alpha = 0$.

Then every surjective elementary map (M, M^*) *of* $\mathfrak{R} \times \mathfrak{R}'$ *is additive.*

For proving Theorem 2 some preparatory material is needed, following same steps as [4]. Firstly, we consider the case when the monomial is of type $f = f(x_1, x_2, x_3)$. We begin with the following lemma.

Lemma 1. *$M(0) = 0$ and $M^*(0) = 0$.*

Proof. $M(0) = M\big(0M^*(0) \cdot 0\big) = M(0)0 \cdot M(0) = 0$. Similarly, we have $M^*(0) = 0$. □

Lemma 2. *M and M^* are bijective.*

Proof. It suffices to prove that M and M^* are injective. We first show that M is injective. Let x_1 and x_2 be in $\mathfrak{R}$ and suppose that $M(x_1) = M(x_2)$. Since $M^*\big(uM(x_i) \cdot v\big) = M^*(u)x_i \cdot M^*(v)$ $(i = 1, 2)$ for all $u, v \in \mathfrak{R}'$, it follows that $M^*(u)x_1 \cdot M^*(v) = M^*(u)x_2 \cdot M^*(v)$. Hence from the surjectivity of M^* and conditions (i) and (ii) we conclude that $x_1 = x_2$. Now we turn to proving the injectivity of M^*. Let u_1 and u_2 be in $\mathfrak{R}'$ and suppose $M^*(u_1) = M^*(u_2)$. Since

$$\begin{aligned} M^*M\big(xM^*(u_i) \cdot y\big) &= M^*\big(M(x)u_i \cdot M(y)\big) \\ &= M^*\big(M(x)MM^{-1}(u_i) \cdot M(y)\big) \\ &= M^*M(x)M^{-1}(u_i) \cdot M^*M(y) \end{aligned}$$

for all $x, y \in \mathfrak{R}$, it follows that

$$M^*M(x)M^{-1}(u_1) \cdot M^*M(y) = M^*M(x)M^{-1}(u_2) \cdot M^*M(y).$$

Noting that M^*M is also surjective, we see that $M^{-1}(u_1) = M^{-1}(u_2)$, by conditions (i) and (ii). Consequently $u_1 = u_2$. □

Lemma 3. *The pair (M^{*-1}, M^{-1}) is an elementary map of $\mathfrak{R} \times \mathfrak{R}'$, that is, the maps $M^{*-1} : \mathfrak{R} \to \mathfrak{R}'$ and $M^{-1} : \mathfrak{R}' \to \mathfrak{R}$ satisfy*

$$\begin{aligned} M^{*-1}\big(aM^{-1}(x) \cdot b\big) &= M^{*-1}(a)x \cdot M^{*-1}(b), \\ M^{-1}\big(xM^{*-1}(a) \cdot y\big) &= M^{-1}(x)a \cdot M^{-1}(y) \end{aligned}$$

for all $a, b \in \mathfrak{R}$ and $x, y \in \mathfrak{R}'$.

Proof. The first equality can be obtained from

$$\begin{aligned} M^*\big(M^{*-1}(a)x \cdot M^{*-1}(b)\big) &= M^*\big(M^{*-1}(a)MM^{-1}(x) \cdot M^{*-1}(b)\big) \\ &= aM^{-1}(x) \cdot b \end{aligned}$$

and the second one follows in a similar way. □

Lemma 4. *Let $s, a, b \in \mathfrak{R}$ such that $M(s) = M(a) + M(b)$. Then*

(i) $M(sx \cdot y) = M(ax \cdot y) + M(bx \cdot y)$ *for* $x, y \in \mathfrak{R}$;
(ii) $M(xy \cdot s) = M(xy \cdot a) + M(xy \cdot b)$ *for* $x, y \in \mathfrak{R}$;
(iii) $M^{*-1}(xs \cdot y) = M^{*-1}(xa \cdot y) + M^{*-1}(xb \cdot y)$ *for* $x, y \in \mathfrak{R}$ *for* $x, y \in \mathfrak{R}$;
(iv) $M(s \cdot xy) = M(a \cdot xy) + M(b \cdot xy)$ *for* $x, y \in \mathfrak{R}$;
(v) $M(x \cdot ys) = M(x \cdot ya) + M(x \cdot yb)$ *for* $x, y \in \mathfrak{R}$;
(vi) $M^{*-1}(x \cdot sy) = M^{*-1}(x \cdot ay) + M^{*-1}(x \cdot by)$ *for* $x, y \in \mathfrak{R}$ *for* $x, y \in \mathfrak{R}$.

