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Generalized classes of suborbital graphs for the congruence subgroups of the modular group

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Abstract. Let Γ be the modular group. We extend a nontrivial Γ-invariant equivalence relation on $\widehat{\mathbb{Q}}$ to a general relation by replacing the group $\Gamma_0(n)$ by $\Gamma_K(n)$, and determine the suborbital graph $\mathcal{F}^K_{u,n}$, an extended concept of the graph $\mathcal{F}_{u,n}$. We investigate several properties of the graph, such as, connectivity, forest conditions, and the relation between circuits of the graph and elliptic elements of the group $\Gamma_K(n)$. We also provide the discussion on suborbital graphs for conjugate subgroups of Γ.

Introduction

Let G be a permutation group acting transitively on a nonempty set X. Then the action of G can be extended naturally on $X \times X$ by

$$g(v, w) = (g(v), g(w)),$$

where $g \in G$ and $v, w \in X$. The orbit $G(v, w)$ is called a *suborbital* of G containing (v, w). A *suborbital graph* $\mathcal{G}(v, w)$ for G on the set X is a graph whose vertex set is the set X and the family of directed edges is

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the suborbital $G(v,w)$. Hence, there exists a directed edge from v_1 to v_2, denoted by $v_1 \to v_2$, if $(v_1,v_2) \in G(v,w)$.

The concept of suborbital graphs was first introduced by Sims [14]. Then Jones, Singerman, and Wicks [8] used this idea to construct the suborbital graphs $\mathcal{G}_{u,n}$ for the modular group Γ acting on the extended set of rational numbers $\widehat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$. To examine the properties of $\mathcal{G}_{u,n}$, they applied the fact that the action of Γ on $\widehat{\mathbb{Q}}$ is *imprimitive*, i.e., there is a Γ-invariant equivalence relation other than the two trivial relations which form the partitions $\{\widehat{\mathbb{Q}}\}$ and $\{\{v\} : v \in \widehat{\mathbb{Q}}\}$. They used the congruence subgroup $\Gamma_0(n)$ to induce the nontrivial Γ-invariant equivalence relation on $\widehat{\mathbb{Q}}$, and studied the subgraphs $\mathcal{F}_{u,n}$ of the graphs $\mathcal{G}_{u,n}$ restricted on the block $[\infty]$, the equivalence class containing ∞. Note that the graph $\mathcal{G}_{u,n}$ is the union of m copies of $\mathcal{F}_{u,n}$, where m is the index of $\Gamma_0(n)$ in Γ. Moreover, if $\mathcal{F}_{u,n}$ contains edges, it is actually a suborbital graph for $\Gamma_0(n)$ on the block $[\infty]$.

There are several studies related to the graphs for the modular group, see [1, 4, 5, 7, 11, 13], and other papers about suborbital graphs for other groups, see [2, 3, 6, 9, 10, 12]. In [11], the authors used the different Γ-invariant equivalence relation obtained from another congruence subgroup $\Gamma_1(n)$ of Γ, and investigated the connectivity of subgraphs of $\mathcal{G}_{u,n}$ on the block containing ∞.

Inspired by the results in [8, 11], we introduce a Γ-invariant equivalence relation using the congruence subgroup $\Gamma_K(n)$ where K is a subgroup of the group of unit $\mathbb{Z}_n^*$. This group is a generalization of $\Gamma_0(n)$ and $\Gamma_1(n)$, so it provides a generalized Γ-invariant equivalence relation of those induced from $\Gamma_0(n)$ and $\Gamma_1(n)$. We denote by $\mathcal{F}_{u,n}^K$ the subgraph of $\mathcal{G}_{u,n}$ on the block $[\infty]_K$ with respect to the group $\Gamma_K(n)$, and demonstrate various properties of $\mathcal{F}_{u,n}^K$, such as, connectivity, forest conditions, including the relation between circuits of the graph and elliptic elements of the group $\Gamma_K(n)$. In the final section we provide a discussion of the relation of suborbital graphs for congruence subgroups. We show that the suborbital graphs for the group $\Gamma^0(n)$ studied in [7] is isomorphic to some graph $\mathcal{F}_{u,n}$. The result is also extended to the case of $\Gamma_K(n)$ and $\Gamma^K(n)$, a generalization of $\Gamma^0(n)$. Moreover, we discuss suborbital graphs for $\Gamma_K(n)$ on $[\infty]_K$ which are more general than $\mathcal{F}_{u,n}^K$.

This work can be restricted to the case of $\Gamma_1(n)$ be replacing the group K by the trivial subgroup $\{\overline{1}\}$ of $\mathbb{Z}_n^*$. This case was studied in [11] already; however, the results in there are different from ours because of the definition of $\Gamma_1(n)$. The differences will be explained in another our publication.

1. Preliminaries

Let Γ be a set of all *linear fractional (Möbius) transformations* on the upper half-plane $\mathbb{H}^2 = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$ of the form

$$z \mapsto \frac{az+b}{cz+d}, \tag{1}$$

where $a, b, c, d \in \mathbb{Z}$, and $ad - bc = 1$. With the composition of functions, Γ forms a group which is called the *modular group*. The group Γ is isomorphic to $\text{PSL}(2, \mathbb{Z})$, the quotient group of the unimodular group $\text{SL}(2, \mathbb{Z})$ by its centre $\{\pm I\}$. Thus, every element of Γ of the form (1) can be referred to as the pair of matrices

$$\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}).$$

For convenience, we may leave the sign of matrices representing elements of the group Γ and identify them with their negative sign.

Let n be any natural number. One can show that

$$\Lambda(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}) : a \equiv 1 \mod n, \text{ and } b \equiv c \equiv 0 \mod n \right\}$$

is a subgroup of $\text{SL}(2, \mathbb{Z})$. The image of $\Lambda(n)$ in $\Gamma = \text{PSL}(2, \mathbb{Z})$ under the quotient mapping is called the *principal congruence subgroup of level n* and denoted by $\Gamma(n)$. We can see easily that

$$\Gamma(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : a \equiv \pm 1 \mod n, \text{ and } b \equiv c \equiv 0 \mod n \right\}.$$

A subgroup of Γ containing $\Gamma(n)$, for some n, is called a *congruence subgroup* of Γ. There are two well-known congruence subgroups of the modular group, that is,

$$\Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0 \mod n \right\},$$

and

$$\Gamma_1(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : a \equiv \pm 1 \mod n, \text{ and } c \equiv 0 \mod n \right\}.$$

These two groups are mainly used in [8] and [11], respectively.

