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RESEARCH ARTICLE

On regular torsionless S-posets

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Abstract. This paper shall be concerned with the notion of regular torsionless in the category of S-posets. Besides elementary basic properties of regular torsionless S-posets, we consider cyclic regular torsionless S-posets and also study when regular torsionless property is preserved under coproducts. Then we characterize pomonoids over which all free or projective S-posets are regular torsionless. Finally, we present conditions on S which follow if all regular torsionless S-posets are principally weakly po-flat, weakly po-flat, strongly flat, or projective.

Introduction

Over the past three decades, an extensive theory of the properties of S-acts has been developed. A comprehensive survey of this area was published in 2000 by Kilp et al. in [3]. The category of S-posets, as the ordered version of the category of S-acts, recently has captured the interest of some mathematicians [1,2]. There are many papers attempting to generalize some properties including projectivity and various kinds of flatness properties from S-acts to S-posets (see, for example, [9–11]). In the category of S-acts, torsionless right acts over a monoid S are acts A_S such that the natural homomorphism φ_A from A_S into its second dual is injective. As far as we know torsionless acts are introduced and considered in [4]. In this paper we introduce (regular) torsionless S-posets as the

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S-posets for which φ_A is a (regular) monomorphism. In Section 1, we give some basic properties of regular torsionless S-posets. In particular, we discuss when regular torsionless property is preserved under coproducts and give the conditions over which the amalgamated coproduct $A(I)$, cyclic S-posets and Rees factor S-posets are regular torsionless. In Section 2, pomonoids over which free or projective S-posets are regular torsionless are characterized. Finally, some necessary conditions for regular torsionless S-posets to satisfy some flatness properties are given.

First we give some preliminaries needed in the sequel. A monoid S endowed with a partial order, compatible with the binary operation, is called a pomonoid. For a pomonoid S a right S-poset is a poset A which is also a right S-act whose action is monotone in both arguments. A right S-subposet of a right S-poset A_S is a nonempty subset of A that is closed under the action of S. Moreover, S-poset morphisms or simply S-morphisms are monotone maps between S-posets which preserve actions. The class of right S-posets and S-morphisms form a category, denoted by POS-S. Let S be a pomonoid and I a nonempty set of S. In the following statements, if we say I is a *right ideal* of S, it only means $IS \subseteq S$. A *right poideal* of a pomonoid S is a subset I of S which is both a right ideal ($IS \subseteq I$) and a poset ideal (that is, $a \leqslant b,\ b \in I$ imply $a \in I$). For $s \in S$, $(sS]$ is called a principal right poideal of S.

Let A be a right S-poset. An S-poset congruence θ on A is a right S-act congruence with the property that the S-act A/θ can be made into an S-poset in such a way that the natural map $A \to A/\theta$ is an S-morphism. For an S-act congruence θ on A we write $a \leqslant_\theta a'$ if the so-called θ-chain

$$a \leqslant a_1 \theta b_1 \leqslant a_2 \theta b_2 \ \ldots \ \leqslant a_n \theta b_n \leqslant a'$$

from a to a' exists in A, where $a_i, b_i \in A,\ 1 \leqslant i \leqslant n$. It can be shown that an S-act congruence θ on a right S-poset A is an S-poset congruence if and only if $a\theta a'$ whenever $a \leqslant_\theta a' \leqslant_\theta a$. For two S-poset congruences θ, σ on A_S, we say that $\theta \leqslant \sigma$ if $x \leqslant_\theta y$ implies $x \leqslant_\sigma y$ for each $x, y \in A_S$. Let $H \subseteq A \times A$. Then $a \leqslant_{\alpha(H)} b$ if and only if $a \leqslant b$ or there exist $n \geqslant 1, (c_i, d_i) \in H, s_i \in S, 1 \leqslant i \leqslant n$ such that

$$a \leqslant c_1 s_1 \ d_1 s_1 \leqslant c_2 s_2 \ \ldots \ d_n s_n \leqslant b.$$

The relation $\nu(H)$ given by $a \ \nu(H) \ b$ if and only if $a \leqslant_{\alpha(H)} b \leqslant_{\alpha(H)} a$ is the S-poset congruence induced by H.

