Algebra and Discrete Mathematics RESEARCH ARTICLE
Volume **26** (2018). Number 2, pp. 247–255

On a graph isomorphic to its intersection graph: self-graphoidal graphs

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Communicated by V. A. Artamonov

Abstract. A graph G is called a graphoidal graph if there exists a graph H and a graphoidal cover ψ of H such that $G \cong \Omega(H, \psi)$. Then the graph G is said to be self-graphoidal if it is isomorphic to one of its graphoidal graphs. In this paper, we have examined the existence of a few self-graphoidal graphs from path length sequence of a graphoidal cover and obtained new results on self-graphoidal graphs.

All graphs considered here are connected, finite, simple and nontrivial. The *order* and the *size* of a graph G are denoted by p and q respectively. For definitions and notations not defined here we refer to [6]. Suppose $P = (v_0, v_1, v_2, \ldots, v_{n-1}, v_n)$ is a path in G. Then the vertices $v_1, v_2, \ldots, v_{n-1}$ are called *internal vertices* of P and the vertices v_0, v_n are called *external vertices* of P. A path of the form $P = (v_0, v_1, v_2, \ldots, v_{n-1}, v_n = v_0)$ is a closed path and v_0 is taken as its only *external vertex*. Let ψ be a collection of internally edge disjoint paths in G. A vertex of G is said to be an internal vertex of ψ if it is an internal vertex of some path(s) in ψ, otherwise it is called an external vertex of ψ. B. D. Acharya and E. Sampathkumar [2] introduced the concept of *graphoidal cover* as follows

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2010 MSC: 05C38, 05C75.

Key words and phrases: graphoidal cover, graphoidal covering number, graphoidal graph, self-graphoidal graph.

1 Definition. *A graphoidal cover of a graph G is a collection ψ of (not necessarily open) paths in G satisfying the following conditions:*

- *Every path in ψ has at least two vertices.*
- *Every vertex of G is an internal vertex of at most one path in ψ.*
- *Every edge of G is in exactly one path in ψ.*

The set of all graphoidal covers of G is denoted by $\mathcal{G}_G$. The minimum cardinality of a graphoidal cover of G is called the *graphoidal covering number* of G and is denoted by $\eta(G)$ or η if G is clear from the context. Clearly, $E(G) \in \mathcal{G}_G$ and hence one has $\eta(G) \leqslant |E(G)| = q$.

E. Marczewski [5] introduced the concept of an *intersection graph* as follows.

2 Definition. *If $\mathcal{F} = \{S_1, S_2, S_3, \ldots, S_n\}$ is a family of distinct nonempty subsets of a set S whose union is S then the intersection graph of $\mathcal{F}$, denoted by $\Omega(\mathcal{F})$, is the graph whose vertex- and edge- sets are given by $V_{\Omega(\mathcal{F})} = \{S_1, S_2, \ldots, S_n\}$ and $E_{\Omega(\mathcal{F})} = \{S_iS_j : i \neq j \text{ and } S_i \cap S_j \neq \varnothing\}$.*

For a graph G and $\psi \in \mathcal{G}_G$, the *intersection graph* on ψ is denoted by $\Omega(G, \psi)$.

3 Definition. *A graph G is called a graphoidal graph if there exists a graph H and $\psi \in \mathcal{G}_H$ such that $G \cong \Omega(H, \psi)$.*

Since $E(G) \in \mathcal{G}_G$ this notion generalizes the notion of well known *line graph*.

Let us denote by $\Theta(G) = \{H : G \cong \Omega(H, \psi)$, for some $\psi \in \mathcal{G}_H\}$. Then G is *graphoidal* if and only if $\Theta(G) \neq \varnothing$. Further, if $\Theta(G) \neq \varnothing$ then $\Theta(G)$ contains infinitely many graphs H.

4 Definition. *A graphoidal graph G is said to be self-graphoidal if it is isomorphic to one of its graphoidal graphs.*

The following problem has been proposed in [10].

Problem 1. *Which graphoidal graphs G satisfy $G \in \Theta(G)$?*

Partial solutions to this problem have been obtained in [8]. In case, all edges in the graph are taken as a graphoidal cover. Then, we get $\Omega(G, \psi) \cong L(G)$, where $L(G)$ is the *line graph* of G.