We now introduce some classes of congruence subgroups of the modular group. Let K be a subgroup of a group of units $\mathbb{Z}_n^*$, and $\overline{a}_n$ denote a congruence class containing an integer a modulo n. Without the confusion, we may leave the subscript n and use $\overline{a}$ instead. One can prove easily that

$$\Lambda_K(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) : \overline{a} \in K, \text{ and } c \equiv 0 \mod n \right\}$$

is a subgroup of $\mathrm{SL}(2,\mathbb{Z})$ containing the group $\Lambda(n)$, so the image in Γ of this group is certainly a congruence subgroup of Γ. We let $\Gamma_K(n)$ denote the congruence subgroup of Γ obtained in this way. Obviously,

$$\Gamma_K(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : \overline{a} \in -K \cup K, \text{ and } c \equiv 0 \mod n \right\},$$

where $-K = \{-\overline{a} : \overline{a} \in K\}$. In the case that K is a trivial subgroup of $\mathbb{Z}_n^*$, $\{\overline{1}\}$ or $\mathbb{Z}_n^*$, $\Gamma_K(n)$ is such $\Gamma_1(n)$ and $\Gamma_0(n)$, respectively.

We see that every coefficient of a transformation in the modular group is an integer. Then the action (1) can be extended to act on $\widehat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$. In [8], the authors represented every element in $\widehat{\mathbb{Q}}$ as reduced fractions $\frac{x}{y} = \frac{-x}{-y}$, where $x, y \in \mathbb{Z}$ and $\gcd(x,y) = 1$. In the case of ∞, it is represented by the fractions $\frac{1}{0} = \frac{-1}{0}$. Now the action (1) of Γ on $\widehat{\mathbb{Q}}$ can be rewritten as follows,

$$\frac{x}{y} \mapsto \frac{ax+by}{cx+dy}.$$

Certainly, $\frac{ax+by}{cx+dy}$ is a reduced fraction. The action of Γ on $\widehat{\mathbb{Q}}$ is absolutely independent from the non-uniqueness of the representations of fractions. Note that the action of Γ on the set $\widehat{\mathbb{Q}}$ is *transitive*, that is, for every $v, w \in \widehat{\mathbb{Q}}$ there exists a transformation $\gamma \in \Gamma$ such that $\gamma(v) = w$, equivalently, for every $v \in \widehat{\mathbb{Q}}$ there exists $\gamma \in \Gamma$ such that $\gamma(\infty) = v$. This means that we can represent every element in $\widehat{\mathbb{Q}}$ by $\gamma(\infty)$, where $\gamma \in \Gamma$.

We see that $\Gamma_\infty < \Gamma_K(n) \leqslant \Gamma$ where Γ_∞ is the stabilizer of ∞, the set of all translations $z \to z + b$ with $b \in \mathbb{Z}$. The second inequality is strict if $n > 1$. Then a nontrivial Γ-invariant equivalence relation on $\widehat{\mathbb{Q}}$, see also [8, page 319] for a general definition, related to the group $\Gamma_K(n)$ is given by

$$\gamma(\infty) \sim \gamma'(\infty) \text{ if and only if } \gamma' \in \gamma\Gamma_K(n),$$

where $\gamma\Gamma_K(n)$ is a left coset of $\Gamma_K(n)$ in Γ. An equivalence class is called a *block* and denoted by $[v]_K$, the block containing an element v of $\widehat{\mathbb{Q}}$.

From the relation obtained above, we see that the block $[\gamma(\infty)]_K$ is the set

$$\{\gamma\Gamma_K(n)\}(\infty) = \{\gamma\gamma_{_K}(\infty) : \gamma_{_K} \in \Gamma_K(n)\}.$$

In particular, the block $[\infty]_K$ is the $\Gamma_K(n)$-orbit,

$$\{\Gamma_K(n)\}(\infty) = \{\gamma_K(\infty) : \gamma_{_K} \in \Gamma_K(n)\}.$$

Therefore, $\Gamma_K(n)$ acts transitively on the block

$$[\infty]_K = \left\{\frac{x}{y} \in \widehat{\mathbb{Q}} : \overline{x} \in -K \cup K, y \equiv 0 \mod n\right\}.$$

Proposition 1. *Let n, m be positive integers, K and K' be subgroups of $\mathbb{Z}_n^*$ and $\mathbb{Z}_m^*$, respectively. Then the following statements are equivalent,*
1) $\Gamma_K(n) \leqslant \Gamma_{K'}(m)$,
2) $[\infty]_K \subseteq [\infty]_{K'}$,
3) $m \mid n$ *and* $\{k \in \mathbb{Z} : \overline{k}_n \in -K \cup K\} \subseteq \{k \in \mathbb{Z} : \overline{k}_m \in -K' \cup K'\}$.

Proof. 1) $\Rightarrow$ 2) It is obvious from the fact that if $H \leqslant G$, the orbit $H(x)$ is always contained in the orbit $G(x)$.

2) $\Rightarrow$ 3) Suppose that $[\infty]_K \subseteq [\infty]_{K'}$, and $a \in \{k \in \mathbb{Z} : \overline{k}_n \in -K \cup K\}$. Then $\frac{a}{n} \in [\infty]_K \subseteq [\infty]_{K'}$. This implies that $m \mid n$ and $\overline{a}_m \in -K' \cup K'$, that is, $a \in \{k \in \mathbb{Z} : \overline{k}_m \in -K' \cup K'\}$.

3) $\Rightarrow$ 1) Suppose that the conditions hold, and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ belongs to $\Gamma_K(n)$. Then $\overline{a}_n \in -K \cup K$ and $c \equiv 0 \mod n$. Since $m \mid n$, we have $c \equiv 0 \mod m$. The remaining condition implies that $\overline{a}_m \in -K' \cup K'$. Hence, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_{K'}(m)$. □

2. The graph $\mathcal{F}^K_{u,n}$

In this section we determine the graph $\mathcal{F}^K_{u,n}$ and describe general properties of the graph, for example, edge conditions and isomorphism results. Let $\mathcal{G}(v, w)$ be a suborbital graph for Γ on $\widehat{\mathbb{Q}}$. The directed graph $\mathcal{G}(v, w)$ and its arrow reversed graph $\mathcal{G}(w, v)$ are called *paired* suborbital graphs. In the case $\mathcal{G}(v, w) = \mathcal{G}(w, v)$, the graphs become undirected, we will call them *self-paired*. Since Γ acts transitively on the set $\widehat{\mathbb{Q}}$, there is a transformation $\gamma \in \Gamma$ mapping v to ∞. Hence, a suborbital graphs $\mathcal{G}(v, w)$ and $\mathcal{G}(\infty, \gamma(w))$ are identical. If $\gamma(w) = \frac{u}{n}$ where $u, n \in \mathbb{Z}$, $n \geqslant 0$ and $\gcd(u, n) = 1$, the graph will be simply denoted by $\mathcal{G}_{u,n}$. This is a notation traditionally used in [8] and the related research. In the case $\frac{u}{n} = \infty$, the graph just contains loop at every vertex, and every vertex is not adjacent to others. This is a trivial case of the suborbital graphs, and need not be studied. We will consider only the nontrivial case, that is, $\frac{u}{n} \neq \infty$. In this case we let all edges be complete geodesics in the upper half-plane $\mathbb{H}^2$ joining between two vertices. We denote by $\mathcal{F}^K_{u,n}$ the subgraph of $\mathcal{G}_{u,n}$ whose vertex set is the block $[\infty]_K$. For the case $\Gamma_K(n) = \Gamma_0(n)$, the graph and the block will be simply denoted by $\mathcal{F}_{u,n}$ and $[\infty]_0$, respectively. The remark below demonstrates a trivial result immediate from Proposition 1 and the definition of $\mathcal{F}^K_{u,n}$.