The subkernel of an S-poset morphism $f : A_S \to B_S$ is defined by $\overrightarrow{\ker} f := \{(a, a') \in A \times A : f(a) \leqslant f(a')\}$. Then $\nu(\overrightarrow{\ker} f) = \ker f :=$

$\{(a,a') \in A \times A : f(a) = f(a')\}$, and in $A_S/\ker f$, $[a]_{\ker f} \leqslant [a']_{\ker f}$ if and only if $f(a) \leqslant f(a')$. An S-morphism $f : A \to B$ is a *regular monomorphism* if it is an *order-embedding*, i.e., $a \leqslant a' \Longleftrightarrow f(a) \leqslant f(a')$, for all $a, a' \in A$. Obviously, f is a regular monomorphism if and only if $\overrightarrow{\ker} f = \xi_A = \{(a,a') \in A \times A | \ a \leqslant a'\}$. S-isomorphism means both monomorphism and epimorphism. A surjective order embedding of S-posets is called an *order isomorphism*.

Now we recall the concepts we use in Section 2, more information on these concepts can be found in [1,10]. A right S-poset A_S is weakly po-flat if $a \otimes s \leqslant a' \otimes t$ in $A_S \otimes S$ implies that the same inequality holds also in $A_S \otimes {}_S(Ss \cup St)$ for $a, a' \in A_S, s, t \in S$. A right S-poset A_S is principally weakly po-flat if $as \leqslant a's$ implies that $a \otimes s \leqslant a' \otimes s$ in $A_S \otimes {}_SSs$ for $a, a' \in A_S, s \in S$. A right S-poset A_S satisfies the condition

(P) if, for all $a, b \in A$ and $s, t \in S$, $as \leqslant bt$ implies $a = a'u, b = a'v$ for some $a' \in A, \ u, v \in S$ with $us \leqslant vt$,

and it satisfies condition

(E) if, for all $a \in A$ and $s, t \in S$, $as \leqslant at$ implies $a = a'u$ for some $a' \in A$, $u \in S$ with $us \leqslant ut$.

A right S-poset is called strongly flat if it satisfies both conditions (P) and (E). Projectivity and freeness are defined in the standard categorical manner.

We recall from [11] that an S-subposet B of an S-poset A is called *strongly convex* if for any $a \in A$ and any $b \in B$, $a \leqslant b$ implies $a \in B$. By [11, Theorem 2.3], every S-poset A is uniquely decomposable into a disjoint union of strongly convex indecomposable S-subposets. Now, using [11, Theorem 3.4], for a projective S-poset P all its strongly convex indecomposable S-subposets are cyclic projective. Moreover, by [11, Proposition 3.2], aS is projective if and only if there exists an idempotent element $e \in S$ such that $a = ae$, and $as \leqslant at$ implies $es \leqslant et$ for each $s, t \in S$, in this case a is called *left e-po-cancellable*.

Concluding this section we give brief results describing when direct products of S are projective, which will be needed in Section 2. If S is a pomonoid, the cartesian product S^I is a right S-poset equipped with the order and the action componentwise where I is a non-empty set. Moreover, $(s_i)_{i \in I} \in S^I$ is denoted simply by $(s_i)_I$ or $\vec{s}$, and the right S-poset $S \times S$ will be denoted by $D(S)$.

We say that two elements a, b of an S-poset A_S are comparable if $a \leqslant b$ or $b \leqslant a$ and denote this relation by $a \nparallel b$.

Definition 1. Let S be a pomonoid and I be a nonempty set. For each $\vec{a} \in S^I$ we define:

$$\underline{L}(\vec{a}) = \{\vec{b} \in S^I \mid b_i a_i \nparallel b_j a_j,\ \forall i, j \in I\}.$$

Note that $\underline{L}(\vec{a})$ is either empty or left S-subposets of S^I.

Proposition 1. *Let S be a pomonoid such that for each $J \neq \varnothing$, S^J is projective right S-poset. Then for every nonempty set I and every $\vec{a} \in S^I$, $\underline{L}(\vec{a})$ is either empty or cyclic left S-poset of S^I.*

Proof. Suppose that $\underline{L}(\vec{a}) \neq \varnothing$ for $\vec{a} \in S^I$. Put $\underline{L}(\vec{a}) = \{\vec{b^j} \mid j \in J\}$, where $\vec{b^j} = ({b_i}^j)_I$ for each $j \in J$. Consider $({b_i}^j)_J$ and $({b_k}^j)_J$ for $i, k \in I$. We have

$$({b_i}^j)_J \ \nparallel \ 1({b_i}^j)_J, \qquad ({b_i}^j)_J a_i \nparallel 1({b_k}^j)_J a_k \quad \text{and} \quad ({b_k}^j)_J \ \nparallel \ 1({b_k}^j)_J.$$