Proof. (i) Let $x, y \in \mathfrak{R}$. Then

$$\begin{aligned} M(sx \cdot y) &= M\big(sM^*M^{*-1}(x) \cdot y\big) \\ &= M(s)M^{*-1}(x) \cdot M(y) \\ &= \big(M(a) + M(b)\big)M^{*-1}(x) \cdot M(y) \\ &= M(a)M^{*-1}(x) \cdot M(y) + M(b)M^{*-1}(x) \cdot M(y) \\ &= M(ax \cdot y) + M(bx \cdot y). \end{aligned}$$

(ii) Let $x, y \in \mathfrak{R}$. Then

$$\begin{aligned} M(xy \cdot s) &= M\big(xM^*M^{*-1}(y) \cdot s\big) \\ &= M(x)M^{*-1}(y) \cdot M(s) \\ &= M(x)M^{*-1}(y) \cdot \big(M(a) + M(b)\big) \\ &= M(x)M^{*-1}(y) \cdot M(a) + M(x)M^{*-1}(y) \cdot M(b) \\ &= M(xy \cdot a) + M(xy \cdot b). \end{aligned}$$

(iii) Let $x, y \in \mathfrak{R}$. By Lemma 2.3

$$\begin{aligned} M^{*-1}(xs \cdot y) &= M^{*-1}(xM^{-1}M(s) \cdot y) \\ &= M^{*-1}(x)M(s) \cdot M^{*-1}(y) \\ &= M^{*-1}(x)\big(M(a) + M(b)\big) \cdot M^{*-1}(y) \\ &= M^{*-1}(x)M(a) \cdot M^{*-1}(y) + M^{*-1}(x)M(b) \cdot M^{*-1}(y) \\ &= M^{*-1}(xa \cdot y) + M^{*-1}(xb \cdot y). \end{aligned}$$

Similarly, we prove (iv), (v) and (vi), which finishes the proof. □

Lemma 5. *The following statements are true:*

(i) $M(a_{11} + a_{12}) = M(a_{11}) + M(a_{12})$;
(ii) $M^{*-1}(a_{11} + a_{12}) = M^{*-1}(a_{11}) + M^{*-1}(a_{12})$.

Proof. By surjectivity of M, there exists $s \in \mathfrak{R}$ such that $M(s) = M(a_{11}) + M(a_{12})$. Now,

$$M(e_1e_1 \cdot s) = M(e_1e_1 \cdot a_{11}) + M(e_1e_1 \cdot a_{12}) = M(s).$$

It follows that $e_1e_1 \cdot s = s$ which implies $s_{21} = s_{22} = 0$. Also

$$M(s \cdot e_1e_1) = M(a_{11} \cdot e_1e_1) + M(a_{12} \cdot e_1e_1) = M(a_{11}).$$

From this equality we get $s \cdot e_1e_1 = a_{11}$ and therefore $s_{11} = a_{11}$.

For an arbitrary $b_{12} \in \mathfrak{R}_{12}$, we obtain

$$M(sb_{12} \cdot e_1) = M(a_{11}b_{12} \cdot e_1) + M(a_{12}b_{12} \cdot e_1) = M(a_{12}b_{12} \cdot e_1),$$

which implies $sb_{12} \cdot e_1 = a_{12}b_{12} \cdot e_1$, or still $(s_{12} - a_{12})b_{12} = 0$. In a similar way, for an arbitrary $b_{21} \in \mathfrak{R}_{21}$, we have

$$M(sb_{21} \cdot e_1) = M(a_{11}b_{21} \cdot e_1) + M(a_{12}b_{21} \cdot e_1) = M(a_{12}b_{21} \cdot e_1).$$

Hence $sb_{21} \cdot e_1 = a_{12}b_{21} \cdot e_1$ and thus $(s_{12} - a_{12})b_{21} = 0$. Finally, for $b_{22} \in \mathfrak{R}_{22}$,

$$\begin{aligned} M^{*-1}(e_1s \cdot b_{22}) &= M^{*-1}(e_1a_{11} \cdot b_{22}) + M^{*-1}(e_1a_{12} \cdot b_{22}) \\ &= M^{*-1}(e_1a_{12} \cdot b_{22}). \end{aligned}$$

As a consequence, $e_1s \cdot b_{22} = e_1a_{12} \cdot b_{22}$ which implies $(s_{12} - a_{12})b_{22} = 0$. From these considerations, $(s_{12} - a_{12})\mathfrak{R} = 0$. According to (i), $s_{12} = a_{12}$. □

Similarly, we prove the lemma below.

