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On the zero forcing number of graphs and their splitting graphs

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Abstract. In [10], the notion of the splitting graph of a graph was introduced. In this paper we compute the zero forcing number of the splitting graph of a graph and also obtain some bounds besides finding the exact value of this parameter. We prove for any connected graph Γ of order $n \geqslant 2$, $Z[S(\Gamma)] \leqslant 2Z(\Gamma)$ and also obtain many classes of graph in which $Z[S(\Gamma)] = 2Z(\Gamma)$. Further, we show some classes of graphs in which $Z[S(\Gamma)] < 2Z(\Gamma)$.

1. Introduction

Throughout this paper we use the notation Γ for the graph $G = (V, E)$ and we consider only simple, undirected and finite graphs. The zero forcing number of a graph $\Gamma = (V, E)$ is a new graph invariant introduced in [5]. In this paper we introduce the concept of zero forcing number of splitting graph of a graph Γ. The splitting graph of a graph Γ is the graph $S(\Gamma)$ obtained by taking a vertex v' corresponding to each vertex $v \in \Gamma$ and join v' to all vertices of Γ adjacent to v (see[10]). The zero forcing number $Z(\Gamma)$ of a graph Γ can be defined as follows:

- Color change rule : Let Γ be a graph with each vertex is colored either white or black. Suppose if u is a black vertex of Γ and exactly one neighbor v of u is white, then change the color of v to black.

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- Given a coloring of Γ, then the *derived coloring* is the result of applying the color-change rule until no more changes are possible.
- A *zero forcing set* Z for a graph Γ is a subset of the vertices of Γ such that if at first the vertices in Z are colored black and the remaining vertices are colored white, then the derived coloring of Γ yields a graph with all black vertices.
- $Z(\Gamma)$ is the minimum $|Z|$ over all zero forcing sets $Z \subseteq V(\Gamma)$ (see [5]).

When the color change rule is applied to a vertex u to change the color of v, we say u forces v and write $u \to v$. The sequence of $v_1 \to u_1$, $v_2 \to u_2$, ..., $v_k \to u_k$ is called a *forcing sequence* for Z (see [9]).

This parameter was found by the AIM Minium Rank Special Graphs Group(see [5]) and they used this parameter $Z(\Gamma)$ to bound the minimum rank for numerous families of graphs. The zero forcing set can also be used as a tool for logic circuits (see [3]).

In this paper, we initiate the study of the zero forcing number of the splitting graph $S(\Gamma)$ of a graph Γ. We start with some preliminary results. For more definitions on graphs we refer to [2] and [7]. We can find the following observation in [8].

Observation 1 ([8])**.** For any connected graph $\Gamma = (V, E)$, $Z(\Gamma) = 1$ if and only if $\Gamma = P_n$ for some $n \geqslant 1$.

It can be noted that if Γ is a connected graph of order $n \geqslant 3$, then $S(\Gamma)$ contains a cycle C_4. Therefore, by using the above observation we have the following.

Proposition 2. Let Γ be a connected graph of order $n \geqslant 3$. Then $Z[S(\Gamma)] \geqslant 2$, and this bound is sharp for the path P_n.

Proposition 3. For any connected graph $\Gamma = (V, E)$, $Z[S(\Gamma)] = 1$ if and only if Γ is the path P_2.

Proof. If $\Gamma = (V, E)$ is the path P_2, then $S(\Gamma)$ is the path P_4 and therefore $Z[S(\Gamma)] = 1$. The converse follows from Observation 1. □

2. Bounds on $Z[S(\Gamma)]$

In this section we prove some bounds on the zero forcing number of $S(\Gamma)$.

Theorem 4. Let Γ be a connected graph of order $n \geqslant 3$. Then $Z[S(\Gamma)] \leqslant 2Z(\Gamma)$.

Proof. Consider any minimum zero forcing set Z of Γ. Let $Z = \{v_1, v_2, \dots, v_k\}$, $1 \leqslant k \leqslant n$ be a minimum zero forcing set of Γ. Now consider the set

$$Z^{'} = \{v_1, v_2, \dots, v_k\} \cup \{v_1', v_2', \dots, v_k'\} \in V[S(\Gamma)],$$

where $\{v_1', v_2', \dots, v_k'\}$ be the copies of the vertices of $\{v_1, v_2, \dots, v_k\}$ in $V[S(\Gamma)]$. Color all vertices in $Z^{'}$ as black.