So by [11, Proposition 2.6], these elements belong to a strongly convex indecomposable S-subposet of S^J. As we mentioned earlier, using Theorem 3.4 and Proposition 3.2 of [11] since S^J is projective, let this indecomposable S-subposet be $\vec{p}S$, where $\vec{p} = (p_j)_J$, and $\vec{p}$ be left e-po-cancellable for some idempotent $e \in S$. Then for each $i \in I$ there exists $u_i \in S$ such that $({b_i}^j)_J = \vec{p}u_i$. Now, we get

$$\vec{p}u_i a_i = ({b_i}^j)_J a_i \nparallel 1({b_k}^j)_J a_k = \vec{p}u_k a_k,$$

which implies that $eu_i a_i \nparallel eu_k a_k$. Thus, $\vec{q} = (eu_i)_I \in \underline{L}(\vec{a})$. Moreover, for $\vec{b^j} \in \underline{L}(\vec{a})$, we have

$$\vec{b^j} = ({b_i}^j)_I = p_j u_i = p_j e u_i = p_j (eu_i)_I = p_j \vec{q}.$$

Therefore, $\underline{L}(\vec{a}) = S\vec{q}$ is cyclic. $\square$

Corollary 1. *Let S be a pomonoid such that for each $J \neq \varnothing$, S^J is projective right S-poset. Then S is right po-cancellative.*

Proof. Let $a, s, t \in S$ be such that $sa \leqslant ta$ and $s \neq t$. Take $\vec{a} = (a)_S \in S^S$. Then $\{s, t\}^S \subseteq \underline{L}(\vec{a})$, which gives that $|\underline{L}(\vec{a})| \geqslant 2^{|S|}$. Thus $\underline{L}(\vec{a})$ is not cyclic, which is a contradiction to the previous proposition. $\square$

1. Regular torsionless S-posets

In this section we introduce regular torsionless S-posets and give some basic results. First we need some preliminaries.

Let A_S be a right S-poset. Then $\hom(A_S, S_S)$, if not empty, is a left S-poset under the left multiplication $(sf)(a) = sf(a)$ and the order $f \leqslant g \iff (\forall a \in A_S)(f(a) \leqslant g(a))$ for every $f, g \in \hom(A_S, S_S)$, $s \in S$ and $a \in A_S$. The left S-poset $\hom(A_S, S_S)$ is called the *dual* of A_S and denoted by $(A_S)^*$. Note that $\hom(A_S, S_S)$ can be considered as an S-subposet of $({}_SS)^I$ where I has the same cardinality as the underlying set of A_S. Moreover, $\hom(-_S, S_S)$ is a contravariant functor from the category of all right S-posets POS-S into the category all left S-posets S-POS. This functor is called the *dual* functor. If $f : B_S \to A_S$ is an S-morphism, then $\hom(f, S_S)$ will be denoted by f^*. Right-left dually, if ${}_SA$ is a left S-poset then $\hom({}_SA, {}_SS)$ is a right S-poset.

If A_S is a right S-poset then $((A_S)^*)^*$ is called the *second dual* of A_S and denoted by $(A_S)^{**}$. If f is an S-morphism of right S-posets then $(f^*)^*$ will be denoted by f^{**}. If A_S is a right S-poset and its second dual exists then the mapping

$$\varphi_A : A_S \to \hom(\hom(A_S, S_S), S_S) = (A_S)^{**}$$

defined by $\varphi_A(a)(f) = f(a)$ for every $a \in A_S$ and $f \in \hom(A_S, S_S)$ is an S-morphism of right S-posets. These S-morphisms determine a natural transformation $\varphi = (\varphi_A)_{A_S \in \text{POS-}S}$,

$$\varphi : \mathrm{Id}_{\text{POS-}S} \to \hom(\hom(-_S, S_S), S_S).$$

In what follows φ_A will be called the natural S-morphism from A_S into $(A_S)^{**}$. Now the preliminaries are prepared to present our main definition.

Definition 2. An S-poset A_S is called

(i) *torsionless* if φ_A is a monomorphism, i.e., $\ker \varphi_A = \Delta_A$,

(ii) *regular torsionless* if φ_A is a regular monomorphism, i.e., $\overrightarrow{\ker}\varphi_A = \xi_A$,

(iii) *dense* if φ_A is surjective,

(iv) *reflexive* if φ_A is an isomorphism, and

(v) *regular reflexive* if φ_A is an order isomorphism.

Notice that since $\varphi_A(a) \in (A_S)^{**}$ for every $a \in A_S$, second duals exist whenever duals exist. We start with two statements which are immediate from the definition.