Remark 1. If $K \leqslant K'$, then $\mathcal{F}^K_{u,n}$ is a subgraph of $\mathcal{F}^{K'}_{u,n}$. In particular, $\mathcal{F}^K_{u,n}$ is a subgraph of $\mathcal{F}_{u,n}$.

For $n = 1$, we obtain $\Gamma_K(n) = \Gamma$ and $[\infty]_K = \widehat{\mathbb{Q}}$. Hence $\mathcal{G}_{1,1} = \mathcal{F}_{1,1} = \mathcal{F}^K_{1,1}$. We call this graph the *Farey graph*. The following are some basic results of suborbital graphs for Γ which were obtained in [8].

Lemma 1. Γ *acts on vertices and edges of* $\mathcal{G}_{u,n}$ *transitively.*

Lemma 2. $\mathcal{G}_{u,n} = \mathcal{G}_{u',n'}$ *if and only if* $n = n'$ *and* $u \equiv u' \mod n$.

Lemma 3. $\mathcal{G}_{u,n}$ *is self-paired if and only if* $u^2 \equiv -1 \mod n$.

Lemma 4. *No edges of* $\mathcal{F}_{1,1}$ *cross in* $\mathbb{H}^2$.

Proposition 2. *There is an edge* $\frac{r}{s} \to \frac{x}{y}$ *in* $\mathcal{G}_{u,n}$ *if and only if one of the following conditions holds,*

1) $x \equiv ur \mod n, y \equiv us \mod n$ *and* $ry - sx = n$,
2) $x \equiv -ur \mod n, y \equiv -us \mod n$ *and* $ry - sx = -n$.

Next we state the first result for the graph $\mathcal{F}^K_{u,n}$, the edge conditions. Let us consider the fractions $\frac{r}{s}$ and $\frac{x}{y}$ in the previous proposition. If they are in the block $[\infty]_K$, then $s \equiv r \equiv 0 \mod n$, so $s \equiv \pm ur \mod n$. We now have the following proposition immediately.

Proposition 3. *There is an edge* $\frac{r}{s} \to \frac{x}{y}$ *in* $\mathcal{F}^K_{u,n}$ *if and only if it satisfies one of the following conditions,*

1) $x \equiv ur \mod n$ *and* $ry - sx = n$,
2) $x \equiv -ur \mod n$ *and* $ry - sx = -n$.

Suppose that $v \to w$ is an edge of $\mathcal{F}^K_{u,n}$. Then there exists a transformation $\gamma \in \Gamma$ such that $\gamma(\infty \to \frac{u}{n}) = v \to w$. Since v are in $[\infty]_K$, we can see easily that $\gamma \in \Gamma_K(n)$, and so $\frac{u}{n} = \gamma^{-1}(w) \in \big(\Gamma_K(n)\big)(\infty) = [\infty]_K$. This means that if $\mathcal{F}^K_{u,n}$ contains edges, it also contains the vertex $\frac{u}{n}$. The converse is also true that if $\frac{u}{n}$ is a vertex of $\mathcal{F}_{u,n}^K$, the graph contains edges including the edge $\infty \to \frac{u}{n}$. We conclude this fact in the following corollary.

Corollary 1. *$\mathcal{F}_{u,n}^K$ contains edges if and only if $\frac{u}{n} \in [\infty]_K$, i.e., $\overline{u} \in -K \cup K$.*

We have known that $\Gamma_K(n)$ acts transitively on the vertex set of $\mathcal{F}_{u,n}^K$, the block $[\infty]_K$. We will show that it also acts transitively on edges of $\mathcal{F}_{u,n}^K$. We may suppose that the graph contains edges. Then Corollary 1 implies that $\frac{u}{n} \in [\infty]_K$. Thus, $\mathcal{F}_{u,n}^K$ is really a suborbital graph for $\Gamma_K(n)$ on the block $[\infty]_K$. We now obtain a trivial consequences coming from [8, Proposition 3.1] that $\Gamma_K(n)$ acts transitively on edges of $\mathcal{F}_{u,n}^K$.

Corollary 2. *$\Gamma_K(n)$ acts on vertices and edges of $\mathcal{F}_{u,n}^K$ transitively.*

The next corollary provides the sufficient and necessary conditions for $\mathcal{F}_{u,n}^K$ to be a self-paired suborbital graph for $\Gamma_K(n)$.

Corollary 3. *$\mathcal{F}_{u,n}^K$ is self-paired if and only if $\overline{u} \in K$ and $u^2 \equiv -1 \mod n$.*

Proof. Suppose that $\mathcal{F}_{u,n}^K$ is self-paired. By using Lemma 1, $\mathcal{G}_{u,n}$ is self-paired. Then Lemma 3 implies that $u^2 \equiv -1 \mod n$, and so, $\overline{u} \in K$ if and only if $-\overline{u} \in K$. Since $\mathcal{F}_{u,n}^K$ contains edges, Corollary 1 implies that $\overline{u} \in -K \cup K$. If $\overline{u} \in -K$, we have $-\overline{u} \in K$, and so, $\overline{u} \in K$. For the converse, Lemma 3 implies that $G_{u,n}$ is self-paired. By Corollary 1, $\mathcal{F}_{u,n}^K$ contains edges, so it is a self-paired suborbital graph on $[\infty]_K$. $\square$

Next we verify the isomorphism results for the graph $\mathcal{F}_{u,n}^K$. The first one shows that the reflection of $\mathcal{F}_{u,n}^K$ across the imaginary axis is another suborbital graph $\mathcal{F}_{-u,n}^K$. For the second one, let us consider the case of the graph $\mathcal{F}_{u,n}$ first. Suppose that n is a multiple of a positive integer m. [8, Lemma 5.3 (ii)] shows that $\mathcal{F}_{u,n}$ is an isomorphic subgraph of $\mathcal{F}_{u,m}$. We know by Remark 1 that $\mathcal{F}_{u,n}^K$ is a subgraph of $\mathcal{F}_{u,n}$. Hence, $\mathcal{F}_{u,n}^K$ becomes an isomorphic subgraph of $\mathcal{F}_{u,m}$. Certainly, the graph $\mathcal{F}_{u,m}$ may not be smallest, so we can find the smaller graph $\mathcal{F}_{u,m}^{K'}$ containing $\mathcal{F}_{u,n}^K$ as an isomorphic subgraph.