We show that the set $Z^{'}$ forms a zero forcing set for $S(\Gamma)$. Now consider the vertices in Γ which has exactly one white neighbor in Γ. Let it be $v_1, v_2, \dots, v_l, l \leqslant k$ and $v_1', v_2', \dots, v_l'$ be the corresponding vertices of $v_1, v_2, \dots, v_l$ in $S(\Gamma)$. Now we can see that in $S(\Gamma)$, $N(v_1')$, $N(v_2')$, …, $N(v_l')$, each one contains exactly one white vertex. Let it be $u_1, u_2, \dots, u_l$. Now clearly $v_1' \to u_1$, $v_2' \to u_2$, …, $v_l' \to u_l$. Again consider the set $\{v_1, v_2, \dots, v_l\}$ in $S(\Gamma)$. At this time we can see that $v_1 \to u_1'$, $v_2 \to u_2'$,…, $v_l \to u_l'$. Consider the white vertices which are adjacent to $u_1, u_2, \dots, u_l$ in Γ. Let it be $w_1, w_2, \dots, w_l$. Clearly $u_1' \to w_1$ $u_1 \to w_1'$ and so on. Therefore the set $Z^{'}$ forms a zero forcing set for $S(\Gamma)$. □

A subset $D \subseteq V(\Gamma)$ is called a dominating set if $V - D$ is adjacent to at least one vertex in D. The minimum number of vertices in such a set D is called the domination number of Γ and is denoted by $\gamma(\Gamma)$. A dominating set which is connected is known as a connected dominating set and the minimum number of vertices in any connected dominating set is called the connected domination number $\gamma_c(\Gamma)$ (see [11]). In [1] Amos et al. (2015) determined the following upper bound on the zero forcing number.

Corollary 5. *(cf [1, Corollary 4.3])* For any connected graph Γ of order $n \geqslant 2$, $Z(\Gamma) \leqslant n - \gamma_c(\Gamma)$.

Characterization of graphs in which $Z(\Gamma) = n - \gamma_c(\Gamma)$ still remains an open problem.

From Theorem 4 and Corollary 5 we conclude the following upper bound.

Proposition 6. For any connected graph Γ of order $n \geqslant 2$, $Z[S(\Gamma)] \leqslant 2[n - \gamma_c(\Gamma)]$, and this inequality is sharp.

Proof. Note that Theorem 4 yields

$$Z[S(\Gamma)] \leqslant 2Z(\Gamma) \tag{1}$$

whereas Corollary 5 yields

$$Z(\Gamma) \leqslant n - \gamma_c(\Gamma). \tag{2}$$

From (1) and (2) the result follows. To see that the bound is sharp, consider cycles of order $n \geqslant 4$. $\square$

3. Families of graphs where $Z[S(\Gamma)] = 2Z(\Gamma)$

It is an open problem to characterize families of graphs does $Z[S(\Gamma)] = 2Z(\Gamma)$. In this section we provide some familiar families of graphs for which the equality $Z[S(\Gamma)] = 2Z(\Gamma)$ holds. We start with paths and cycles.

Proposition 7. If Γ is the path P_n on $n \geqslant 3$ vertices, then $Z[S(\Gamma)] = 2 = 2[Z(\Gamma)]$.

Proposition 8. If Γ is the cycle C_n on $n \geqslant 4$ vertices, then $Z[S(\Gamma)] = 4 = 2[Z(\Gamma)]$.

Proof. Let $v_1, v_2, ..., v_n$ be the vertices of C_n and let $v_1', v_2', ..., v_n'$ be the corresponding vertices of C_n in $S(C_n)$. Consider the set $Z = \{v_1, v_2', v_2, v_3'\}$. Color these vertices with the color black. Now $v_2' \to v_3$, $v_3' \to v_4$ and $v_3 \to v_4'$ and so on. Therefore the set Z forms a zero forcing set and hence $Z[S(\Gamma)] \leqslant 4$. We can easily verify that with 3 black vertices it is not possible to change the color of all other vertices to black. Hence $Z[S(\Gamma)] = 4$. $\square$

If Γ is the graph K_3 on 3 vertices, then we can choose the black vertices depicted in Figure 1 as the zero forcing set of $S(\Gamma)$. Therefore, $Z[S(K_3)] = 3$.

Proposition 9. If Γ is the star $K_{1,n}$ on $n+1$ vertices, then $Z[S(\Gamma)] = 2n - 2 = 2(n-1) = 2[Z(\Gamma)]$.