Lemma 1. *An S-poset A_S, $|A_S| > 1$, is regular torsionless (torsionless) if and only if for every $x, y \in A_S$, $x \nleq y (x \neq y)$, there exists $f \in \hom(A_S, S_S)$ such that $f(x) \nleq f(y)$ $(f(x) \neq f(y))$.*

Lemma 2. *The dual of the one-element right S-poset Θ_S exists and Θ_S is regular torsionless if and only if S contains a left zero. In this case all right S-posets have duals.*

Lemma 3. *The following hold for a pomonoid S_S.*

(i) *S_S is regular torsionless.*

(ii) *Every S-subposet of a regular torsionless S-poset is regular torsionless.*

(iii) *A retract of a regular torsionless S-poset is regular torsionless.*

(iv) *Let $\{A_i\}_{i\in I}$ be a family of regular torsionless S-posets. If $\coprod_{i\in I} A_i$ is regular torsionless, then each A_i is regular torsionless.*

(v) *Let $\{A_i\}_{i\in I}$ be a family of regular torsionless S-posets , then $\prod_{i\in I} A_i$ is regular torsionless.*

Proof. (i) If $x, y \in S_S$, $x \nleq y$, then for $\mathrm{id_S} \in \mathrm{hom(S_S, S_S)}$ we have $\mathrm{id_S(x)} \nleq \mathrm{id_S(y)}$. So S_S is regular torsionless by Lemma 1.

(ii) Let B_S be an S-subposet of A_S. If $x, y \in B_S$, $x \nleq y$, then there exists $f \in \hom(A_S, S_S)$ such that $f(x) \nleq f(y)$. Now $f|_{B_S} \in \hom(B_S, S_S)$ and $f|_{B_S}(x) \nleq f|_{B_S}(y)$. Hence B_S is regular torsionless by Lemma 1.

(iii) and (iv) immediately follow from (ii).

(v) Let $\pi_i : \prod_{i\in I} A_i \to A_i$, $i \in I$, be the projections. If $x = (x_i)_I, y = (y_i)_I \in \prod_{i\in I} A_i$, $x \nleq y$, then there exists $j \in I$ such that $x_j \nleq y_j$. Since A_j is regular torsionless, there exists $f \in \hom(A_j, S_S)$ such that $f(x_j) \nleq f(y_j)$. Now $f\pi_j \in \hom(\prod_{i\in I} A_i, S_S)$ and $(f\pi_j)(x) \nleq (f\pi_j)(y)$, and the result follows. $\square$

The following proposition shows that A_S is regular torsionless if and only if it can be embedded into some direct power of S.

Proposition 2. *An S-poset A_S is regular torsionless if and only if A_S can be embedded into S^I for some non-empty set I.*

Proof. Necessity. If A_S is regular torsionless then $\varphi_A : A_S \to (A_S)^{**}$ is a regular monomorphism. As we know $(A_S)^{**} = \hom(\hom(A_S, S_S), S_S)$ can be considered as an S-subposet of S^I where I has the same cardinality as the underlying set of the left S-poset $\hom(A_S, S_S)$, and so A_S can be an embedded into S^I.

Sufficiency. Suppose A_S is an S-subposet of S^I for some non-empty set I. By Lemma 2, parts (i, ii, v), S_S is regular torsionless, S^I is regular torsionless, and so A_S is regular torsionless. $\square$

The following proposition shows that from an S-poset which has a dual one can always construct a regular torsionless S-poset.

Proposition 3. *If an S-poset A_S has a dual, then $A_S/\ker\varphi_A$ is regular torsionless. Moreover, $\ker\varphi_A$ is the smallest congruence with this property, i.e., if A_S/ρ is regular torsionless for some congruence ρ on A_S then $\ker\varphi_A \leqslant \rho$.*

Proof. Notice that for each $x, y \in A_S$

$$(x, y) \in \overrightarrow{\ker}\varphi_A \Longleftrightarrow (\forall f \in (A_S)^*)(f(x) \leqslant f(y)).$$

That is, $\overrightarrow{\ker}\varphi_A = \cap_{f\in A^*}\overrightarrow{\ker}f$. Suppose $[a] \nleq [b]$ in $A_S/\ker\varphi_A$ for $a, b \in A_S$. Then there exists $f \in (A_S)^*$ such that $f(a) \nleq f(b)$. Define a mapping $\overline{f} : A_S/\ker\varphi_A \to S_S$ by $\overline{f}([x]) = f(x)$ for every $x \in A_S$. Since $\overrightarrow{\ker}\varphi_A \leqslant \overrightarrow{\ker}f$, $\overline{f}$ is a well-defined S-morphism. As $\overline{f}([a]) = f(a) \nleq f(b) = \overline{f}([b])$, $A_S/\ker\varphi_A$ is regular torsionless by Lemma 1.