Lemma 6. *The following statements are true:*
(i) $M(a_{11} + a_{21}) = M(a_{11}) + M(a_{21})$;
(ii) $M^{*-1}(a_{11} + a_{21}) = M^{*-1}(a_{11}) + M^{*-1}(a_{21})$.

Lemma 7. *The following statements are true:*
(i) $M(a_{11} + a_{12} + a_{21} + a_{22}) = M(a_{11}) + M(a_{12}) + M(a_{21}) + M(a_{22})$;
(ii) $M^{*-1}(a_{11}+a_{12}+a_{21}+a_{22}) = M^{*-1}(a_{11})+M^{*-1}(a_{12})+M^{*-1}(a_{21})+M^{*-1}(a_{22})$.

Proof. By surjectivity of M, there exists $s \in \mathfrak{R}$ such that $M(s) = M(a_{11}) + M(a_{12}) + M(a_{21}) + M(a_{22})$. Now,

$$M(e_1e_1 \cdot s) = M(e_1e_1 \cdot a_{11}) + M(e_1e_1 \cdot a_{12}) = M(a_{11} + a_{12}).$$

It follows from this equality that $e_1e_1 \cdot s = a_{11} + a_{12}$ which implies $s_{11} = a_{11}$ and $s_{12} = s_{12}$. Also

$$M(s \cdot e_1e_1) = M(a_{11} \cdot e_1e_1) + M(a_{21} \cdot e_1e_1) = M(a_{11} + a_{21}),$$

from where we get $s \cdot e_1e_1 = a_{11} + a_{21}$, or still $s_{21} = a_{21}$.

For arbitrary $b_{22}, c_{22} \in \mathfrak{R}_{22}$, we have

$$\begin{aligned} M^{*-1}(b_{22}s \cdot c_{22}) &= M^{*-1}(b_{22}a_{11} \cdot c_{22}) + M^{*-1}(b_{22}a_{12} \cdot c_{22}) \\ &\quad + M^{*-1}(b_{22}a_{21} \cdot c_{22}) + M^{*-1}(b_{22}a_{22} \cdot c_{22}) \\ &= M^{*-1}(b_{22}a_{22} \cdot c_{22}). \end{aligned}$$

Hence $b_{22}s \cdot c_{22} = b_{22}a_{22} \cdot c_{22}$ which implies $b_{22}(s_{22} - a_{22}) \cdot c_{22} = 0$. Now, for an arbitrary $c_{21} \in \mathfrak{R}_{21}$, we have

$$\begin{aligned} M^{*-1}(b_{22}s \cdot c_{21}) &= M^{*-1}(b_{22}a_{11} \cdot c_{21}) + M^{*-1}(b_{22}a_{12} \cdot c_{21}) \\ &\quad + M^{*-1}(b_{22}a_{21} \cdot c_{21}) + M^{*-1}(b_{22}a_{22} \cdot c_{21}) \\ &= M^{*-1}(b_{22}a_{21} \cdot c_{21} + b_{22}a_{22} \cdot c_{21}). \end{aligned}$$

Thus $b_{22}s \cdot c_{21} = b_{22}a_{21} \cdot c_{21} + b_{22}a_{22} \cdot c_{21}$ and therefore $b_{22}(s_{22} - a_{22}) \cdot c_{21} = 0$. As a consequence it follows that $b_{22}(s_{22} - a_{22}) \cdot \mathfrak{R} = 0$ which implies $b_{22}(s_{22} - a_{22}) = 0$. Finally, for an arbitrary $b_{12} \in \mathfrak{R}_{12}$, we have

$$\begin{aligned} M(e_1 \cdot b_{12}s) &= M(e_1 \cdot b_{12}a_{11}) + M(e_1 \cdot b_{12}a_{12}) \\ &\quad + M(e_1 \cdot b_{12}a_{21}) + M(e_1 \cdot b_{12}a_{22}) \\ &= M(e_1 \cdot b_{12}a_{21} + e_1 \cdot b_{12}a_{22}). \end{aligned}$$

Hence $e_1 \cdot b_{12}s = e_1 \cdot b_{12}a_{21} + e_1 \cdot b_{12}a_{22}$ which implies $b_{12}(s_{22} - a_{22}) = 0$. Consequently $\mathfrak{R}(s_{22} - a_{22}) = 0$. By condition (i), we have $s_{22} = a_{22}$.