Proof. Assume that we have a zero forcing set Z consisting of $2n-3$ vertices. Then the number of white vertices in Z is $2n+2-(2n-3) = 5$. Consider the five white vertices in $S(\Gamma)$. Consider the case when either two of them will be in A-part or two of them will be in B-part. We can easily verify that in this case the color changing rule is not possible, a contradiction. Therefore we need at least $2n-2$ black vertices in any zero forcing set of $S(\Gamma)$ and hence

$$Z[S(\Gamma)] \geqslant 2n - 2 \tag{3}$$

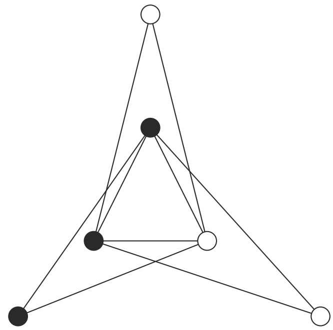

Figure 1. The splitting graph of K_3 with $Z[S(K_3)] = 3$.

Consider the 4-white vertices as depicted in Figure 2. Consider one black vertex from A-part, this black vertex forces the vertex u to black. Change the color of u to black. Again consider one black vertex from B-part, this black vertex forces the vertex w to black. Again change the color of w to black. If we consider the vertex w, then there is exactly one neighbor which is white. Change the color of this vertex to black. In a similar manner we can change the color of other vertex to black. Now we get a derived coloring of $S(\Gamma)$ with all vertices colored black. This implies,

$$Z[S(\Gamma)] \leqslant 2n - 2. \tag{4}$$

From (3) and (4) the result follows. $\square$

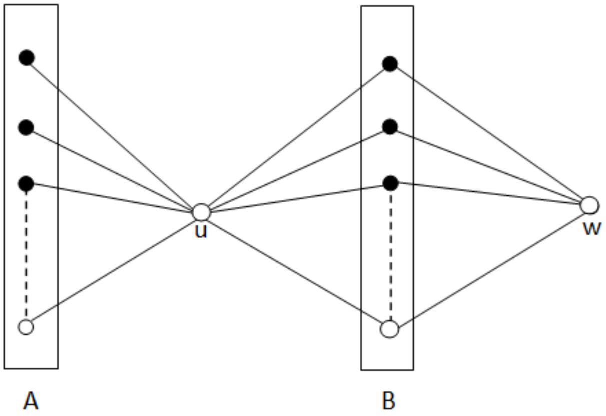

Figure 2.

Proposition 10 ([5])**.** For any graph Γ, $Z(\Gamma) \geqslant \delta(\Gamma)$, where $\delta(\Gamma)$ denote the minimum degree of the graph Γ.

Proposition 11. Let Γ be a connected graph with $Z(\Gamma) = k = \delta(\Gamma)$ and let $\hat{\Gamma}$ be the graph obtained from Γ by adding a single vertex v and joining it to all other vertices of G. Then $Z(\hat{\Gamma}) = Z(\Gamma) + 1$.

Proof. Since Γ is a graph with $Z(\Gamma) = \delta(\Gamma)$ and we have from Proposition 10 $\delta(\hat{\Gamma}) \leqslant Z(\hat{\Gamma})$. Let v be a vertex in Γ with $\delta(\Gamma) = k$. In $\hat{\Gamma}$, $\delta(\hat{\Gamma}) = k + 1 = \delta(\Gamma) + 1$. Therefore, $\delta(\Gamma) + 1 \leqslant Z(\hat{\Gamma})$. This implies,

$$Z(\Gamma) + 1 \leqslant Z(\hat{\Gamma}). \tag{5}$$

Now color the vertex v which is connected to all other vertices of Γ by black. Now $Z(\Gamma) \cup \{v\}$ forms a zero forcing set for $Z(\hat{\Gamma})$. This implies

$$Z(\hat{\Gamma}) \leqslant Z(\Gamma) + 1. \tag{6}$$

From (5) and (6) the result follows. □

A wheel graph is a graph obtained by connecting a single vertex to all vertices of a cycle graph C_{n-1}. If Γ is the cycle graph, then $Z(\Gamma) = 2$ (see [9]). By using Proposition 11 we can easily verify that if Γ is the wheel graph, then $Z(\Gamma) = 3$.

Proposition 12. Let Γ be the wheel graph with $n \geqslant 5$ vertices obtained by connecting a single vertex to all vertices of the cycle graph C_{n-1}. Then $Z[S(\Gamma)] = 6$.