Moreover, suppose that A_S/ρ is regular torsionless for a congruence ρ on A_S and $x \leqslant_{\ker\varphi_A} y$, $x, y \in A_S$. Let $\pi : A_S \to A_S/\rho$ be the canonical epimorphism. Then $(g\pi)(x) \leqslant (g\pi)(y)$ and thus $g([x]_\rho) \leqslant g([y]_\rho)$ for every homomorphism $g : A_S/\rho \to S_S$. Since A_S/ρ is regular torsionless, Lemma 1 implies $[x]_\rho \leqslant [y]_\rho$, i.e. $x \leqslant_\rho y$. Hence $\ker\varphi_A \leqslant \rho$. $\square$

We use the notion $A(I)$ for the amalgamated coproduct of two copies of a pomonoid S over a proper right ideal I. The definition of $A(I)$ for S-posets appeared in [2] and in [7] it is proved that $A(I)$ is a right S-poset. Let I be a right ideal of a pomonoid S, x, y, z not belonging to S, and $A(I) = (\{x, y\} \times (S \setminus I)) \cup (\{z\} \times I)$. Define a right S-action on $A(I)$ by

$$(w, u)s = \begin{cases} (w, us) & \text{if } us \notin I,\ w \in \{x, y\} \\ (z, us) & \text{if } us \in I. \end{cases}$$

The order on $A(I)$ is defined as:

$$\begin{aligned}(w_1, s) \leqslant (w_2, t) \Longleftrightarrow & (w_1 = w_2 \text{ and s} \leqslant \text{t}) \\ & \text{or} \quad (w_1 \neq w_2,\ s \leqslant i \leqslant t \quad \text{for some } i \in I).\end{aligned}$$

In what follows we allow the core I also to be empty. This gives us the coproduct of two copies S, as follows

$$S\coprod S=\{(w,s)|\ s\in S, w\in\{x,y\}\},$$

where $(w_1,s)\leqslant(w_2,t)$ if and only if $w_1=w_2,\ s\leqslant t$.

Proposition 4. *Let $A_S=S_S\coprod S_S$. Then*

$$\overrightarrow{\ker}\varphi_A=\xi_A\cup\{((w_1,u),(w_2,v))|w_1\neq w_2\in\{x,y\},\ (\forall s,t\in S)(su\leqslant tv)\}.$$

Proof. Suppose that $(a,b)\in\overrightarrow{\ker}\varphi_A$. If $a=(w,u)$ and $b=(w,v)$, $u,v\in S$, $w\in\{x,y\}$, a mapping $f:S_S\coprod S_S\to S_S$ defined by $f(c)=p$ if $c=(w,p), w\in\{x,y\}$ is a well-defined S-morphism and thus $f(a)\leqslant f(b)$ by the definition of $\overrightarrow{\ker}\varphi_A$. Hence $u\leqslant v$, and $(a,b)\in\xi_A$. Now suppose that $a=(w_1,u)$ and $b=(w_2,v), w_1\neq w_2, u,v\in S$. Then for $t,s\in S$ the mappings $f:S_S\coprod S_S\to S_S$ defined by

$$f(c)=\begin{cases} sp & \text{if } c=(w_1,p),\\ tp & \text{if } c=(w_2,p)\end{cases}$$

is a well-defined S-morphism. Since $(a,b)\in\overrightarrow{\ker}\varphi_A$, $su=f(w_1,u)\leqslant f(w_2,v)=tv$. Hence

$$\overrightarrow{\ker}\varphi_A\subseteq\xi_A\cup\{((w_1,u),(w_2,v))|w_1\neq w_2\in\{x,y\},\ (\forall s,t\in S)(su\leqslant tv)\}.$$

Now suppose that $(w_1,u),(w_2,v)\in S_S\coprod S_S$ such that for each $w_1\neq w_2, s,t\in S,\ su\leqslant tv$. Let $f\in\hom(A_S,S_S)$. It is easily checked that there exist elements $s,t\in S$ such that

$$f((w,p))=\begin{cases} sp & \text{if } w=w_1,\\ tp & \text{if } w=w_2.\end{cases}$$

We have $f((w_1,u))=su\leqslant tv=f((w_2,v))$ which means that

$$\xi_A\cup\{((w_1,u),(w_2,v))|w_1\neq w_2\in\{x,y\},\ (\forall s,t\in S)(su\leqslant tv)\}\subseteq\overrightarrow{\ker}\varphi_A.$$

□

Corollary 2. *$S_S\coprod S_S$ is regular torsionless if and only if*

$$\{((w_1,u),(w_2,v))|w_1\neq w_2\in\{x,y\},\ (\forall s,t\in S)(su\leqslant tv)\}=\varnothing.$$

The following theorem is concerned with coproducts of regular torsionless right S-posets.