Let $K' = \{\overline{k}_m : \overline{k}_n \in K\}$. It is not difficult to see that K' is closed under the multiplication modulo m, so $K' \leqslant \mathbb{Z}_m^*$. We use K' to obtain the general version of [8, Lemma 5.3 (ii)]. We are now ready to prove the proposition.

Proposition 4.

1) $\mathcal{F}_{u,n}^{K}$ *is isomorphic to* $\mathcal{F}_{-u,n}^{K}$ *by the mapping* $v \mapsto -v$.
2) *If* $m \mid n$, *then* $\mathcal{F}_{u,n}^{K}$ *is an isomorphic subgraph of* $\mathcal{F}_{u,m}^{K''}$ *by the mapping* $v \mapsto \frac{nv}{m}$, *where* K'' *is a supergroup of* $K' = \{\overline{k}_m : \overline{k}_n \in K\}$. *In particular,* $\mathcal{F}_{u,n}^{K}$ *is an isomorphic subgraph of* $\mathcal{F}_{u,m}^{K'}$

Proof. 1) We can see easily that $\frac{r}{s} \in [\infty]_K$ if and only if $\frac{-r}{s} \in [\infty]_K$. Clearly, the mapping is bijective. We need to check that the mapping is edge-preserving so that it is an isomorphism. Let $\frac{r}{s} \to \frac{x}{y}$ be an edge in $\mathcal{F}_{u,n}^{K}$. Then by Proposition 3, $x \equiv \pm ur \mod n$ and $ry - sx = \pm n$. This implies that $-x \equiv \mp(-u)(-r) \mod n$ and $(-r)y - s(-x) = \mp n$. Therefore, $\frac{-r}{s} \to \frac{-x}{y}$ is an edge of $\mathcal{F}_{-u,n}^{K}$.

2) We will prove only the particular case, the general case will be obtained directly after applying Remark 1 which implies that $\mathcal{F}_{u,m}^{K'}$ is a subgraph of $\mathcal{F}_{u,m}^{K''}$. Let $m \mid n$, and $v = \frac{r}{sn}$ be a vertex of $\mathcal{F}_{u,n}^{K}$, $s \in \mathbb{Z}$. We then have $v \mapsto \frac{nv}{m} = \frac{r}{sm}$. Since $\gcd(r, sn) = 1$ and $m \mid n$, $\gcd(r, sm) = 1$. Since $\overline{r}_n \in -K \cup K$, we have $\overline{r}_m \in -K' \cup K'$. Thus, $\frac{r}{sm}$ is a vertex of $\mathcal{F}_{u,n}^{K'}$. The injective property is obvious, so we prove only the edge-preserving property. Suppose that $\frac{r}{sn} \to \frac{x}{yn}$ be an edge of $\mathcal{F}_{u,n}^{K}$. Proposition 3 implies that $x \equiv \pm ur \mod n$, and $r(yn) - (sn)x = \pm n$. Since $m \mid n$, $x \equiv \pm ur \mod m$. We see that $ry - sx = \pm 1$, so $r(ym) - (sm)x = \pm m$. Therefore, there is an edge $\frac{r}{sm} \to \frac{x}{ym}$ in $\mathcal{F}_{u,n}^{K_m}$. □

Corollary 4. *No edges of* $\mathcal{F}_{u,n}^{K}$ *cross in* $\mathbb{H}^2$

Proof. By using the second result of Proposition 4 with $m = 1$, $\mathcal{F}_{u,n}^{K}$ becomes an isomorphic subgraph of $\mathcal{F}_{1,1}$. Lemma 4 said that there are no edges of $\mathcal{F}_{1,1}$ crossing in $\mathbb{H}^2$. Then so does $\mathcal{F}_{u,n}^{K}$. □

3. Connectivity of graphs

In this section we investigate connectivity of the graph $\mathcal{F}_{u,n}^{K}$. The goal of this section is to show the following theorem.

Theorem 1. *The graph* $\mathcal{F}_{u,n}^{K}$ *is connected if and only if* $n \leqslant 4$.

To prove this theorem we consider each case of n. Proposition 5 and Proposition 7 will result the conclusion when $n \leqslant 4$ and $n \geqslant 5$, respectively. Now let us consider the graph $\mathcal{F}_{u,n}^{K}$. We have known from Remark 1 that $\mathcal{F}_{u,n}^{K}$ is a subgraph of $\mathcal{F}_{u,n}$. The connectivity of $\mathcal{F}_{u,n}$ was already concluded in [8, Theorem 5.10]. However, results for the subgraph does not depend on its supergraph in general. Thus, it is worth examining the connectivity of $\mathcal{F}^K_{u,n}$. One can verify that $\mathcal{F}^K_{u,n} = \mathcal{F}_{u,n}$ if and only if $-K \cup K = \mathbb{Z}^*_n$, that is, $\Gamma_K(n) = \Gamma_0(n)$. Then we prove only the case $-K \cup K \subset \mathbb{Z}^*_n$. The cases $n \leqslant 4$ or $n = 6$ need not be proved since $\mathcal{F}^K_{u,n} = \mathcal{F}_{u,n}$ for every subgroup K of $\mathbb{Z}^*_n$. For completeness, we conclude them again in the proposition below using the notation $\mathcal{F}^K_{u,n}$, and then prove the remaining cases.

Proposition 5. *$\mathcal{F}^K_{u,6}$ is not connected, but $\mathcal{F}^K_{u,n}$ is connected for every $n \leqslant 4$.*

Lemma 5. *Let $\frac{j}{k}$ be a reduced fraction where $k \mid n$. Then there are not adjacent vertices v and w of $\mathcal{F}^K_{u,n}$ such that $v < \frac{j}{k} < w$.*

Proof. We assume by contrary that v and w are adjacent vertices of $\mathcal{F}^K_{u,n}$. By using Proposition 4 with $m = 1$, the vertices nv and nw are adjacent in $\mathcal{F}_{1,1}$. Then the edge joining these two vertices crosses an edge $\frac{nj}{k} \to \infty$ of $\mathcal{F}_{1,1}$ in $\mathbb{H}^2$ that provides a contradiction to Lemma 4. Thus, v and w cannot be adjacent □

Lemma 6. *Let $a, b, k \in \mathbb{Z}$, and $b \neq 0 \neq k$. Then $\frac{1+2abk}{4b^2k}$ is a reduced fraction.*