Proof. From the above note $Z[\Gamma] = 3$ and from Theorem 3, $Z[S(\Gamma)] \leqslant 2\ Z(\Gamma)$, we can conclude the following

$$Z[S(\Gamma)] \leqslant 6. \tag{7}$$

To prove the reverse part assume $Z[S(\Gamma)] = 5$. Divide the graph $S(\Gamma)$ into three parts as shown in Figure 3.

Let $v_1, v_2, \ldots, v_{n-1}$ be the vertices in $S(\Gamma)$ with $\deg(v_i) = 6$, $1 \leqslant i \leqslant n-1$, $v'_1, v'_2, \ldots, v'_{n-1}$ be the vertices in $S(\Gamma)$ with $\deg(v'_i) = 3$, $1 \leqslant i \leqslant n-1$ and let v_n be the vertex which is adjacent to $\{v_1, v_2, \ldots, v_{n-1}\} \cup \{v'_1, v'_2, \ldots, v'_{n-1}\}$ and v'_n be the vertex which is adjacent to $\{v_1, v_2, \ldots, v_{n-1}\}$ with $\deg(v_n) = 2n-2$ and $\deg(v'_n) = n-1$.

Case 1. $\{v_n, v'_n\} \in Z$. Now $S(\Gamma) - \{v_n, v'_n\} = S(C_n)$. We know that from proposition 8 $Z[S(C_n)] = 4$. This implies $Z[S(\Gamma)] = 4 + 2 = 6 \neq 5$, a contradiction.

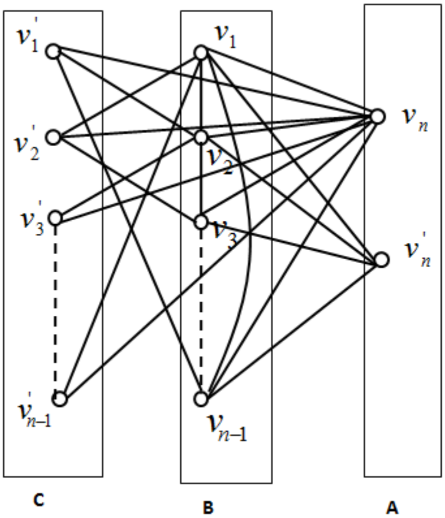

FIGURE 3.

Case 2. Suppose $v_n \in Z$ and $v'_n \notin Z$. Now we have four vertices remains in Z. If we use these four vertices to begin the color changing rule, then we can observe that with these 4-vertices we can change the color of at most two vertices to black, not all. A contradiction.

Case 3. Suppose $v_n \notin Z$ and $v'_n \in Z$. Now we have four vertices remains in Z. If we use these four vertices to begin the color changing rule, then we can observe that with these 4-vertices we can change the color of at most two vertices to black, not all. A contradiction. □

We now prove one more additional family of graphs in which $Z[S(\Gamma)] = 2Z(\Gamma)$. The following definition can be found in [1].

Definition 13 ([1])**.** A connected graph $\Gamma = (V, E)$ is defined as a cycle-path graph (CP-graph) if it contains r vertex disjoint cycles that are connected by $r-1$ edges of the path P_r. Thus a CP-graph with n vertices contains $m = n + r - 1$ edges and edge between two cycles is a cut edge.

Example 14. Let Γ be the graph depicted in Figure 4. Then Γ represents the CP-graph with the cycle C_4 and the path P_3. That is the graph Γ is the C_4P_3-graph.

Proposition 15. Let Γ be the CP-graph with r vertex disjoint cycles. Then $Z(\Gamma) = r + 1$.

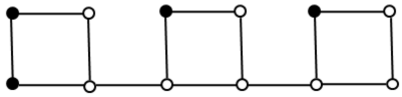

FIGURE 4.

Proof. We proceed by induction on the number of cycles r. Assume that $r = 1$. In this case Γ is a cycle, $Z(\Gamma) = 2 = r + 1$. Assume the theorem is true for all CP-graphs with $r - 1$ cycles, where $r \geqslant 2$. Let C be an end-cycle that is a cycle connected to rest of the graph by a unique edge $e = \{u, v\}$, where $u \in V(\Gamma) - C$ and $v \in C$. The induced subgraph $< \Gamma[V - C] >$ is a CP-graph with $r - 1 < r$ cycles. Assume, the result is true for $< \Gamma[V - C] >$, that is, $Z(< \Gamma[V - C] >) = r - 1 + 1 = r$.