Theorem 1. *The following assertions are equivalent for a pomonoid S:*

(i) *if A_i, $i \in I$, are regular torsionless right S-posets then $\coprod_{i\in I} A_i$ is regular torsionless;*

(ii) *$S_S \coprod S_S$ is regular torsionless;*

(iii) *for each $u, v \in S$ there exist $s, t \in S$ such that $su \nleqslant tv$.*

Proof. (i) $\to$ (ii) is obvious.

(ii) $\to$ (iii) follows from the previous corollary.

(iii) $\to$ (i). Suppose A_i, $i \in I$, are regular torsionless right S-posets and let $x, y \in \coprod_{i\in I} A_i$ such that $x \nleqslant y$. If $x, y \in A_i$ for some $i \in I$, then there exists an S-morphism $f_i : A_i \to S_S$ such that $f_i(x) \nleqslant f_i(y)$. Define $f : \coprod_{i\in I} A_i \to S_S$ by $f|_{A_j}$, $j \in I, j \neq i$, is an arbitrary S-morphism and $f|_{A_i} = f_i$. Clearly f is an S-morphism for which $f(x) \nleqslant f(y)$.

Otherwise, suppose that $x \in A_i, y \in A_j$, $i, j \in I$, $i \neq j$. Take an S-morphism $f : \coprod_{i\in I} A_i \to S_S$ for which $f|_{A_k} = f_k$, $k \in I$, is an arbitrary S-morphism. If now $f_i(x) \leqslant f_j(y)$ then, by assumption for $f_i(x), f_j(y) \in S$ there exist $s, t \in S$ such that $sf_i(x) \nleqslant tf_j(y)$. Then define $g : \coprod_{i\in I} A_i \to S_S$ for which $g|_{A_k} = f_k$, $k \in I$, $k \neq i, j$, $g|_{A_i} = sf_i$, and $g|_{A_j} = tf_j$. Obviously, g is an S-morphism for which $g(x) \nleqslant g(y)$. Thus $\coprod_{i\in I} A_i$ is regular torsionless by Lemma 1. $\square$

Proposition 5. *Let S be a pomonoid, $e \in S$ an idempotent such that $e \nparallel 1$ and $I = es$. Then the amalgamated coproduct $A(I)$ is regular torsionless.*

Proof. Let $a, b \in A(I)$ and $a \nleqslant b$. If $a = (w_1, u)$, $b = (w_2, v)$, $u, v \in S$, $w_i \in \{x, y, z\}$, and $u \nleqslant v$, then a mapping $f : A(I) \to S_S$ defined by $f(w, s) = s$, for $w \in \{x, y, z\}$, is a well-defined S-morphism and $f(a) \nleqslant f(b)$. It suffices to check only the case that $a = (w_1, u)$ and $b = (w_2, v)$, $u \leqslant v$, $w_1 \neq w_2 \in \{x, y\}$. If $e \leqslant 1$, then a mapping $f : A(I) \to S_S$ defined by

$$f(c) = \begin{cases} s & \text{if } c = (w, s),\ w \neq w_2, \\ es & \text{if } c = (w_2, s). \end{cases}$$

is a well-defined S-morphism. Now if $u = f(a) \leqslant ev = f(b)$, then $u \leqslant ev \leqslant v$ which means that $a \leqslant b$, a contradiction. If $1 \leqslant e$, f can be defined analogously. Therefore $f(a) \nleqslant f(b)$ and $A(I)$ is regular torsionless by Lemma 1. $\square$

Similar to the argument of the previous proposition, the following result can be proved.

Proposition 6. *Let S be a pomonoid, $e \in S$ an idempotent and $I = (es] \neq S$; the principal right poideal generated by e or $I = [es) \neq S$. Then the amalgamated coproduct $A(I)$ is regular torsionless.*

Now, we turn our attention to cyclic S-posets.

