Algebra and Discrete Mathematics
Volume **27** (2019). Number 1, pp. 12–19
© Journal "Algebra and Discrete Mathematics"
RESEARCH ARTICLE

On indices and eigenvectors of quivers*

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Communicated by A. V. Zhuchok

Abstract. We study formulas for eigenvectors of strongly connected simply laced quivers in terms of eigenvalues. The relation of these formulas to the isomorphism of quivers is investigated.

1. Introduction

In this work we study a possibility to use indices and eigenvectors of strongly connected simply laced quivers as characteristics, which can provide the conclusion whether quivers are isomorphic or not.

Following P. Gabriel and [1] we use the term *quiver* for an oriented graph. The term "quiver" was introduced in [3], which is devoted to finite dimensional algebras over an algebraically closed field, with zero square radical (see details in [4, §11.10]).

Recall that a quiver is called strongly connected, if for every two vertices of it there exists an oriented path from one to other. A quiver is called simply laced, if it has no loops and multiple arrows.

The maximum root of the characteristic polynomial of the adjacency matrix of a quiver is called its *index*. In this work we use the terms *eigenvector*, *eigenvalue*, *index* and *characteristic polynomial* of a quiver, meaning the notions, which correspond to its adjacency matrix. An attempt to reduce the question about the isomorphism of quivers to the properties of their characteristic polynomials and eigenvalues was already made in [2]. We call two vectors with n coordinates *permutationally equivalent* (or, simply, *equivalent*), if they are equal up to multiplication by a constant and permutation of coordinates. Also we call a non-zero vector *normalized*,

*This work was partially supported by FAPESP (Brazil), project 13/11350-2.
2010 MSC: 15A18.
Key words and phrases: strongly connected quiver, eigenvector.

if its Euclidean norm is 1 and the first positive coordinate is non-negative. The unique normalized eigenvector of a strongly connected quiver, which corresponds to the index, will be called an *index-vector*. We write *SCSL-quiver* for a strongly connected simply laced quiver.

The importance of the assumption that a quiver is strongly connected, is provided by the following classical fact reformulated using the notion a quiver.

Theorem 1 (Frobenius theorem, Theorem 6.5.2 in [5])**.** *The adjacency matrix of a strongly connected quiver has a positive eigenvalue r which is a simple root of the characteristic polynomial. This vector is the unique positive eigenvector up to multiplication by a constant. The absolute values of all the other eigenvalues do not exceed r. To the maximal eigenvalue r there corresponds an eigenvector with all positive coordinates.*

Theorem 2 ([2])**.** *SCSL-quivers with four vertices are isomorphic if and only if their characteristic polynomials are equal and right and left index-vectors are permutationally equivalent.*

The number of vertices of a quiver cannot be increased in Theorem 2. The matrices

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

provide a counterexample. They are non-equivalent. Nevertheless, their characteristic polynomials are equal, their right eigenvectors coincide, and their left eigenvectors coincide, too.

Also the equivalence of left index-vectors cannot be removed from Theorem 2. This shows the following example.

Example 1. There are two non-isomorphic SCSL-quivers, whose normalized right index-vectors and characteristic polynomials coincide [2].

Let

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

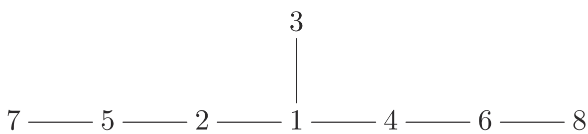
be adjacency matrices of quivers Q_1 and Q_2, respectively. These quivers are not isomorphic, because Q_1 has a vertex, which is a head of three arrows, and Q_2 does not have such a vertex.

The characteristic polynomials of A_1 and A_2 are equal to $\lambda^4 - 2\lambda^2 - 2\lambda - 1$. Right index-vectors of Q_1 and Q_2 quivers coincide and are approximately equal to $(0.314,\ 0.577,\ 0.484,\ 0.577)^t$. The approximate values of left index-vectors of A_1 and A_2 are $(0.307,\ 0.366,\ 0.565,\ 0.673)$ and $(0.314,\ 0.577,\ 0.577,\ 0.484)$ respectively. This fact shows that the condition of the equivalence of left index-vectors is necessary.

Remark 1. Example 1, which shows that the equivalence of left index-vectors cannot be removed from Theorem 2, is unique (up to isomorphism of quivers).