Proof. Let $p = \gcd(1 + 2abk, 4b^2k)$ and $q = \gcd(p, 2bk)$. Then $q \mid 1 + 2abk$ and $q \mid 2bk$. Thus $q \mid 1$, so $q = 1$. Hence, $p = \gcd(p, 4b^2k) = 1$. □

Proposition 6. *If $n \geqslant 5$, the graph $\mathcal{F}^K_{u,n}$ with $u \equiv \pm 1 \mod n$ is not connected.*

Proof. The case $n = 6$ is concluded in Proposition 5. Then we suppose that $n \neq 6$. By using Lemma 2 and Lemma 4, we can consider only the case $u = 1$. We see that the block $[\infty]_K$ always contains all fractions $\frac{r}{s}$ with $r \equiv \pm 1 \mod n$ and $s \equiv 0 \mod n$. If the block $[\infty]_K$ contains another fraction $\frac{x}{y}$ with $x \not\equiv \pm 1 \mod n$, by using Proposition 3, $\frac{x}{y}$ is never joined to $\frac{r}{s}$. This provides disconnectedness of the graph. Next we suppose that the block $[\infty]_K$ contains only fractions $\frac{r}{s}$ where $r \equiv \pm 1 \mod n$ and $s \equiv 0 \mod n$. Since $n \geqslant 5$ and $n \neq 6$, there are at least two proper fractions $\frac{z}{n}$ and $\frac{z'}{n}$ such that $\frac{1}{n} < \frac{z}{n} < \frac{z'}{n} < \frac{n-1}{n}$. We will show that the interval $(\frac{z}{n}, \frac{z'}{n})$ contains some vertices of $\mathcal{F}^K_{u,n}$. Certainly, every vertex of the graph in this interval is not adjacent to ∞. By using Lemma 6 with $a = z + z'$ and $b = n$, we obtain that $\frac{1+2(z+z')nk}{4n^2k}$ is a reduced fraction. Obviously, it is contained in $[\infty]_K$ for every $k \in \mathbb{N}$. If we consider this fraction as an infinite sequence over the index k, the sequence converges to the fraction $\frac{z+z'}{2n}$, the middle value of the open interval $(\frac{z}{n}, \frac{z'}{n})$. Thus, the interval contains vertices of $\mathcal{F}^K_{u,n}$. We now replace $\frac{j}{k}$ in Lemma 5 by $\frac{z}{n}$ and $\frac{z'}{n}$. Hence, vertices of $\mathcal{F}^K_{u,n}$ in the interval $(\frac{z}{n}, \frac{z'}{n})$ is separated from others outside the interval providing disconnectedness of the graph. □

Lemma 7. *If $u \not\equiv \pm 1 \mod n$, then there are not adjacent vertices v and w of $\mathcal{F}^K_{u,n}$ such that $v < \frac{1}{2} < w$.*

Proof. The case that n is even follows from Lemma 5. We then suppose that n is odd. Assume that v is adjacent to w. Then Lemma 4 implies that nv and nw are adjacent vertices in $\mathcal{F}_{1,1}$. By using [8, Lemma 4.1], nv and nw are adjacent term in some $\mathcal{F}_m$, the Farey sequence of order m. Since $nv < \frac{n}{2} < nw$, we obtain $m = 1$. Then $nv = (n-1)/2$ and $nw = (n+1)/2$, so $v = \frac{(n-1)/2}{n}$ and $w = \frac{(n+1)/2}{n}$. If $v \to w$ is an edge in $\mathcal{F}^K_{u,n}$, Proposition 3 implies that $(n+1)/2 \equiv -u(n-1)/2 \mod n$. Then $1 \equiv -u(-1) \equiv u \mod n$ which contradicts to the assumption. For the case that $w \to v$ is an edge of $\mathcal{F}^K_{u,n}$, we will obtain $u \equiv -1 \mod n$. This also provides a contradiction. Therefore, v and w are not adjacent in $\mathcal{F}^K_{u,n}$. □

Proposition 7. *$\mathcal{F}^K_{u,n}$ is not connected for every $n \geqslant 5$.*

Proof. In this proposition we prove the remaining cases. Here, we can assume that $-K \cup K \subset \mathbb{Z}^*_n$ and $u \not\equiv \pm 1 \mod n$. Since $-K \cup K \subset \mathbb{Z}^*_n$, there exists $\frac{t}{n} \in (0,1)$ such that $\frac{t}{n} \notin [\infty]_K$. By using Proposition 3, one can compute that there are at most two vertices of $\mathcal{F}^K_{u,n}$ in the interval $(0,1)$ adjacent to ∞. Hence, there is at least one interval $(\frac{r}{s}, \frac{x}{y})$, where $\frac{r}{s}, \frac{x}{y} \in \{0, \frac{1}{2}, \frac{t}{n}, 1\}$, not containing these two vertices. We now put $a = ry + sx, b = sy$, and apply Lemma 6 with the same step used in Proposition 6. We finally obtain at least one vertex of $\mathcal{F}^K_{u,n}$ contained in $(\frac{r}{s}, \frac{x}{y})$. Certainly, every vertex of $\mathcal{F}^K_{u,n}$ in $(\frac{r}{s}, \frac{x}{y})$ is not adjacent to ∞. By applying Lemma 5, some cases may require Lemma 7, the vertices in $(\frac{r}{s}, \frac{x}{y})$ is not adjacent to other vertices this interval. Thus $\mathcal{F}^K_{u,n}$ is not connected. □

4. Circuits of graphs

This section discusses circuits of the graph $\mathcal{F}^K_{u,n}$. A *circuit* of $\mathcal{F}^K_{u,n}$ is a sequence of $m \geqslant 3$ different vertices $v_1, v_2, \ldots, v_m \in \mathcal{F}^K_{u,n}$ such that $v_1 \to v_2 \to \cdots \to v_m \to v_1$ and some arrows may be reversed. If $m = 3$, we call it a *triangle*. A *directed triangle* is a triangle whose arrows are in the same direction. Otherwise, called an *anti-directed triangle*. The next two statements, Proposition 8 and Remark 2, provide sufficient and necessary conditions for the graph $\mathcal{F}_{u,n}^K$ to contain triangles.