Proposition 7. *Let $\rho \neq \Delta_S$ be a right congruence on a pomonoid S. The cyclic right S-poset S/ρ is regular torsionless if and only if for all $s, t \in S$, If $s \nleqslant_\rho t$, there exists $u \in S$ such that $us \nleqslant ut$ and $x \leqslant_\rho y$, $x, y \in S$, implies $ux \leqslant uy$.*

Proof. Necessity. Suppose S/ρ is regular torsionless and $s, t \in S$ so that $s \nleqslant_\rho t$. Then there exists $f \in \hom(S/\rho, S_S)$ such that $f([s]_\rho) \nleqslant f([t]_\rho)$, by Lemma 1. Let $u = f([1])$. Then $us = f([1])s = f([s]) \nleqslant f([t]) = f([1])t = ut$. But for any $x, y \in S$, $x \leqslant_\rho y$, $ux = f([1])x = f([x]) \leqslant f([y]) = f([1])y = uy$.

Sufficiency. By the assumption, if $s \nleqslant_\rho t$, there exists $u \in S$ such that $us \nleqslant ut$ and $x, y \in S$, $x \leqslant_\rho y$, implies $ux \leqslant uy$. Now a mapping $f : S/\rho \to S_S$ defined by $f([x]) = ux$ is a well-defined S-morphism such that $f(s) \nleqslant f(t)$. Thus S/ρ is regular torsionless. □

Corollary 3. *Let K_S be a non-trivial convex right ideal of a pomonoid S. The Rees factor S-poset S/K_S is regular torsionless if and only if for every $s, t \in S$, $[s] \nleqslant [t]$ implies the existence of $u, z \in S$, z a left zero, such that $us \nleqslant ut$ and $uk = z$ for every $k \in K_S$.*

Proof. Repeat the argument of the proof of Proposition 7 and take $z = f([k])$, then $uk = f([1])k = f([k]) = z$ for every $k \in K_S$. □

Lemma 4. *Let S be a pomonoid. If all Rees factor S-posets of the form $S/[sS]$, $s \in S$, are regular torsionless, then for every right cancellable element c of S there exist $u, v \in S$ such that $cu \leqslant 1 \leqslant cv$.*

Proof. Suppose $c \in S$ is right cancellable if $S = [cS]$ the result follows. Otherwise, $[cS]$ is a proper convex right ideal of S. Since the Rees factor S-poset $S/[cS]$ is regular torsionless and $1 \notin [cS]$ without loss of generality we can suppose that $[1] \nleqslant [ct]$ for some $t \in S$. So by Corollary 3, there exist $u, z \in S$, z a left zero, such that $u \nleqslant uct$ and $uk = z$ for every $k \in [cS]$. In particular, $uct = uc = z$. Since z is a left zero, one has $uc = zc$. Using right cancellability of c we get $u = z = uct$, a contradiction. □

2. On homological classification

In this section we present some results on homological classification. We start with questions when some property implies regular torsionless. Then we give some necessary conditions for regular torsionless S-posets to satisfy some flatness properties.

Theorem 2. *The following are equivalent for a pomonoid S:*

(i) *every free right S-poset is regular torsionless.*

(ii) *every projective right S-poset is regular torsionless.*

(iii) *for each $u, v \in S$ there exist $s, t \in S$ such that $su \not\leq tv$.*

Proof. (i) $\Longleftrightarrow$ (ii) follows from Lemma 3 since every projective S-poset is an S-subposet of free S-poset.

(i) $\Longleftrightarrow$ (iii) follows from Theorem 1 since S_S is regular torsionless and every free right S-poset is a coproduct of copies of S_S. □

Lemma 5. *If all principally weakly (po-)flat right S-posets over a pomonoid S are regular torsionless then S contains at least two different left zeros.*

Proof. Since the one-element right S-poset is principally weakly flat and coproducts of principally weakly flat S-posets are principally weakly flat, a right S-poset $A_S = \{a\} \coprod \{b\}$ is regular torsionless where a, b are two zeros. Then there exists $f \in \hom(A_S, S_S)$ such that $f(a) \not\leq f(b)$ by Lemma 1. But, being images of zeros, $f(a)$ and $f(b)$ are left zeros of S. □

Now, we discuss the converse direction of two previous results.

Proposition 8. *All regular torsionless right S-posets are po-torsionfree.*

Proof. Suppose A_S is regular torsionless. Then A_S is isomorphic to an S-subposet of S^I for some non-empty set I by Lemma 3. But S_S is po-torsionfree, products of po-torsionfree S-posets are po-torsionfree and S-subposets of po-torsionfree S-posets are po-torsionfree and so A_S is po-torsionfree. □

Let S be a pomonoid. We recall from [5] that a finitely generated left S-poset ${}_SB$ is called *finitely definable (FD)* if the S-morphism $S^\Gamma \otimes B \to B^\Gamma$ is order-embedding for all non-empty sets Γ.