The proof of (ii) is similar, since the pair $(M^{*-1}; M^{-1})$ is also an elementary map of $\mathfrak{R} \times \mathfrak{R}'$. □

Lemma 8. *The following statements are true:*

(i) $M(a_{12} + b_{21}c_{21}) = M(a_{12}) + M(b_{21}c_{21})$;
(ii) $M(a_{12} + b_{12}a_{22}) = M(a_{12}) + M(b_{12}a_{22})$;
(iii) $M(a_{11} + a_{12}a_{21}) = M(a_{11}) + M(a_{12}a_{21})$;
(iv) $M(a_{21} + a_{22}b_{21}) = M(a_{21}) + M(a_{22}b_{21})$.

Proof. (i) Observing that $a_{12} + b_{21}c_{21} = e_1 \cdot (e_1 + b_{21})(a_{12} + c_{21})$, we get

$$\begin{aligned}
M(a_{12} + b_{21}c_{21}) &= M\big(e_1 \cdot (e_1 + b_{21})(a_{12} + c_{21})\big)\\
&= M(e_1) \cdot M^{*-1}(e_1 + b_{21})M(a_{12} + c_{21})\\
&= M(e_1) \cdot \big(M^{*-1}(e_1) + M^{*-1}(b_{21})\big)\big(M(a_{12}) + M(c_{21})\big)\\
&= M(e_1) \cdot M^{*-1}(e_1)M(a_{12}) + M(e_1) \cdot M^{*-1}(e_1)M(c_{21})\\
&\quad + M(e_1) \cdot M^{*-1}(b_{21})M(a_{12}) + M(e_1) \cdot M^{*-1}(b_{21})M(c_{21})\\
&= M(e_1 \cdot e_1a_{12}) + M(e_1 \cdot e_1c_{21}) + M(e_1 \cdot b_{21}a_{12}) + M(e_1 \cdot b_{21}a_{21})\\
&= M(a_{12}) + M(b_{21}a_{21}).
\end{aligned}$$

(ii) From Lemma 7(i) and (ii) we have

$$\begin{aligned}
M(a_{12} + b_{12}a_{22}) &= M\big(e_1 \cdot (e_1 + b_{12})(a_{12} + a_{22})\big)\\
&= M(e_1) \cdot M^{*-1}(e_1 + b_{12})M(a_{12} + a_{22})\\
&= M(e_1) \cdot \big(M^{*-1}(e_1) + M^{*-1}(b_{12})\big)\big(M(a_{12}) + M(a_{22})\big)\\
&= M(e_1) \cdot M^{*-1}(e_1)M(a_{12}) + M(e_1) \cdot M^{*-1}(e_1)M(a_{22})\\
&\quad + M(e_1) \cdot M^{*-1}(b_{12})M(a_{12}) + M(e_1) \cdot M^{*-1}(b_{12})M(a_{22})\\
&= M(e_1 \cdot e_1a_{12}) + M(e_1 \cdot e_1a_{22}) + M(e_1 \cdot b_{12}a_{12}) + M(e_1 \cdot b_{12}a_{22})\\
&= M(a_{12}) + M(b_{12}a_{22}).
\end{aligned}$$

So (ii) follows. Observing that $a_{11} + a_{12}a_{21} = (a_{11} + a_{12})(e_1 + a_{21}) \cdot e_1$ and $a_{21} + a_{22}b_{21} = (a_{21} + a_{22})(e_1 + b_{21}) \cdot e_1$, then (iii) and (iv) can be proved similarly. □

Lemma 9. $M(a_{21}a_{12} + a_{22}b_{22}) = M(a_{21}a_{12}) + M(a_{22}b_{22})$.