5 Theorem ([8])**.** *If G is self-graphoidal then the number of paths in a graphoidal cover of G is equal to the number of vertices in G.*

6 Corollary. *No tree is a self-graphoidal graph.*

7 Theorem ([8])**.** *$K_{2,2}$ is the only complete bipartite graph which is self-graphoidal.*

The converse of the theorem 5 is not true. Consider the complete bipartite graph $K_{3,6}$ in which the number of paths in the graphoidal cover is equal to the number of its vertices. But, it is not self-graphoidal by theorem 7.

8 Theorem ([1])**.** *For any graph G with $\delta \geqslant 3$, $\eta(G) = q - p$.*

9 Definition. *The path length sequence $l_1, l_2, \ldots, l_n$, where l_i, $i = 1, 2, \ldots, n$ is the length of its paths in an edge disjoint path decomposition usually written in nonincreasing order, as $l_1 \geqslant l_2 \geqslant \ldots \geqslant l_n$, such that $\sum l_i = q$.*

Problem 2. *Given nonnegative integers with $l_1 \geqslant \ldots \geqslant l_n$, does there exist a self-graphoidal graph with the path length sequence $\langle l_1, l_2, \ldots, l_n \rangle$?*

10 Theorem ([8])**.** *Every cycle is self-graphoidal.*

11 Theorem ([8])**.** *There exists a 3-regular self-graphoidal graph on $p \equiv 0 \pmod 4$ vertices.*

12 Theorem ([8])**.** *There exists a 4-regular self-graphoidal graph.*

13 Lemma. *If any path length sequence of a regular graph G has at least one path of length greater than 3 then G is not self-graphoidal.*

Proof. Let ψ be any graphoidal cover of G. Then, by § 5, we get

$$|\psi| = p \Rightarrow \eta = p \Rightarrow d = 4.$$

Hence, G is self-graphoidal if $d \leqslant 4$. Also, by § 10, § 11 and § 12, we get $2 \leqslant d \leqslant 4$. Again, we know that each vertex of degree d, which is an internal vertex of one path, appears as an external vertex of $d - 2$ other paths in ψ. Let ψ be a graphoidal cover of G satisfying § 5 and P be a path of length 4 in ψ. Then P has three internal vertices and so, the degree of P as a vertex in the intersection graph $\Omega(G, \psi)$ is at least 3 if G is a 3-regular graph and greater than 4 if G is a 4-regular graph. Thus in either case $\Omega(G, \psi)$ will not be regular. Again, there does not exist any path of length $\geqslant 4$ in a 2-regular graph satisfying § 5. Hence the lemma follows. □

The following results are the consequence of § 5, § 11, § 12 and § 13.

14 Theorem. *If a 3-regular graph G on $p \equiv 0 (mod\ 4)$ vertices has a path length sequence $\langle l_i \rangle$, $i = 1, 2, \dots, n$ such that $\sum l_i = q$ then G is self-graphoidal.*

Proof. Let the vertices of G be labeled in the cyclic order as $v_1, v_2, \dots, v_p$.

If $p = 4$ then, by § 5 and § 13, the only possible path length sequences are (i) $(3, 1, 1, 1)$ and (ii) $(2, 2, 1, 1)$.

Corresponding to the path length sequence (i) construct the path cover as $\psi = \{(v_1, v_2, v_3, v_1), (v_3, v_4), (v_2, v_4), (v_4, v_1)\}$. Then ψ is a graphoidal cover of G and $\Omega(G, \psi) \cong G$.

For the path length sequence (ii), we can make G self-graphoidal by changing one pair of path lengths 3 and 1 to 2 and 2 as $\psi = \{(v_2, v_1, v_3), (v_2, v_3, v_4), (v_2, v_4), (v_4, v_1)\}$.

If $p = 8$ then, by § 5 and § 13, the path length sequences are (i) $(3, 3, 1, 1, 1, 1, 1, 1)$, (ii) $(3, 2, 2, 1, 1, 1, 1, 1)$ and (iii) $(2, 2, 2, 2, 1, 1, 1, 1)$.

Corresponding to the path length sequence (i) construct the path cover as $\psi = \{(v_1, v_2, v_3, v_1), (v_5, v_6, v_7, v_5), (v_3, v_4), (v_2, v_4), (v_4, v_5), (v_7, v_8), (v_6, v_8), (v_8, v_1)\}$. Then ψ is a graphoidal cover of G and $\Omega(G, \psi) \cong G$.