Proposition 8. *$\mathcal{F}_{u,n}^K$ contains directed triangles if and only if $\frac{u}{n} \in [\infty]_K$ and $u^2 \pm u + 1 \equiv 0 \mod n$.*

Proof. Let $\mathcal{F}_{u,n}^K$ contains directed triangles. Then so does $\mathcal{F}_{u,n}$ since $\mathcal{F}_{u,n}^K$ is a subgraph $\mathcal{F}_{u,n}$. By [8, Theorem 5.11] we have $u^2 \pm u + 1 \equiv 0 \mod n$. Since $\mathcal{F}_{u,n}^K$ contains edges, Corollary 1 implies that $\frac{u}{n} \in [\infty]_K$. For the converse implication, we suppose that the conditions hold. Then $\overline{u} \in -K \cup K$. Since $u^2 \pm u + 1 \equiv 0 \mod n$, $\overline{u \pm 1} = \mp \overline{u}^2 \in -K \cup K$. We now obtain $\frac{u \pm 1}{n} \in [\infty]_K$. By Proposition 3, one can easily check that the graph $\mathcal{F}_{u,n}^K$ contains the directed triangle of the form $\infty \to \frac{u}{n} \to \frac{u \pm 1}{n} \to \infty$. □

It is not difficult to see that $\mathcal{F}_{u,1} = \mathcal{F}_{1,1}$ is a self-paired graph containing directed triangles. Then, it contains anti-directed triangles. [8, Theorem 5.11 (ii)] said that $\mathcal{F}_{u,n}$ contains no anti-directed triangles if $n \geqslant 1$. Since $\mathcal{F}_{u,n}^K$ is a subgraph of $\mathcal{F}_{u,n}$ and they are identical if $n = 1$, it is worth to remark that,

Remark 2. $\mathcal{F}_{u,n}^K$ contains anti-directed triangles if and only if $n = 1$.

The next proposition was proved in [1, Theorem 10] for the case of $\mathcal{F}_{u,n}$ that the graph is a *forest*, a graph contains no circuits, if and only if it contains no triangles. The general case can be proved by using this fact together with Proposition 8.

Theorem 2. *$\mathcal{F}_{u,n}^K$ is a forest if and only if it contains no triangles, i.e., $\frac{u}{n} \notin [\infty]_K$ or $u^2 \pm u + 1 \not\equiv 0 \mod n$.*

Proof. The forward implication is clear by the definition of a forest. For the converse we assume the contrary that $\mathcal{F}_{u,n}^K$ contains circuits. Then so does $\mathcal{F}_{u,n}$. By the proof of [1, Theorem 10], $\mathcal{F}_{u,n}$ contains triangles. Thus, we have $u^2 \pm u + 1 \equiv 0 \mod n$. Since there is an edge in $\mathcal{F}_{u,n}^K$, Corollary 1 implies that $\frac{u}{n} \in [\infty]_K$. By Proposition 8, $\mathcal{F}_{u,n}^K$ contains triangles. □

We know from Theorem 1 that $\mathcal{F}_{u,n}^K$ is connected if and only if $n \leqslant 4$. Combine with Theorem 2, we obtain the following corollary.

Corollary 5. *$\mathcal{F}_{u,n}^K$ is a tree if and only if $n = 2, 4$.*

In Section 2 we have proved that $\Gamma_K(n)$ acts transitively on vertices and edges of $\mathcal{F}^K_{u,n}$, see Corollary 2. This situation also occurs for directed triangles. The proof can be done by using the transitivity of the action of $\Gamma_K(n)$ on edges of $\mathcal{F}^K_{u,n}$.

Proposition 9. $\Gamma_K(n)$ *acts on directed triangles of* $\mathcal{F}^K_{u,n}$ *transitively.*

Proof. By Proposition 8, we see that if $\mathcal{F}^K_{u,n}$ contains triangles, it always contains the triangle $\frac{1}{0} \to \frac{u}{n} \to \frac{u\pm1}{n} \to \frac{1}{0}$. Suppose that $v_1 \to v_2 \to v_3 \to v_1$ is an arbitrary directed triangle in $\mathcal{F}^K_{u,n}$. It is sufficient to show that there is a transformation $\gamma \in \Gamma_K(n)$ such that $\gamma(\infty \to \frac{u}{n} \to \frac{u\pm1}{n} \to \infty) = v_1 \to v_2 \to v_3 \to v_1$.

Since $v_1 \to v_2$ is an edge of the graph $\mathcal{F}^K_{u,n}$, Corollary 2 implies that there is an element $\gamma \in \Gamma_K(n)$ such that $\gamma(\infty \to \frac{u}{n}) = v_1 \to v_2$. One can verify that γ is unique. Next we prove $\gamma(\frac{u\pm1}{n}) = v_3$. Since $v_3 \to v_1$ and $v_2 \to v_3$ are edges of $\mathcal{F}^K_{u,n}$, we obtain that $\gamma^{-1}(v_3 \to v_1) = \gamma^{-1}(v_3) \to \infty$ and $\gamma^{-1}(v_2 \to v_3) = \frac{u}{n} \to \gamma^{-1}(v_3)$ are edges of $\mathcal{F}^K_{u,n}$. First, we apply edge conditions, Proposition 3, to the first identity and obtain $\gamma^{-1}(v_3) = \frac{x}{n}$ for some $x \in \mathbb{Z}$. Next we replace $\gamma^{-1}(v_3)$ in the second identity by $\frac{x}{n}$ and apply Proposition 3 again. Then $un - xn = \pm n$, and so $x = u \pm 1$. Thus, $\gamma(\frac{u\pm1}{n}) = v_3$. The proof is now complete. □

In the proof of the previous proposition, the triangle $v_1 \to v_2 \to v_3 \to v_1$ is arbitrary. If we replace v_1, v_2 and v_3 by $\frac{u}{n}$, $\frac{u\pm1}{n}$ and ∞, respectively, then there is a unique transformation $\gamma_1 \in \Gamma_K(n)$ rotating the triangle $\infty \to \frac{u}{n} \to \frac{u\pm1}{n} \to \infty$ in such a way that $\gamma_1(\infty \to \frac{u}{n} \to \frac{u\pm1}{n} \to \infty) = \frac{u}{n} \to \frac{u\pm1}{n} \to \infty \to \frac{u}{n}$. One can show easily that

$$\gamma_1 = \begin{pmatrix} u & -(u^2 \pm u + 1)/n \\ n & -(u \pm 1) \end{pmatrix}.$$

Therefore, γ_1 and γ in the proof above induce a unique transformation $\gamma\gamma_1\gamma^{-1}$ rotating another given directed triangle in $\mathcal{F}^K_{u,n}$. We provide the lemma below after concluding this result with more precisely.

Lemma 8. *Let* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ *is an* elliptic element *of the modular group* Γ, *that is* $|a+d| < 2$. *if* $|a+d| = 0$, *then* γ *has order* 2, *otherwise,* γ *has order* 3.

Proof. Suppose that $|a+d| = 0$. Then $\gamma = \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right)$, so $-a^2 - bc = 1$. We see that

$$\gamma^2 = \begin{pmatrix} a^2+bc & ab-ab \\ ac-ac & a^2+bc \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

becomes the identity transformation. Hence, γ has order 2.