Proof. We first claim that $M(a_{21}a_{12} \cdot c_{22} + a_{22}b_{22} \cdot c_{22}) = M(a_{21}a_{12} \cdot c_{22}) + M(a_{22}b_{22} \cdot c_{22})$ holds for all $c_{22} \in \mathfrak{R}_{22}$. Indeed, from Lemma 7(i) and (ii), we obtain

$$\begin{aligned}
M(a_{21}a_{12} \cdot c_{22} + a_{22}b_{22} \cdot c_{22}) &= M\big((a_{21} + a_{22})(a_{12} + b_{22}) \cdot c_{22}\big)\\
&= M(a_{21} + a_{22})M^{*-1}(a_{12} + b_{22}) \cdot M(c_{22})\\
&= \big(M(a_{21}) + M(a_{22})\big)\big(M^{*-1}(a_{12}) + M^{*-1}(b_{22})\big) \cdot M(c_{22})\\
&= M(a_{21})M^{*-1}(a_{12}) \cdot M(c_{22}) + M(a_{21})M^{*-1}(b_{22}) \cdot M(c_{22})\\
&\quad + M(a_{22})M^{*-1}(a_{12}) \cdot M(c_{22}) + M(a_{22})M^{*-1}(b_{22}) \cdot M(c_{22})\\
&= M(a_{21}a_{12} \cdot c_{22}) + M(a_{21}b_{22} \cdot c_{22}) + M(a_{22}a_{12} \cdot c_{22})\\
&\quad + M(a_{22}b_{22} \cdot c_{22})\\
&= M(a_{21}a_{12} \cdot c_{22}) + M(a_{22}b_{22} \cdot c_{22}),
\end{aligned}$$

as desired. Now let $s \in \mathfrak{R}$ such that $M(s) = M(a_{21}a_{12}) + M(a_{22}b_{22})$, which existence is ensured by surjectivity. Then

$$M(e_1e_1 \cdot s) = M\big(e_1e_1 \cdot (a_{21}a_{12})\big) + M\big(e_1e_1 \cdot (a_{22}b_{22})\big) = 0.$$

Hence $e_1e_1 \cdot s = 0$, or still $s_{11} = s_{12} = 0$. Similarly, we prove $s_{21} = 0$.

For an arbitrary element $x_{21} \in \mathfrak{R}_{21}$, it follows from Lemma 4-(iv) that

$$\begin{aligned} M(sx_{21}) = M(s \cdot x_{21}e_1) &= M\big((a_{21}a_{12}) \cdot x_{21}e_1\big) + M\big((a_{22}b_{22}) \cdot x_{21}e_1\big) \\ &= M(a_{21}a_{12} \cdot x_{21} + a_{22}b_{22} \cdot x_{21}), \end{aligned}$$

from where we get

$$\big(s - (a_{21}a_{12} + a_{22}b_{22})\big)x_{21} = 0. \tag{1}$$

As a next step we prove that

$$\big(s - (a_{21}a_{12} + a_{22}b_{22})\big)x_{22} = 0 \tag{2}$$

holds for every $x_{22} \in \mathfrak{R}_{22}$. First, for y_{21}, by Lemma 4-(i)

$$\begin{aligned} M(sx_{22} \cdot y_{21}) &= M\big((a_{21}a_{12})x_{22} \cdot y_{21}\big) + M\big((a_{22}b_{22})x_{22} \cdot y_{21}\big) \\ &= M\big((a_{21}a_{12})x_{22} \cdot y_{21} + (a_{22}b_{22})x_{22} \cdot y_{21}\big), \end{aligned}$$

which implies that $sx_{22} \cdot y_{21} = (a_{21}a_{12})x_{22} \cdot y_{21} + (a_{22}b_{22})x_{22} \cdot y_{21}$. Therefore $\big(s - (a_{21}a_{12} + a_{22}b_{22})\big)x_{22} \cdot y_{21} = 0$.