Let S be a minimum zero forcing set of $< \Gamma[V - C] >$ and let w be a neighbor of v on C. Consider the set $Z = S \cup \{w\}$. Since $\{u, v\}$ is the only cut edge between $< \Gamma[V - C] >$ and C, therefore we can start the color changing of the vertices of $< \Gamma[V - C] >$ with S vertices. Since u is a black vertex and the only white vertex which is adjacent to u is v therefore, $u \to v$ to black. Now in C we can see that $\{u, w\}$ forms zero forcing set, where $u \in Z(< \Gamma[V - C] >)$. Therefore by induction hypothesis $Z(\Gamma) = Z(< \Gamma[V - C] >) + |\{w\}| = r + 1$. □

Proposition 16. Let Γ be the CP-graph with r-vertex disjoint cycles C_n of order $n \geqslant 4$. Then $Z[S(\Gamma)] = 2(r + 1)$.

Proof. We prove the result by induction on the number of cycles r on the CP-graph. Assume that $r = 1$. In this case Γ is a cycle, we have from Proposition 8, $Z[S(\Gamma)] = 2(1 + 1) = 4$. Assume the result is true for all CP-graphs with $r - 1$ cycles C_n, where $r \geqslant 2$. Let C be an end-cycle that is a cycle connected to rest of the CP-graph by a unique edge $e = \{u, v\}$, where $u \in V(\Gamma) - C$ and $v \in C$ and let $S(C)$ be the splitting graph of the cycle C in $S(\Gamma)$. Now $S(C)$ is connected to the rest of $S(\Gamma)$ by three edges. Let these edges be $X = \{u_1v_1, u_1v_2, u_2v_1\}$, where $\{u_1, u_2\} \in < V[S(\Gamma)] - V[S(C)] >$ (that is the subgraph induced by $V[S(\Gamma)] - V[S(C)]$)and $\{v_1, v_2\} \in S(C)$. Assume, the result is true for $< V[S(\Gamma)] - V[S(C)] >$, that is, $Z\{< V[S(\Gamma)] - V[S(C)] >\} = 2[(r - 1) + 1] = 2r$.

Let U be a minimum zero forcing set of $< V[S(\Gamma)] - V[S(C)] >$, let w_1 be the neighbor of v_1 in $V[S(C)]$ and w_1' be the corresponding vertex of w_1 in $V[S(C)]$. Consider the set $Z = U \cup \{w_1, w_1'\}$. Since X is a cut set between $S(\Gamma) - S(C)$ and $S(C)$ therefore, the set U forces the vertices v_1

and v_2 to black. Now $\{w_1, w'_1\}$ is in Z. Therefore the set $\{v_1, v_2, w_1, w'_1\}$ forms a zero forcing set of $S(C)$ in $S(\Gamma)$. Therefore by induction hypothesis $Z[S(\Gamma)] =< V[S(\Gamma)] - V[S(C)] > +|\{w_1, w'_1\}| = 2r + 2$. □

The *Cartesian product* $G\Box H$ of two graphs G and H is the graph with vertex set equal to the Cartesian product $V(G) \times V(H)$ and where two vertices (u, v) and (u', v') are adjacent in $G\Box H$ if $u = u'$ and $vv' \in E(H)$ or $uu' \in E(G)$ and $v = v'$. Ladder graph is the graph obtained by taking the Cartesian product of P_n with P_2. In [5], it was proved that if Γ is the ladder graph, then $Z(\Gamma) = 2$. We now prove one more additional family of graphs in which $Z[S(\Gamma)] = 2\ Z(\Gamma)$.

Proposition 17. If Γ is the splitting graph of the ladder graph, then $Z(\Gamma) = 4$.

Proof. Consider the graph Γ depicted in Figure 5. The set of left black vertices of the graph Γ forms a zero forcing set of Γ. It can be easily verified that with three vertices we cannot form a zero forcing set. Therefore, $Z(\Gamma) = 4$. □

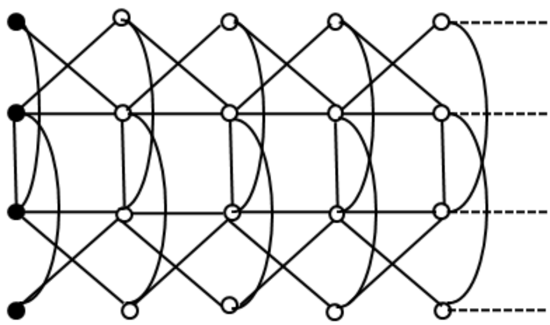

Figure 5.