Next suppose that $|a+d|=1$. Then $d=-(a\pm 1)$. We will prove only the case $d=-a-1$. The other case can be proved similarly. Now we have $\gamma = \left(\begin{smallmatrix} a & b \\ c & -a-1 \end{smallmatrix}\right)$ and $-a^2-a-bc=1$. Consider

$$\gamma^2 = \begin{pmatrix} a^2+bc & ab-ab-b \\ ac-ac-c & a^2+2a+1+bc \end{pmatrix} = \begin{pmatrix} -a-1 & -b \\ -c & a \end{pmatrix}.$$

We see that γ^2 is the inverse transformation of γ. Then γ has order 3. □

Remark 3. Elements of Γ which are conjugate to elliptic elements are elliptic, so $\gamma\gamma_1\gamma^{-1}$ is elliptic.

Corollary 6. *There is a unique elliptic element γ of order 3 in $\Gamma_K(n)$ rotating a triangle $v_1 \to v_2 \to v_3 \to v_1$ of $\mathcal{F}^K_{u,n}$ in such a way that $\gamma(v_1 \to v_2 \to v_3 \to v_1) = v_2 \to v_3 \to v_1 \to v_2$.*

The above corollary and the two consequences below are all about relations between elliptic elements in the group $\Gamma_K(n)$ and its suborbital graph $\mathcal{F}^K_{u,n}$. All of them were proved already in [1] for the version of $\Gamma_0(n)$ and $\mathcal{F}_{u,n}$. The proofs of the two results below follow from the former.

Theorem 3. *$\Gamma_K(n)$ contains an elliptic element of order 3 if and only if there exists $\overline{u} \in -K \cup K$ such that $\mathcal{F}^K_{u,n}$ contains a triangle.*

Proof. Suppose that

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is an elliptic element of order 3 contained in $\Gamma_K(n)$. Then Lemma 8 implies that $|a+d|=1$. Since $\gamma \in \Gamma_K(n)$, $ad \equiv 1 \mod n$. Thus $a^2 \pm a + 1 \equiv 0 \mod n$. Certainly, $\overline{a} \in -K \cup K$. If we choose $u \equiv a \mod n$, Theorem 8 implies that $\mathcal{F}^K_{u,n}$ contains a triangle. Conversely, suppose that the conditions hold. Again, by using Theorem 8, we obtain $u^2 \pm u + 1 \equiv 0 \mod n$. Now let γ be the transformation

$$\gamma_1 = \begin{pmatrix} u & -(u^2 \pm u + 1)/n \\ n & -(u \pm 1) \end{pmatrix}$$

defined before Lemma 8. It is certainly an elliptic element of order 3 in $\Gamma_K(n)$. □

Theorem 4. *$\Gamma_K(n)$ contains an elliptic element of order 2 if and only if there exists $\overline{u} \in K$ such that $\mathcal{F}^K_{u,n}$ is self-paired.*

Proof. Suppose that

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is an elliptic element of order 2 contained in $\Gamma_K(n)$. Then Lemma 8 implies that $a + d = 0$. Since $\gamma \in \Gamma_K(n)$, $ad \equiv 1 \mod n$. Then $a^2 \equiv -1 \mod n$. Certainly, $\overline{a} \in -K \cup K$. If $\overline{a} \in K$, we choose $u \equiv a \mod n$. If $\overline{a} \in -K$, we choose $u \equiv -a \mod n$. Thus, we have $\overline{u} \in K$ and $u^2 \equiv -1 \mod n$. Now apply Corollary 3, we obtain that $\mathcal{F}^K_{u,n}$ is self-paired. Conversely, suppose that the conditions hold. Again, by using Corollary 3, we obtain $u^2 \equiv -1 \mod n$, that is, $u^2 + 1 \equiv 0 \mod n$. Hence by computation, the transformation

$$\begin{pmatrix} u & -(u^2+1)/n \\ n & -u \end{pmatrix}$$

belongs to $\Gamma_K(n)$. Lemma 8 implies that it is an elliptic element of order 2. □

5. Graphs for conjugate subgroups of Γ

This section is inspired by [5, 7] which studied suborbital graphs for the groups $\Gamma_0(n)$ and $\Gamma^0(n)$, respectively. As subgroups of the modular group Γ, they are considered to act on $\widehat{\mathbb{Q}}$, and their specific suborbital graphs were determined on their orbits whom they act transitively. We extend the topic to the case of $\Gamma_K(n)$ and $\Gamma^K(n)$. The discussion shows that we can study only the suborbital graph $\mathcal{F}^K_{u,n}$ to conclude some general properties of a suborbital graph for $\Gamma^K(n)$ through a graph isomorphism.

We start with the spacial case of the groups $\Gamma_0(n)$ and $\Gamma^0(n)$. The group $\Gamma^0(n)$ is another congruence subgroup of Γ determined by,

$$\Gamma^0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : b \equiv 0 \mod n \right\}.$$

It is conjugate to the group $\Gamma_0(n)$. More precisely, $\Gamma^0(n) = \gamma\Gamma_0(n)\gamma^{-1}$ where $\gamma = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in \Gamma$. In [5], the authors determined the suborbital graphs of $\Gamma_0(n)$ on its orbit containing ∞, $\big(\Gamma_0(n)\big)(\infty) = \{\frac{x}{y} \in \widehat{\mathbb{Q}} : y \equiv 0 \mod n\}$. They assumed and studied for the case that n is a prime number p. Suborbital graphs whom they studied are, in fact, the graph $\mathcal{F}_{u,p}$ on the block $[\infty]_0$. Likewise, in [7], the authors studied suborbital graphs for $\Gamma^0(p)$. In this case the graphs were determined on the orbit of 0, $\big(\Gamma^0(p)\big)(0) = \{\frac{x}{y} \in \widehat{\mathbb{Q}} : x \equiv 0 \mod p\}$. We shall roughly denote it by

$\bar{\mathcal{F}}_{p,u}$, the suborbital graph for $\Gamma^0(p)$ whose edges from the suborbital $\big(\Gamma^0(p)\big)(0, \frac{p}{u})$. What is the relation between the graphs $\mathcal{F}_{u,p}$ and $\bar{\mathcal{F}}_{p,u}$?

We see that $\big(\Gamma^0(p)\big)(0, \frac{p}{u}) = \big(\gamma\Gamma_0(n)\big)(\infty, \frac{-u}{p})$. Then $\bar{\mathcal{F}}_{p,u}$ is actually a subgraph of $\mathcal{G}_{-u,p}$ on the block $[0]_0 = [\gamma(\infty)]_0$. It is certainly isomorphic to the graph $\mathcal{F}_{-u,p}$, and so, isomorphic to the graph $\mathcal{F}_{u,p}$ after applying Proposition 4. This fact can be directly extended to the case of $\Gamma_K(n)$ and $\Gamma^K(n)$ where $\Gamma^K(n)$ is a congruence subgroup of Γ defined by

$$\Gamma^K(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : \overline{a} \in -K \cup K, \text{ and } b \equiv 0 \mod n \right\}.$$

Certainly, $\Gamma^0(n) = \gamma\Gamma_0(n)\gamma^{-1}$. We let $\bar{\mathcal{F}}^K_{n,u}$ denote the suborbital graph for $\Gamma^K(n)$ on the orbit $\big(\Gamma^K(n)\big)(0) = [0]_K$ where the suborbital $\big(\Gamma^K(p)\big)(0, \frac{n}{u})$ is the set of edges.