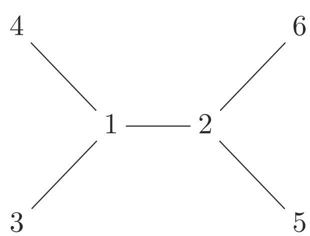
We improve Theorem 2, using the method of computing of eigenvectors and characteristic polynomials, which has been presented in [1]. Let A be an adjacency matrix of a quiver with n vertices. Any characteristic vector v of A can be treated as a solution of the equation $A - \lambda E = 0$, considered as a system of linear equations with respect to unknown coordinates of v. Thus, the formulas for coordinates of $v_1, \dots, v_n$ in terms of the eigenvalues λ can be found, and this computation is algorithmically equivalent to solving a system of homogeneous linear equations. The formulas for $v_1, \dots, v_n$ in terms of an arbitrary characteristic number λ of A can be considered as a new characteristic of a quiver (of the adjacency matrix).

Example 2 ([1, p. 94])**.** For the Dynkin graph $\widetilde{E}_7$,



the general formulas for the eigenvector $v = (v_1, \dots, v_8)^t$ in terms of eigenvalues are $v_1 = \lambda^3 - 2\lambda$, $v_2 = \lambda^4 - 4\lambda^2 + 3$, $v_3 = \lambda^2 - 2$, $v_4 = \lambda^2 - 1$, $v_5 = \lambda^5 - 5\lambda^3 + 5\lambda$, $v_6 = \lambda$, $v_7 = \lambda^4 - 5\lambda^2 + 5$, $v_8 = 1$.

Example 3 ([1, p. 97])**.** For the Dynkin graph $\widetilde{D}_5$,



the general formulas for the eigenvector in terms of eigenvalues are $v_1 = \lambda$, $v_2 = \lambda^2 - 2$, $v_3 = v_4 = 1$, $v_5 = v_6 = \frac{\lambda^3 - 3\lambda}{2}$.

2. Gauss-DGFKK expressions

We show that for any strongly connected simply laced quiver there exist expressions for the coordinates of eigenvectors in terms of eigenvalues. Recall the following fact.

Remark 2 (Remark 6.5.2 in [5]). A permutationally irreducible matrix $A \geqslant 0$ cannot have two linearly independent non-negative eigenvectors corresponding to the same eigenvalue.

Lemma 1. *Let Q be a strongly connected simply laced quiver on n vertices with a characteristic polynomial ξ. For any eigenvalue λ_0 there exist elements $f_1, \ldots, f_n$ of the field of fractions of $\mathbb{R}[x]$, such that $v^t = (f_1(\lambda_0), \ldots, f_n(\lambda_0))^t$ for the eigenvector v of Q, which corresponds to λ_0.*

Proof. If $\lambda_0 \in \mathbb{R}$, then v is also real, so its coordinates can be considered as the required polynomials $f_1, \ldots, f_n$, whence the lemma is trivial.

Suppose that $\lambda_0 \in \mathbb{C} \setminus \mathbb{R}$. Denote by $\overline{\lambda}_0$ the complex conjugate to λ_0 and let $f_0(x) = (x - \lambda_0)(x - \overline{\lambda}_0) \in \mathbb{R}[x]$. Denote by F_0 the field of fractions of the factor-ring $\mathbb{R}[x]/(f_0)$. Apply the classical Gaussian elimination algorithm of diagonalization to the matrix $A - \lambda E$ over the field F_0. This process leads to the general form of $g \in F_0^n$ such that $(A - \lambda E)g = 0 \in F_0$. By Remark 2, each of vectors $g(\lambda_0)$ and $g(\overline{\lambda}_0)$ is an eigenvector of Q, and they correspond to λ_0 and $\overline{\lambda}_0$, respectively. □

Lemma 2. *For any strongly connected simply laced quiver there exist real fractional-polynomial formulas, which express the coordinates of an eigenvector in terms of eigenvalues. In other words, let $v^{(1)}, \ldots, v^{(k)}$ be all linearly independent vectors of Q, which, by Remark 2, correspond to eigenvalues $\lambda_1, \ldots, \lambda_k$. Then there exist functions $f_1, \ldots, f_n$ from the field of fractions of $\mathbb{R}[x]$ such that*

$$(v^{(s)})^t = (f_1(\lambda_s), \ldots, f_n(\lambda_s))$$

for every s, $1 \leqslant s \leqslant k$.