Proposition 9. *If all regular torsionless right S-posets are principally weakly po-flat then all principal left ideals of S are finitely definable (FD).*

Proof. By Lemma 3, S^I is regular torsionless for any non-empty set I, and so principally weakly po-flat by the assumption. So $S^I \otimes Ss \to (Ss)^\Gamma$ is order-embedding for each $s \in S$. Using Proposition 2.4 of [5] it follows that all principal left ideals of S are finitely definable. $\square$

Note that for every $s, t \in S$ we have $H(s,t) = \{(as, a't) | \; as \leqslant a't\}$ and $\widehat{S(p,q)} = \{(u,v) \in D(S) | \; \exists w \in S, u \leqslant wp, wq \leqslant v\}$.

Proposition 10. *If all regular torsionless right S-posets are weakly po-flat then all principal left ideals of S are finitely definable and for every for every $s, t \in S$, if $Ss \cap (St] \neq \varnothing$, then $H(s,t)$ is a subset of $\widehat{S(p,q)}$ for some $(p,q) \in H(s,t)$.*

Proof. The S-poset S^I is regular torsionless for each non-empty set I and so weakly po-flat by the assumption. Thus Theorem 2.7 of [5] implies the result. $\square$

Recall from [5] that the set $L(a,b) := \{(u,v) \in D(S) | \; ua \leqslant vb\}$ is a left S-subposet of $D(S)$, and the set $l(a,b) := \{u \in S | \; ua \leqslant ub\}$ is a left ideal of S. Moreover, a pomonoid S is called a left PSF pomonoid if all principal left ideals of a pomonoid S are strongly flat.

Proposition 11. *If all regular torsionless right S-posets are strongly flat then*

(i) *for each idempotent $e \in S$ there exists $s \in S$ such that $es \nparallel 1$ and e is incomparable with* 1,
(ii) *for all $(s,t) \in D(S)$, $L(s,t) = \varnothing$ or is a cyclic left S-poset,*
(iii) *for all $(s,t) \in D(S)$, $l(s,t) = \varnothing$ or is a principal left ideal of S,*
(iv) *S is a right PSF pomonoid.*

Proof. Suppose $e \in S$ is an idempotent such that es is incomparable with 1 for each $s \in S$. Then for $I = (eS]$ the amalgamated coproduct $A(I)$ is regular torsionless by Proposition 6. Similarly if $e \nparallel 1$ Proposition 5 implies that $A(eS)$ is regular torsionless, and so strongly flat by the assumption which is a contradiction by Lemma 2.4 of [7]. Thus S satisfies condition (i). The S-posets S^I for any non-empty set I are regular torsionless and so strongly flat. Form Corollary 3.5 of [5] it follows that S satisfies conditions (ii) and (iii). We have all principal right ideals of S are regular torsionless by Lemma 3, and so strongly flat by the assumption. Thus S satisfies conditions (iv). $\square$

Using Theorem 3.3 of [5], similar to the proof of the previous proposition we imply two following results.

Proposition 12. *If all regular torsionless right S-posets satisfy Condition* (P)*, then*

(i) *for each idempotent $e \in S$ there exists $s \in S$ such that $es \nparallel 1$ and e is incomparable with* 1*,*

(ii) *for all $(s,t) \in D(S)$, $L(s,t) = \varnothing$ or is a cyclic left S-poset.*

Proposition 13. *If all regular torsionless right S-posets satisfy Condition* (E)*, then for all $(s,t) \in D(S)$, $L(s,t) = \varnothing$ or is a cyclic left S-poset.*

A pomonoid S is called a *rpp pomonoid* if the S-subposet sS is projective for each $s \in S$.

Proposition 14. *For a pomonoid S if all regular torsionless right S-posets are projective, then*

(i) *S is po-cancellative,*

(ii) *incomparable principal right ideals of S are disjoint,*

(iii) *S satisfies the ascending chain condition (ACC) for principal right ideals.*

Proof. By Lemma 3, the S-poset S^I is regular torsionless for each nonempty set I and so projective by the assumption. Corollary 1 implies that S is right po-cancellative. Using Lemma 3, all right ideals of S are regular torsionless, and so projective by the assumption. So S is a rpp pomonoid and satisfies conditions (ii) and (iii) from Theorem 2.8 of [8]. Suppose $sx \leqslant sy$, $s, x, y \in S$. Since S is a rpp pomonoid there exists an idempotent $e \in S$ such that $ex \leqslant ey$. But since S is right po-cancellative, $e = 1$. Thus $x \leqslant y$, and S is po-cancellative. □

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