4. Families of graphs where $Z[S(\Gamma)] < 2Z(\Gamma)$

We start this section with a CP-graph family in which $Z[S(\Gamma)] < 2Z(\Gamma)$. Let us consider the CP-graph C_3P_r, where C_3 is the cycle graph on 3 vertices and P_r is the path on $r > 1$ vertices.

Proposition 18. Let Γ be the C_3P_r - graph. Then $Z[S(\Gamma)] \leqslant 2r < 2r+2$.

We now obtain a formula for the zero forcing number of the friendship graph F_n. The friendship graph F_n can be obtained from the wheel graph by deleting the alternate edges of the cycle C_{n-1} where n is odd. Also F_n can be obtained by coalescing k copies of the cycle graph C_3 with a common vertex (see [4]).

The following Lemma can be found in [9].

Lemma 19 ([9])**.** Let $\Gamma = (V, E)$ be a graph with cut-vertex $v \in V(\Gamma)$. Let $X_1, \ldots, X_k$ be the vertex sets for the connected components of $\Gamma - v$, and for $1 \leqslant i \leqslant k$, let $\Gamma_i = \Gamma[X_i \cup \{v\}]$. Then $Z(\Gamma) \geqslant \sum\limits_{i=1}^{k} Z(\Gamma_i) - k + 1$.

Theorem 20. Let F_n be the friendship graph with k copies of the cycle graph C_3. Then $Z(F_n) = \lfloor n/2 \rfloor + 1$.

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of F_n and v_n be the central vertex. The cycle graph C_3 is a complete graph of order three. Therefore, $Z(C_3) = 2$. Since v_n is a cut vertex, $G - v_n$ will have $\lfloor n/2 \rfloor$ components. Lemma 19 gives

$$Z(F_n) \geqslant 1 - \lfloor n/2 \rfloor + 2\lfloor n/2 \rfloor = \lfloor n/2 \rfloor + 1. \tag{8}$$

To establish the reverse inequality consider the following set of black vertices

$$Z = \{v_1, v_3, \ldots, v_{n-2}\} \cup \{v_n\}.$$

Clearly the vertices v_1 and v_n are black therefore, the vertex $v_1 \to v_2$ to black. The vertices v_3 and v_n are black therefore, the vertex $v_3 \to v_4$ to black, and so on. Similarly the vertices v_{n-2} and v_n are black therefore, the vertex $v_{n-2} \to v_{n-1}$ to black. Now we get a derived coloring with the zero forcing set Z. The number of vertices in Z is $\lfloor n/2 \rfloor + 1$ and hence

$$\lfloor n/2 \rfloor + 1 \geqslant Z(F_n). \tag{9}$$

Therefore from (8)and (9) the result follows. □

Lemma 21. Let $S(F_n)$ be the splitting graph of F_n and let

$$A_l = \{v_k, v'_k, v_j, v'_j\}, 1 \leqslant l \leqslant \lfloor n/2 \rfloor$$

(v_k, v_j is an edge in F_n and $i, j \neq n$) be the set of vertices of $S(F_n)$ obtained by deleting the vertices v_n and v'_n from $S(F_n)$. Then atleast one vertex from the set A_l will be in any optimal zero forcing set of $S(F_n)$.

Proof. On the contrary assume that non of them belongs to any Z that is, $v_i, v'_i, v_j, v'_j \notin Z$. In any color changing rule v_n and v'_n will never force the vertices in A to black since $N(v_n)$ and $N(v'_n)$ have two white neighbors in A. Therefore at lest one vertex from the set A will be in Z. □

Theorem 22. Let F_n be the friendship graph with $\lfloor n/2 \rfloor$ copies of the cycle graph C_3. Then $Z[S(F_n)] = \lfloor n/2 \rfloor + 2$, where $S(F_n)$ denote the splitting graph of the friendship graph.

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of F_n and let v_n be the common vertex obtained by coalescing k copies of the cycle graph C_3. Let $v'_1, v'_2, \ldots, v'_n$ be the copies of the the vertices $v_1, v_2, \ldots, v_n$ of $S(F_n)$.

Consider the set $Z = \{v_n, v'_n, v'_1, v'_3, v'_5, \ldots, v'_{n-2}\}$. Also let T_1 be the triangle in F_n with $V(T_1) = \{v_1, v_2, v_n\}$ and $V(T'_1) = \{v'_1, v'_2, v'_n\}$ be the copies of the vertices of T_1 in $S(F_n)$.