Proposition 10. *$\mathcal{F}^K_{u,n}$, $\mathcal{F}^K_{-u,n}$, $\bar{\mathcal{F}}^K_{n,u}$ and $\bar{\mathcal{F}}^K_{n,-u}$ are isomorphic.*

Next we discuss the more general suborbital graph for $\Gamma_K(n)$ on $[\infty]_K$. We have known that if $\mathcal{F}^K_{u,n}$ contains edges, it is certainly a suborbital graph for $\Gamma_K(n)$. However, not all suborbital graphs for $\Gamma_K(n)$ can be represented by some $\mathcal{F}^{K'}_{u,m}$. We need to introduce some notations before clarifying this claim by a trivial example on $\Gamma_0(2)$.

Notation. We denote by $\mathcal{F}^K_n(\infty, v)$ the suborbital graph for $\Gamma_K(n)$ on $[\infty]_K$ whose edges from the suborbital $\big(\Gamma_K(n)\big)(\infty, v)$, and denote by $\bar{\mathcal{F}}^K_n(0, v)$ the suborbital graph for $\Gamma^K(n)$ on $[0]_K$ whose edges from the suborbital $\big(\Gamma^K(n)\big)(0, v)$. For the case of $\Gamma_0(n)$ and $\Gamma^0(n)$ we will leave the letter K for the notation of graphs and replace K by 0 for the notation of blocks.

Let us consider the block $[\infty]_0$ of the group $\Gamma_0(2)$. It certainly contains the fraction $\frac{1}{4}$. We show that $\mathcal{F}_2(\infty, \frac{1}{4})$ cannot be written as the graph $\mathcal{F}^K_{u,n}$ for some $\frac{u}{n} \in \widehat{\mathbb{Q}}$, and some $K \leqslant \mathbb{Z}^*_n$. If $\mathcal{F}_2(\infty, \frac{1}{4}) = \mathcal{F}^K_{u,n}$, then $\frac{1}{4} \in [\infty]_K$, and so $n \mid 4$. Thus $n = 1, 2, 4$. It is obvious that $n \neq 1$ and $n \neq 4$ because they provide vertex sets which are larger and smaller than $[\infty]_0$, respectively. For the remaining case, it is clear that $\mathcal{F}_2(\infty, \frac{1}{4}) \neq \mathcal{F}_{1,2}$ since $\mathcal{F}_{1,2}$ does not contain the edge $\infty \to \frac{1}{4}$. Surely, the same situation occurs on the graphs for $\Gamma^K(n)$. However, we see that $\mathcal{F}^K_n(\infty, \frac{u}{m})$ and $\bar{\mathcal{F}}^K_n(0, -\frac{m}{u})$ are subgraphs of $\mathcal{G}_{u,m}$ restricted on the blocks $[\infty]_K$ and $[0]_K$, respectively. Then the following result is still true.

Proposition 11. *$\mathcal{F}^K_n(\infty, \frac{u}{m})$ and $\bar{\mathcal{F}}^K_n(0, -\frac{m}{u})$ are isomorphic.*

We have shown that some suborbital graph for $\Gamma_K(n)$ on the block $[\infty]_K$ can not be written as $\mathcal{F}^{K'}_{u,m}$. However, the graph is, in fact, the disjoint union of copies of some graph $\mathcal{F}^{K'}_{u,m}$. This is the reason why we can study only the graph which is represented by $\mathcal{F}^{K}_{u,n}$ to obtain the results for this general case.

Let us consider the graph $\mathcal{F}^K_n(\infty, \frac{u}{m})$. Certainly, $n \mid m$ and $\overline{u}_n \in -K \cup K$. We may assume that $\overline{u}_n \in K$, and define $K' = \langle \overline{u}_m \rangle$, the cyclic subgroup of $\mathbb{Z}^*_m$ generated by $\overline{u}_m$. One can verify easily that the union of all congruence classes in K' is a subset of the union of those congruence classes in K. Thus, Proposition 1 implies that $\Gamma_{K'}(m) \leqslant \Gamma_K(n)$. We now have $\Gamma_K(n)_\infty < \Gamma_{K'}(m) \leqslant \Gamma_K(n)$, where $\Gamma_K(n)_\infty$ is the stabilizer subgroup of $\Gamma_K(n)$ fixing ∞. Similar to the case of Γ and its congruence subgroup, this provides the $\Gamma_K(n)$-invariant equivalence relation on the block $[\infty]_K$ related to $\Gamma_{K'}(m)$, and the partition $\big\{\big(\gamma\Gamma_{K'}(m)\big)(\infty) : \gamma \in \Gamma_K(n)\big\}$ on $[\infty]_K$ is formed. We see that the orbit $\big(\Gamma_{K'}(m)\big)(\infty)$ is, in fact, the block $[\infty]_{K'}$ and the restriction of the graph $\mathcal{F}^K(\infty, \frac{u}{m})$ on $[\infty]_{K'}$ is actually the graph $\mathcal{F}^{K'}_{u,m}$. Therefore $\mathcal{F}(\infty, \frac{u}{m})$ is the disjoint union of j copies of the graph $\mathcal{F}^{K'}_{u,m}$ where $j = |\Gamma_{K'}(m) : \Gamma_K(n)|$, the index of $\Gamma_{K'}(m)$ in $\Gamma_K(n)$. Since $\mathcal{F}^{K'}_{u,m}$ and $\mathcal{F}^{K'}_{-u,m}$ are isomorphic, then $\mathcal{F}(\infty, \frac{u}{m})$ and $\mathcal{F}(\infty, \frac{-u}{m})$ are isomorphic. After applying this result together with the previous proposition, we now have the following consequences immediately.

Theorem 5. *$\mathcal{F}^K(\infty, \frac{u}{m})$, $\mathcal{F}^K(\infty, -\frac{u}{m})$, $\bar{\mathcal{F}}^K(0, \frac{m}{u})$, $\bar{\mathcal{F}}^K(0, -\frac{m}{u})$ are isomorphic.*

Corollary 7. *$\mathcal{F}(\infty, \frac{u}{m})$, $\mathcal{F}(\infty, -\frac{u}{m})$, $\bar{\mathcal{F}}(0, \frac{m}{u})$, $\bar{\mathcal{F}}(0, -\frac{m}{u})$ are isomorphic.*

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