For the path length sequence (ii), we can make G self-graphoidal by changing one pair of path lengths 3 and 1 to 2 and 2 as $\psi = \{(v_5, v_6, v_7, v_5), (v_2, v_1, v_3), (v_2, v_3, v_4), (v_2, v_4), (v_4, v_5), (v_7, v_8), (v_6, v_8), (v_8, v_1)\}$. Similarly, we can also make the path length sequence (iii) self-graphoidal.

In general, the path cover of G may be constructed as

$$\psi = \{(v_i, v_{i+1}, v_{i+2}, v_i), (v_{i+2}, v_{i+3}), (v_{i+1}, v_{i+3}), (v_{i+3}, v_{i+4}) \pmod{p}\},$$

where $i = 2n + 1$, $n = 0, 2, 4, \dots$ and $i < p$. Then ψ is a graphoidal cover of G and $\Omega(G, \psi) \cong G$.

The remaining path length sequences can be made self-graphoidal by changing one pair of path lengths 3 and 1 to 2 and 2 as

$$((v_i, v_{i+1}, v_{i+2}, v_i), (v_{i+2}, v_{i+3})) \quad \text{to} \quad ((v_{i+1}, v_i, v_{i+2}), (v_{i+1}, v_{i+2}, v_{i+3})).$$

We continue this process till there is a path of length 3 in ψ. □

15 Theorem. *If a 4-regular graph G has a path length sequence $\langle l_i \rangle$, $i = 1, 2, \dots, n$ such that $\sum l_i = q$ then G is self-graphoidal.*

Proof. Let the vertices of G be labeled in cyclic order as $v_1, v_2, \dots, v_p$.

If $p = 5$ then, by § 5 and § 13, the path length sequences are (i) $(3, 3, 2, 1, 1)$, (ii) $(3, 2, 2, 2, 1)$ and (iii) $(2, 2, 2, 2, 2)$.

Corresponding to path length sequence (i) construct the path cover as $\psi = \{(v_4, v_2, v_3, v_4), (v_1, v_4, v_5, v_1), (v_3, v_1, v_2), (v_3, v_5), (v_5, v_2)\}$. Then ψ is a graphoidal cover of G and $\Omega(G, \psi) \cong G$.

For the path length sequence (ii), we can make G self-graphoidal by changing one pair of path lengths 3 and 1 to 2 and 2 as $\psi = \{(v_1, v_4, v_5, v_1), (v_3, v_1, v_2), (v_4, v_2, v_3), (v_4, v_3, v_5), (v_5, v_2)\}$. Similarly, we can also make the path length sequence (iii) self-graphoidal.

If $p = 6$ then, by § 5 and § 13, the path length sequences are (i) $(3, 3, 3, 1, 1, 1)$, (ii) $(3, 3, 2, 2, 1, 1)$, (iii) $(3, 2, 2, 2, 2, 1)$ and (iv) $(2, 2, 2, 2, 2, 2)$.

Corresponding to path length sequence (i) construct the path cover as $\psi = \{(v_2, v_3, v_4, v_2), (v_4, v_5, v_6, v_4), (v_6, v_1, v_2, v_6), (v_3, v_5), (v_5, v_1), (v_1, v_3)\}$. Then ψ is a graphoidal cover of G and $\Omega(G, \psi) \cong G$.

For the path length sequence (ii), we can make G self-graphoidal by changing one pair of path lengths 3 and 1 to 2 and 2 as $\psi = \{v_6, v_5, v_4, v_6), (v_2, v_1, v_6, v_2), (v_3, v_2, v_4), (v_4, v_3, v_5), (v_5, v_1), (v_1, v_3)\}$.

Similarly, we can also make the remaining path length sequences self-graphoidal.

In general, construct the path cover of G as follows:

For $p =$ even,

$$\psi = \{(v_{i+1}, v_{i+2}, v_{i+3}, v_{i+1}) \pmod p, (v_{i+2}, v_{i+4}) \pmod p\},$$

where $i =$ odd and $i < p$. Then ψ is a graphoidal cover of G and $\Omega(G, \psi) \cong G$.

The remaining path length sequences can be made self-graphoidal by changing one pair of path lengths 3 and 1 to 2 and 2 as

$$((v_{i+1}, v_{i+2}, v_{i+3}, v_{i+1}) \pmod p, (v_{i+2}, v_{i+4}) \pmod p)$$

to

$$((v_{i+2}, v_{i+1}, v_{i+3}) \pmod p, (v_{i+3}, v_{i+2}, v_{i+4}) \pmod p).$$

We continue this process till there is a path of length 3.