Proof. Denote by $\xi = \xi_1 \cdot \ldots \cdot \xi_m$ the decomposition of the characteristic polynomial of Q into the product of indecomposable multipliers over $\mathbb{R}[x]$. By Lemma 1, for every i, $1 \leqslant i \leqslant m$ there exist elements $f_1^{(i)}, \ldots, f_n^{(i)}$ of the field of fractions of $\mathbb{R}[x]$ such that for every root λ of ξ_i the corresponding eigenvector $v(\lambda)$ can be expressed as $v(\lambda) = (f_1^{(i)}(\lambda), \ldots, f_n^{(i)}(\lambda))^t$. Thus,

$$v = \sum_{i=1}^{m} \sum_{\lambda:\, \xi_i(\lambda)=0} v(\lambda) \cdot \frac{\xi}{\xi_i},$$

which is the required vector by the construction. □

The proof of Lemma 2 motivates the following construction.
1. For a strongly connected simply laced quiver Q with the adjacency matrix A consider the matrix $B = A - \lambda E$ as one over the field of fractions of the ring of polynomials $\mathbb{R}[\lambda]$.
2. Apply the Gaussian elimination algorithm to reduce the matrix B to the upper diagonal form $\widetilde{B} = (\widetilde{b}_{ij})$.
3. Take the matrix $C = (c_{ij})$ such that $c_{nn} = 0$ and $c_{ij} = \widetilde{b}_{ij}$ otherwise.
4. Use the matrix C, as the matrix of a system of linear homogeneous equations to express variables $x_1, \ldots, x_{n-1}$ in terms of x_n. Thus, we obtain a vector $v(\lambda) = (x_1(\lambda), \ldots, x_{n-1}(\lambda),\, x_n)^t$, which provides a characteristic of Q.

Since Lemma 2 was motivated by [1], the vector $v(\lambda)$, constructed above, will be called the Gauss-DGFKK expression for eigenvectors of Q.

Example 4. Find Gauss-DGFKK expressions for the eigenvectors of the quiver $1 \longleftrightarrow 2$. The necessary transformations of the matrix are

$$\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} \Rightarrow \begin{pmatrix} -\lambda & 1 \\ 1 + \frac{-\lambda}{\lambda} & -\lambda + \frac{1}{\lambda} \end{pmatrix} = \begin{pmatrix} -\lambda & 1 \\ 0 & \frac{1-\lambda^2}{\lambda} \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -\lambda \\ 0 & \lambda^2 - 1 \end{pmatrix}.$$

Claim now that $\lambda^2 - 1$ is the characteristic polynomial of our quiver. Then the equation $x_1 - \lambda x_2 = 0$ with the assumption $x_2 = 1$ gives the Gauss-DGFKK expression $v = (\lambda,\, 1)$.

The following example provides an improvement of Theorem 2 by replacing the condition of equality of left eigenvectors to the condition of equality of the Gauss-DGFKK expressions.

Example 5. Gauss-DGFKK expressions for left eigenvectors of quivers in Example 1 are different.

Our computations show, that the Gauss-DGFKK expression for the right eigenvector of A_1 from Example 1 is $\left(\frac{1}{\lambda},\, \frac{\lambda^2+\lambda+1}{\lambda^3},\, \frac{\lambda+1}{\lambda^2},\, 1\right)^t$. Analogously, the Gauss-DGFKK expression for the right eigenvector of A_2 is $\left(\frac{1}{\lambda},\, \frac{-1-\lambda^2}{\lambda(1-\lambda^2)},\, \frac{-2}{1-\lambda^2},\, 1\right)^t$.

Nevertheless, note that numerical values of given eigenvectors are equal if λ is an eigenvalue of the corresponding matrix. Moreover, the numerical values of these expressions are equal if and only if λ is the eigenvalue of the quiver.

The next theorem follows from Theorem 2 and Remark 1.

Theorem 3. *If SCSL-quivers with four vertices have either different characteristic polynomials, or distinct Gauss-DGFKK expressions for the right eigenvector, then they are non-isomorphic.*

The following example shows that Theorem 3 is not a criterion.

Example 6. There are equivalent SCSL-quivers with four vertices, whose Gauss-DGFKK expressions for the right eigenvectors are different.

Let

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

be the adjacency matrix of a SCSL-quiver and B the adjacency matrix of the quiver, obtained from the SCSL-quiver by renumbering of vertices 3 and 4.