Similarly, for $y_{22} \in \mathfrak{R}_{22}$, using Lemma 4(i)

$$\begin{aligned} M(sx_{22} \cdot y_{22}) &= M\big((a_{21}a_{12})x_{22} \cdot y_{22}\big) + M\big((a_{22}b_{22})x_{22} \cdot y_{22}\big) \\ &= M\big((a_{21}a_{12})x_{22} \cdot y_{22}\big) + M\big((a_{22}b_{22})x_{22} \cdot y_{22}\big) \\ &= M\big(a_{21}(a_{12}x_{22}) \cdot y_{22}\big) + M\big((a_{22}b_{22})x_{22} \cdot y_{22}\big) \\ &= M\big(a_{21}(a_{12}x_{22}) \cdot y_{22} + (a_{22}b_{22})x_{22} \cdot y_{22}\big) \\ &= M\big((a_{21}a_{12})x_{22} \cdot y_{22} + (a_{22}b_{22})x_{22} \cdot y_{22}\big) \end{aligned}$$

yielding that $sx_{22} \cdot y_{22} = (a_{21}a_{12} + a_{22}b_{22})x_{22} \cdot y_{22}$. Thus $\big(s - (a_{21}a_{12} + a_{22}b_{22})\big)x_{22} \cdot y_{22} = 0$ and therefore we obtain that

$$\big(s - (a_{21}a_{12} + a_{22}b_{22})\big)x_{22} \cdot \mathfrak{R} = 0. \tag{3}$$

Equation 3 follows from Theorem 2(i).

From Equation 1 and 2, we get that $\big(s - (a_{21}a_{12} + a_{22}b_{22})\big)\mathfrak{R} = 0$. Hence $s = a_{21}a_{12} + a_{22}b_{22}$ due to Theorem 2(i). $\square$

Taking Lemma 3 into account, we point out that Lemma 9 can still be obtained when M is replaced by M^{*-1}, as states the following lemma.

Lemma 10. *The following are true:*
(i) $M^{*-1}(a_{12}+b_{12}a_{22}) = M^{*-1}(a_{12}) + M^{*-1}(b_{12}a_{22})$.
(ii) $M^{*-1}(a_{11}+a_{12}a_{21}) = M^{*-1}(a_{11}) + M^{*-1}(a_{12}a_{21})$.
(iii) $M^{*-1}(a_{21}+a_{22}b_{21}) = M^{*-1}(a_{21}) + M^{*-1}(a_{22}b_{21})$.
(iv) $M^{*-1}(a_{21}a_{12}+a_{22}b_{22}) = M^{*-1}(a_{21}a_{12}) + M^{*-1}(a_{22}b_{22})$.

Lemma 11. $M(a_{12}+b_{12}) = M(a_{12}) + M(b_{12})$.

Proof. Let $s \in \mathfrak{R}$ be such that $M(s) = M(a_{12}) + M(b_{12})$. Then $M(e_1e_1 \cdot s) = M(e_1e_1 \cdot a_{12}) + M(e_1e_1 \cdot b_{12}) = M(s)$ and $M(s \cdot e_1e_1) = M(a_{12} \cdot e_1e_1) + M(b_{12} \cdot e_1e_1) = 0$ which implies $e_1e_1 \cdot s = s$ and $s \cdot e_1e_1 = 0$, respectively. It follows that $s_{21}+s_{22} = 0$ and $s_{11}+s_{21} = 0$, respectively. Thus $s_{11} = s_{21} = s_{22} = 0$.

For $x_{21} \in \mathfrak{R}_{21}$, applying Lemma 4-(iv),

$$\begin{aligned} M(sx_{21}) &= M(s \cdot x_{21}e_1) = M(a_{12} \cdot x_{21}e_1) + M(b_{12} \cdot x_{21}e_1) \\ &= M(a_{12}x_{21}) + M(b_{12}x_{21}) = M(a_{12}x_{21} + b_{12}x_{21}). \end{aligned}$$

These above equations show that $sx_{21} = (a_{12}+b_{12})x_{21}$. Hence

$$\big(s - (a_{12}+b_{12})\big)x_{21} = 0. \tag{4}$$

For all $x_{22} \in \mathfrak{R}_{22}$

$$\begin{aligned} M^{*-1}(sx_{22}) &= M^{*-1}(e_1 \cdot sx_{22}) = M^{*-1}(e_1 \cdot a_{12}x_{22}) + M^{*-1}(e_1 \cdot b_{12}x_{22}) \\ &= M^{*-1}(a_{12}x_{22}) + M^{*-1}(b_{12}x_{22}) = M^{*-1}(a_{12}x_{22} + b_{12}x_{22}) \end{aligned}$$

which implies that

$$\big(s - (a_{12}+b_{12})\big)x_{22} = 0. \tag{5}$$

We now want to prove that $\big(s - (a_{12}+b_{12})\big)x_{12} = 0$, for all $x_{12} \in \mathfrak{R}_{12}$. Indeed, by Lemma 4-(vi), for $y_{12} \in \mathfrak{R}_{12}$

$$\begin{aligned} M^{*-1}(y_{12} \cdot sx_{12}) &= M^{*-1}(y_{12} \cdot a_{12}x_{12}) + M^{*-1}(y_{12} \cdot b_{12}x_{12}) \\ &= M^{*-1}(y_{12} \cdot a_{12}x_{12} + y_{12} \cdot b_{12}x_{12}). \end{aligned}$$