For $p =$ odd,

$$\psi = \{(v_{i+3}, v_{i+1}, v_{i+2}, v_{i+3}) \pmod p, (v_{i+2}, v_{i+4}) \pmod p, (v_3, v_1, v_2)\},$$

where $i =$ odd and $i < p$. Then ψ is a graphoidal cover of G and $\Omega(G, \psi) \cong G$.

The remaining path length sequences can be made self-graphoidal by changing one pair of path lengths 3 and 1 to 2 and 2 as

$$((v_{i+3}, v_{i+1}, v_{i+2}, v_{i+3}) \pmod p, (v_{i+2}, v_{i+4}) \pmod p)$$

to

$$((v_{i+3}, v_{i+1}, v_{i+2}) \pmod p), (v_{i+3}, v_{i+2}, v_{i+4}) \pmod p)).$$

We continue this process till there is a path of length 3 in ψ. □

16 Theorem ([8]). *A complete graph K_p is self-graphoidal if and only if $3 \leqslant p \leqslant 5$.*

From §5, § 13, § 14, § 15 and § 16, we have

17 Corollary. *If a complete graph K_p has a path length sequence $\langle l_i \rangle$, $i = 1, 2, \ldots, n$ such that $\sum l_i = q$ then K_p is self-graphoidal.*

18 Theorem ([8]). *Wheel W_p is self-graphoidal if and only if $p = 4, 5$.*

19 Lemma. *If any path length sequence of a wheel W_p has at least one path of length greater than 3 then W_p is not self-graphoidal.*

Proof. Without loss of generality, label the vertices of W_p as $v_1, v_2, \ldots, v_p$ with degree $v_1 = p - 1$ and the corresponding vertices of $\Omega(W_p, \psi)$ as $P_1, P_2, \ldots, P_p$ with degree $P_1 = p - 1$. Suppose there exits a path(cycle) P of length 4 in ψ.

If $p = 4$, then there does not exist any path length sequence of length 4 satisfying § 5. So, by § 18, we take $p = 5$.

Let P be a path in W_5. Then v_1 is either internal or external vertex in P since $\deg v_1 = 4$. Hence, the intersection graph $\Omega(W_5, \psi)$ contains a complete subgraph K_4, which contradicts the fact that no wheel contains complete subgraph other than K_2 and K_3.

Let P be a cycle in W_5. Then either $v_1 \notin P$ or $v_1 \in P$. If $v_1 \notin P$ then $\Omega(W_5, \psi)$ again contains a complete subgraph K_4 leading to a contradiction. So, take the cycle as $P = (v_2, v_3, v_4, v_1, v_2)$. Then the vertices v_1 and v_3 will occur respectively in three and two paths of ψ. But the vertex P_2 corresponding to path $P_2 = (v_1, v_3)$ will be of degree 2 in the intersection graph $\Omega(W_5, \psi)$, which is again a contradiction. □

20 Theorem. *If a wheel W_p has a path length sequence $\langle l_i \rangle$, $i = 1, 2, \ldots, n$, such that $\sum l_i = q$ then W_p is self-graphoidal.*

Proof. Without loss of generality, label the vertices of W_p as $v_1, v_2, \ldots, v_p$ with degree $v_1 = p - 1$ and the corresponding vertices of $\Omega(W_p, \psi)$ as $P_1, P_2, \ldots, P_p$ with degree $P_1 = p - 1$. Then, by § 18, we get $p = 4, 5$.

If $p = 4$ then W_4 is same as K_4 which is known to be self-graphoidal.

If $p = 5$ then, by § 5 and § 19, the only possible path length sequences are (i) $(3, 2, 1, 1, 1)$ and (ii) $(2, 2, 2, 1, 1)$.

Case (i). There are two subcases to consider.

Subcase(a). Let $P_1 = (v_5, v_1, v_3, v_4)$ be a path of length 3 in ψ. Then for a path length 2, we have to make one of the remaining vertices v_2, v_4 or v_5 as internal vertex. But, we know that W_5 contains nonadjacent vertices in pairs. So, we have to make v_2 as internal vertex and the remaining paths in ψ are the single edges.

Subcase(b). Let $P = (v_2, v_3, v_1, v_2)$ be a cycle of length 3 in ψ then neither v_4 nor v_5 can be made internal vertices so that $\Omega(W_5, \psi)$ contains nonadjacent vertices in pairs. Thus, we have to take $P_1 = (v_5, v_1, v_4)$ and $P_2 = (v_1, v_2, v_3, v_1)$ along with remaining single edges in ψ.