We can see that in color changing rule the vertex v'_1 forces the vertex v_2 to black and then the vertex v_2 forces the vertex v_1 to black and then the vertex v_1 forces the vertex v'_2 to black. Clearly the set $\{v_n, v'_n, v'_1\}$ forms a zero forcing set of $V(T_1) \cup V(T'_1)$. In a similar manner we can prove that $\{v_n, v'_n, v'_3, v'_1\}$ forms a zero forcing set of $[V(T_1) \cup V(T'_1)] \cup [v(T_2) \cup v(T'_2)]$, and so on. Therefore, the set $Z = \{v_n, v'_n, v'_1, v'_3, v'_5, \ldots, v'_{n-2}\}$ forms a zero forcing set of $Z[S(F_n)]$ and hence

$$Z[S(F_n)] \leqslant \lfloor n/2 \rfloor + 2. \tag{10}$$

To prove the reverse part assume there exist a zero forcing set consisting of $\lfloor n/2 \rfloor + 1$ vertices. Now we consider the following cases.

Case 1. The vertex $v'_n \notin Z$. Since $\deg(v_i) = 4$ for $i \neq n$ and in $S(F_n)$, the vertex v'_n is adjacent to all vertices of the friendship graph F_n except the vertex v_n. Therefore, in any color changing rule to force v'_n we need two more vertices from the set A_l, a contradiction. If we take two more vertices from the set A_l then we get a zero forcing set. Therefore, it is clear from lemma 21 that $Z[S(F_n)] \geqslant \lfloor n/2 \rfloor + 2$.

Case 2. The vertex $v'_n \in Z$. We have from the lemma 21 that we need at least one vertex from A_l to get a zero forcing set. With out loss of generality assume that $B = \{v'_1, v'_3, \ldots, v'_{n-2}\}$ are the black vertices of $S(F_n)$. $B \cup v'_n$ will never force v_n to black, a contradiction. Therefore we need at least one more vertex from A_l to get a zero forcing set of $S(F_n)$. Hence $Z[S(F_n)] \geqslant \lfloor n/2 \rfloor + 2$. □

The generalized friendship graph F^*_p is the graph obtained by joining k copies of the cycle graph $C_n, n \geqslant 3$ and $k \geqslant n$ with a common vertex v.

The following theorem provides the zero forcing number of the generalized friendship graph F_p^*. Here p denotes the number of vertices in F_p^* that is $p = k(n-1)+1$.

Proposition 23. Let F_p^* be the graph obtained by joining k copies of the cycle graph $C_n, n \geqslant 4$ and $k \geqslant n$ with a common vertex v. Then $Z(F_p^*) = k+1$.

Proof. Let $v_1, v_2, \ldots, v_p$ be the vertices of F_p^* and $v_p = v$ be the central vertex. It is known that for the cycle graph C_n, $Z(C_n) = 2$. Now lemma 19 yields

$$Z(F_p^*) \geqslant \sum_{i=1}^{k} Z(C_n) - k + 1 = 2k - k + 1 = k + 1. \tag{11}$$

To establish the reverse inequality consider one vertex from each cycle graph C_n which is adjacent to the central vertex v. Denote the cycles $C_1, C_2, \ldots, C_k$ in F_p^* as follows.

$$\begin{aligned} C_1 &= v_1^1, v_2^1, \ldots, v_p, v_1^1 \\ C_2 &= v_1^2, v_2^2, \ldots, v_p, v_1^2 \\ &\vdots \qquad \vdots \qquad \vdots \\ C_k &= v_1^k, v_2^k, \ldots, v_p, v_1^k. \end{aligned}$$

Consider the set of black vertices $Z = \{v_1^1, v_1^2, \ldots, v_1^k\} \cup \{v_p\}$. Now we can see that $N(v_1^1)$ contains only one white vertex v_2^1. Therefore, $v_1^1 \to v_2^1$, $v_2^1 \to v_3^1$ and so on. Similarly we can see that $N(v_1^2)$ contains only one white vertex v_2^2. Therefore, $v_1^2 \to v_2^2$, $v_2^2 \to v_3^2$ and so on. In the cycle C_k we can see that $N(v_1^k)$ contains only one white vertex v_2^k. Therefore, $v_1^k \to v_2^k$, $v_2^k \to v_3^k$ and so on. Now the set Z forms a zero forcing set and hence

$$Z(F_p^*) \leqslant k + 1. \tag{12}$$

Then the result is an immediate consequence of (11) and (12). □

Proposition 24. Let F_p^* be the graph obtained by joining k copies of the cycle graph $C_n, n \geqslant 4$ and $k \geqslant n$ with a common vertex v. Then $Z(S(F_p^*)) \leqslant 2k+2$.