We then get that $y_{12} \cdot sx_{12} = y_{12} \cdot (a_{12}x_{12} + b_{12}x_{12})$ which implies $y_{12} \cdot \big(s - (a_{12}+b_{12})\big)x_{12} = 0$.

For $y_{21} \in \mathfrak{R}_{21}$, from Lemma 4-(vi)

$$\begin{aligned} M^{*-1}(y_{21} \cdot sx_{12}) &= M^{*-1}(y_{21} \cdot a_{12}x_{12}) + M^{*-1}(y_{21} \cdot b_{12}x_{12}) \\ &= M^{*-1}(y_{21} \cdot a_{12}x_{12} + y_{21} \cdot b_{12}x_{12}). \end{aligned}$$

As a consequence $y_{21} \cdot sx_{12} = y_{21} \cdot a_{12}x_{12} + y_{21} \cdot b_{12}x_{12}$ which implies $y_{21} \cdot \big(s - (a_{12} + b_{12})\big)x_{12} = 0$.

Now, for $y_{22} \in \mathfrak{R}_{22}$, from Lemma 4-(vi)

$$\begin{aligned} M^{*-1}(y_{22} \cdot sx_{12}) &= M^{*-1}(y_{22} \cdot a_{12}x_{12}) + M^{*-1}(y_{22} \cdot b_{12}x_{12}) \\ &= M^{*-1}(y_{22} \cdot a_{12}x_{12} + y_{22} \cdot b_{12}x_{12}). \end{aligned}$$

From this, $y_{22} \cdot sx_{12} = y_{22} \cdot a_{12}x_{12} + y_{22} \cdot b_{12}x_{12}$ which implies $y_{22} \cdot \big(s - (a_{12} + b_{12})\big)x_{12} = 0$.

Hence $\mathfrak{R} \cdot \big(s - (a_{12} + b_{12})\big)x_{12} = 0$, and therefore

$$\big(s - (a_{12} + b_{12})\big)x_{12} = 0, \tag{6}$$

according to Theorem 2(ii). Moreover, from Equations 4, 5 and 6, we get that $\big(s-(a_{12}+b_{12})\big)\mathfrak{R} = 0$. Due to Theorem 2(i) we have $s = a_{12}+b_{12}$. □

Lemma 12. $M(a_{11} + b_{11}) = M(a_{11}) + M(b_{11})$.

Proof. Choose $s = s_{11} + s_{12} + s_{21} + s_{22} \in \mathfrak{R}$ such that $M(s) = M(a_{11}) + M(b_{11})$. Using Lemma 4-(ii) and (iv) we have $M(e_1e_1 \cdot s) = M(s)$ and $M(s \cdot e_1e_1) = M(s)$, which implies $s_{21} + s_{22} = 0$ and $s_{12} + s_{22} = 0$, respectively. Hence $s_{12} = s_{21} = s_{22} = 0$.

These equations show that $s = s_{11}$ and so $s - (a_{11} + b_{11}) \in \mathfrak{R}_{11}$. Next let $x_{12} \in \mathfrak{R}_{12}$ arbitrarily chosen. Applying Lemma 4-(iv) we get that

$$\begin{aligned} M(sx_{12}) &= M(s \cdot e_1x_{12}) = M(a_{11} \cdot e_1x_{12}) + M(b_{11} \cdot e_1x_{12}) \\ &= M(a_{11} \cdot e_1x_{12} + b_{11} \cdot e_1x_{12}) = M(a_{11}x_{12} + b_{11}x_{12}) \end{aligned}$$

which yields $sx_{12} = (a_{11} + b_{11})x_{12}$. Therefore $\big(s - (a_{11} + b_{11})\big)\mathfrak{R}_{12} = 0$ or still $\big(s - (a_{11} + b_{11})\big) \cdot \mathfrak{R}(1 - e_1) = 0$. Since $s - (a_{11} + b_{11}) \in \mathfrak{R}_{11}$, it follows from Theorem 2(iii) that $s = a_{11} + b_{11}$. □