Case (ii). Let $P_1 = (v_2, v_1, v_3)$ be a path of length 2 with v_1 as internal vertex then the path nonadjacent to P_1 in the intersection graph $\Omega(W_5, \psi)$ is $P_2 = (v_4, v_5)$ which are vertex disjoint. In such case, we cannot find another pair of paths which are vertex disjoint in the $\Omega(W_5, \psi)$ and satisfy § 5. So, take $P_1 = (v_2, v_1, v_3)$ and corresponding to $P_2 = (v_2, v_3)$ we can make $P_3 = (v_5, v_4, v_1)$ vertex disjoint. Also the remaining paths are vertex disjoint and satisfy § 5.

Hence, in either case, we can define a mapping $v_i \leftrightarrow P_i, 1 \leqslant i \leqslant 5$ and $v_i v_j \leftrightarrow P_i P_j, 1 \leqslant i \leqslant j \leqslant 5$, each of which is a one-to-one correspondence between wheel W_5 and $\Omega(W_5, \psi)$. □

21 Theorem. *Let G be a connected triangle-free graph. Then G is self-graphoidal if and only if G is a cycle of length at least 4.*

Proof. Suppose G is a triangle-free self-graphoidal graph. Let $P_1,\ P_2$ be any two edge-disjoint paths in G such that $P_1 \cap P_2 \neq \varnothing$ then there does not exit any path P_3 edge disjoint from P_1 and P_2 such that $P_1 \cap P_3 \neq \varnothing$ and $P_2 \cap P_3 \neq \varnothing$, otherwise there would be a triangle formed by P_1, P_2 and P_3 in $\Omega(G, \psi)$ contradicting the choice of G. Then for any vertex v in G with $\deg v > 2$, which is an internal vertex of one path, there exist $(\deg v - 1)$ paths in ψ containing v and these paths form a complete subgraph in $\Omega(G, \psi)$. Hence, $\Delta(G) \leqslant 3$. From § 5 and § 8, we get $\eta < |\psi|$, i.e., there exists some vertices which are not internal to any path in ψ. This gives a contradiction to the choice of G. Hence, G is a cycle of length greater than 3.

The converse part follows from § 10. □

Suppose $e = uv$ is an edge of G and w is not a vertex of G, then e is subdivided when it is replaced by the edges uw and wv. If every edge of G is subdivided, the resulting graph is the *subdivision graph* $S(G)$.

22 Corollary. *The subdivision graph $S(G)$ of any self-graphoidal graph G is self-graphoidal if and only if G is a cycle.*

23 Theorem. *If a graph G is self-graphoidal then $\delta \geqslant 2$.*

Proof. Suppose G is a graph with a pendant vertex, say v. Let u be the unique vertex adjacent to v. Let P_1 and P_2 be the two paths in $\Omega(G, \psi)$ corresponding to vertices u and v respectively in G. Then $P_2 \subseteq P_1$ or some elements of P_2 belong to P_1 and the remaining elements of P_2 does not belong to any other paths in ψ.

If G is a graph with a subgraph $K_{1,3}$ or $K_4 - e$ or bull graph then we can find a path P_2 in G. Hence $\kappa'(G - v) \leqslant 3$.

Now, we have to examine if there exists a path P_2 such that $\Omega(G, \psi) \cong G$.

If $\kappa'(G) = 1$ then by § 5 we get a contradiction to the choice of G.

If $\kappa'(G - v) = 2$ then $G - v$ is a cycle or a Whitney-Robbins synthesis from a cycle (see [7, Theorem 5.2.4, p.224]). Now, if $G - v$ is a cycle, then by adding v back to $G - v$ we get a unicyclic graph with unique pendant vertex v and a cut vertex u. Also G satisfies § 5. Let x, y and z be the three edges incident to u. But these edges create a cycle in $\Omega(G, \psi)$ and $|E(G)| + 1 = E|\Omega(G, \psi)|$, which is a contradiction. Similarly, we can show that the graph G, when $G - v$ is obtained from a cycle by a Whitney-Robbins synthesis, is also not self-graphoidal.

If $\kappa'(G - v) = 3$ then $G - v$ contains $K_4 - e$ or bull graph as a subgraph and the length of P_1 in $G - v$ is 4 and by § 13 and § 19, there does not exist any ψ such that $\Omega(G, \psi) \cong G$. □

24 Corollary. *If a graph G is self-graphoidal then G does not contain any pendant vertices.*

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Received by the editors: 21.01.2016
and in final form 06.11.2018.