Proof. First we note that Proposition 23 yields $Z(F_p^*) = k+1$, and Theorem 3 yields $Z[S(F_p^*)] \leqslant 2\ Z(F_p^*)$. By applying these two results we get

$$Z(S(F_p^*)) \leqslant 2k + 2. \tag{13}$$

□

5. $Z(\Gamma)$ and $P(\Gamma)$ of the splitting graph of a graph

A path covering of a graph Γ is a set of vertex disjoint paths of Γ containing all the vertices of Γ. The minimum number of paths in any minimal path cover of Γ is called the path covering number of Γ and is denoted by $\mathbf{P}(\Gamma)$.

Proposition 25 ([6])**.** For any connected graph Γ, $P(\Gamma) \leqslant Z(\Gamma)$.

We can find the following open question in [9].

Problem 26. For what families of graphs does $Z(\Gamma) = \mathbf{P}(\Gamma)$?

For brevity let us call these families of graphs as **ZP**-graphs. A graph Γ is said to be **ZP** if $Z(\Gamma) = \mathbf{P}(\Gamma)$. Now the characterization of **ZP**-graphs is an open problem. Trees and unicyclic graphs are **ZP**-graphs (see[9]). In this section we prove more families of **ZP**-graphs.

Proposition 27. If Γ is the splitting graph of the path P_n on $n \geqslant 3$ vertices, then $Z(\Gamma) = 2 = \mathbf{P}(\Gamma)$.

Proposition 28. If Γ is the splitting graph of the star $K_{1,n}$ on $n+1$-vertices, then $Z(\Gamma) = 2n-2 = \mathbf{P}(\Gamma)$.

Proof. With out loss of generality we can assume that Γ is the splitting graph of the star $K_{1,n}$. By Proposition 9 we have $Z(\Gamma) = 2n-2$. Now we prove $\mathbf{P}(\Gamma) = 2n-2$. We consider the following three cases.

Case 1. Suppose if we take two vertex disjoint path of length 1 (that is the complete graph K_2) to cover the graph Γ, then it must include the vertices u and w (refer Figure 2). If we include u and w in these vertex disjoint paths, then their remains $2n-2$ uncovered vertices. To count these vertices in the path covering we have to choose each of them as independent paths. In this case the total number of paths we need to cover the entire vertices in Γ is $2n-2+2 = 2n$.

Case 2. Suppose if we take two vertex disjoint paths of length 2 (that is the graph P_3, the path on three vertices) to cover the graph Γ (Take two vertices from part-A and the vertex u as the path P_1. Similarly take any two vertices from part-B and the vertex w as the path P_2 (refer Figure 2)). As in Case-1, the total number of paths we need to cover the entire vertices in Γ is $2n+2-6+2 = 2n-2$.

Case 3. Suppose if we consider a path of length 3 (that is the graph P_4, the path on four vertices) as a path to cover the graph Γ, then it is not possible to choose a path of length 2 or 3 as a path to cover the vertices.

Now as in Case-1, the total number of paths we need to cover the entire vertices in Γ is $2n + 2 - 4 + 1 = 2n - 1$.

From the above three cases, we can conclude that the minimum number of vertex disjoint paths possible to cover the vertices in Γ is occurred in Case-2 and is $2n - 2$. Therefore, $\mathbf{P}(\Gamma) = 2n - 2$. $\square$

6. Conclusion and Open Problems

In the paper we address the problem of determining the zero forcing number of graphs and their splitting graphs. In Section 2, we give upper bounds on the zero forcing number of the splitting graph of a graph. In Section 3, we have found several classes of graphs in which $Z[S(\Gamma)] = 2Z(\Gamma)$. Section 4 provides classes of graphs in which $Z[S(\Gamma)] < 2Z(\Gamma)$. In Section 5, we have proved more families of graphs does $Z(\Gamma) = P(\Gamma)$.

There are few questions that remains open, for example see the following.

Problem 29. Characterize the graphs Γ for which $2Z(\Gamma) = Z[S(\Gamma)]$?

We know that the above equality is true for many classes of graphs. For example, consider the paths and the cycles. Another challenging question which we have not proved is the following.

Problem 30. Characterize the graphs Γ for which $P[S(\Gamma)] = Z[S(\Gamma)]$?

We have proved that for the paths and the star the above equality holds.

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