Lemma 13. *M is additive on $e_1\mathfrak{R} = \mathfrak{R}_{11} + \mathfrak{R}_{12}$.*

Proof. The proof is the same as that of Martindale III (1969, Lemma 5) and is included for the sake of completeness. Indeed, let $a_{11}, b_{11} \in \mathfrak{R}_{11}$

and $a_{12}, b_{12} \in \mathfrak{R}_{12}$. According to Lemmas 7, 11 and 12

$$\begin{aligned} M\big((a_{11}+a_{12})+(b_{11}+b_{12})\big) &= M\big((a_{11}+b_{11})+(a_{12}+b_{12})\big) \\ &= M(a_{11}+b_{11})+M(a_{12}+b_{12}) \\ &= M(a_{11})+M(b_{11})+M(a_{12})+M(b_{12}) \\ &= M(a_{11}+a_{12})+M(b_{11}+b_{12}), \end{aligned}$$

as desired. □

Proof of Theorem 2. Suppose that $a, b \in \mathfrak{R}$ and choose $s \in \mathfrak{R}$ such that $M(s) = M(a) + M(b)$. For all $\alpha \in \Lambda$, M is additive on $e_\alpha\mathfrak{R}$ due to Lemma 13. Thus, for every $r \in \mathfrak{R}$, we have

$$\begin{aligned} M(e_\alpha \cdot rs) &= M(e_\alpha) \cdot M^{*-1}(r)M(s) = M(e_\alpha) \cdot M^{*-1}(r)\big(M(a)+M(b)\big) \\ &= M(e_\alpha) \cdot M^{*-1}(r)M(a) + M(e_\alpha) \cdot M^{*-1}(r)M(b) \\ &= M(e_\alpha \cdot ra) + M(e_\alpha \cdot rb) = M(e_\alpha \cdot ra + e_\alpha \cdot rb) \\ &= M\big(e_\alpha \cdot r(a+b)\big), \end{aligned}$$

and therefore $e_\alpha \cdot rs = e_\alpha \cdot r(a+b)$. Hence $e_\alpha \cdot \mathfrak{R}\big(s-(a+b)\big) = 0$, for every $\alpha \in \Lambda$. We then conclude that $s = a + b$ from Theorem 2(ii). This shows that M is additive on $\mathfrak{R}$.

In order to prove the additivity of M^*, let $x, y \in \mathfrak{R}'$. For $a, b \in \mathfrak{R}$, by using the additivity of M, we have

$$\begin{aligned} M\big(a\big(M^*(x)+M^*(y)\big) \cdot b\big) &= M\big(aM^*(x) \cdot b\big) + M\big(aM^*(y) \cdot b\big) \\ &= M(a)x \cdot M(b) + M(a)y \cdot M(b) \\ &= M(a)(x+y) \cdot M(b) \\ &= M\big(aM^*(x+y) \cdot b\big). \end{aligned}$$

It follows that $a\big(M^*(x)+M^*(y)-M^*(x+y)\big) \cdot b = 0$, for all $a, b \in \mathfrak{R}$, which forces $M^*(x+y) = M^*(x) + M^*(y)$ according to Theorem 2, completing the proof. □

Corollary 1. *Let $\mathfrak{R}$ be a 2 and 3-torsion free prime alternative ring containing a non-trivial idempotent ($\mathfrak{R}$ need not have an identity element), and let $\mathfrak{R}'$ be an arbitrary alternative ring. Then every surjective elementary map (M, M^*) of $\mathfrak{R} \times \mathfrak{R}'$ is additive.*

Proof. The result follows directly from Theorem 2 and [2, Theorem 2.2]. □

Corollary 2. *Let $\mathfrak{A}$ be a prime non-degenerate alternative algebra on a field of characteristic $\neq 2$ containing a idempotent ($\mathfrak{A}$ need not have an identity element), and let $\mathfrak{A}'$ be an arbitrary alternative algebra. Then every surjective elementary map (M, M^*) of $\mathfrak{A} \times \mathfrak{A}'$ is additive.*

Proof. The result follows directly from Theorem 2 and [1, Theorem 1]. □

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