The Gauss-DGFKK expression for the right eigenvector of A is $\left(\frac{1}{\lambda}, \frac{1}{\lambda}, \frac{1}{\lambda^2}, 1\right)^t$. At the same time, the Gauss-DGFKK expression for the right eigenvector of B is $\left(\frac{1}{\lambda^2-1}, \frac{1}{\lambda^2-1}, \frac{\lambda}{\lambda^2-1}, 1\right)^t$.

Example 7. There exist quivers with 5 vertices, whose characteristic polynomials are different, but both left and right eigenvectors can be expressed by the same formulas.

Consider quivers, given by their adjacency matrices

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

Their characteristic polynomials are

$$f_1(\lambda) = \lambda^4 - \lambda - 1 \quad \text{and} \quad f_2(\lambda) = \lambda^4 - \lambda^2 - \lambda - 1$$

respectively. The Gauss-DGFKK formulas for the right eigenvectors are

$$v_r^{(1)} = \left(\frac{1}{\lambda}, \frac{1}{\lambda^3}, \frac{1}{\lambda^2}, 1\right)^t \quad \text{and} \quad v_r^{(2)} = \left(\frac{1}{\lambda}, \frac{1}{\lambda^3}, \frac{1}{\lambda^2}, 1\right)^t.$$

The Gauss-DGFKK formulas for the left eigenvectors are

$$v_l^{(1)} = \left(\frac{\lambda+1}{\lambda^3}, \frac{1}{\lambda}, \frac{\lambda+1}{\lambda^2}, 1\right) \quad \text{and} \quad v_l^{(2)} = \left(\frac{\lambda^2+\lambda+1}{\lambda^3}, \frac{1}{\lambda}, \frac{\lambda+1}{\lambda^2}, 1\right).$$

Nevertheless, formulas for vectors $v_l^{(1)}$ and $v_l^{(2)}$ can be rewritten as

$$v_l = \left(\lambda, \frac{1}{\lambda}, \frac{\lambda+1}{\lambda^2}, 1\right),$$

because $\lambda^4 = \lambda + 1$ for each solution of the equation $f_1(\lambda) = 0$, and $\lambda^4 = \lambda^2 + \lambda + 1$ for each solution of the equation $f_2(\lambda) = 0$.

Example 8. There are non-isomorphic quivers such that their characteristic polynomials are equal and Gauss-DGFKK expressions for right eigenvectors are also equal.

Let SCSL-quivers Q_1 and Q_2 be given by their adjacency matrices

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

These quivers are non-conjugated, because Q_1 has a vertex, such that for all other vertices there are arrows to it and Q_2 has not such a vertex.

Characteristic polynomials of quivers coincide and are equal to

$$-\lambda^5 + 3\lambda^3 + 4\lambda^2 + 2\lambda.$$

Gauss-DGFKK expressions for right eigenvectors v_1 and v_2 of A_1 and A_2 are

$$v_1 = v_2 = \left(\frac{\lambda+1}{\lambda^3-\lambda}, \frac{\lambda+1}{\lambda^3-\lambda}, \frac{1}{\lambda-1}, \frac{1}{\lambda-1}, 1\right)^t.$$

Acknowledgement

The authors would like to thank the referee for various useful comments and remarks, which have helped to improve the manuscript.

References

[1] M.A. Dokuchaev, N.M. Gubareni, V.M. Futorny, M.A. Khibina and V.V. Kirichenko, Dynkin Diagrams and Spectra of Graphs, *São Paulo Journal of Mathematical Sciences*, **7**, No. 1, 83-104 (2013).

[2] I. Dudchenko, V. Kirichenko and M. Plakhotnyk, Simply laced quivers and their properties, *Ukrainian Mathenatical Journal*, **64**, 2012, p. 291-306.

[3] P. Gabriel, Unzerlegbare Darstellungen 1, *Man. Math.*, 1972, v.6, p.71-103;

[4] M. Hazewinkel, N. Gubareni and V.V. Kirichenko, Algebras, Rings and Modules, V.I, Mathematics and Its Applications, Vol. **575**, Kluwer Academic Publishers, 2004, 380 p.

[5] M. Hazewinkel, N. Gubareni and V.V. Kirichenko, Algebras, Rings and Modules, V.II, Mathematics and Its Applications, Vol. **586**, Springer, 2007, 400 p.

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Received by the editors: 11.07.2018
and in final form 19.03.2019.