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RESEARCH ARTICLE

Unimodality polynomials and generalized Pascal triangles

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ABSTRACT. In this paper, we show that if $P(x) = \sum_{k=0}^{m} a_k x^k$ is a polynomial with nondecreasing, nonnegative coefficients, then the coefficients sequence of $P(x^s + \cdots + x + 1)$ is unimodal for each integer $s \geqslant 1$. This paper is an extension of Boros and Moll's result *"A criterion for unimodality"*, who proved that the polynomial $P(x+1)$ is unimodal.

Introduction

Unimodal polynomials arise often in combinatorics, geometry and algebra. We refer the reader to [8, 15] for surveys of diverse techniques used to establish that a polynomial is unimodal.

A finite sequence of real numbers $\{a_0, \ldots, a_m\}$ is said to be unimodal if there exists an index $0 \leqslant m^* \leqslant m$, called the mode of the sequence, such that a_k increases up to $k = m^*$ and decreases from then on, that is, $a_0 \leqslant a_1 \leqslant \cdots \leqslant a_{m^*}$ and $a_{m^*} \geqslant a_{m^*+1} \geqslant \cdots \geqslant a_m$. A polynomial is said to be unimodal if its sequence of coefficients is unimodal.

A sequence of nonnegative real numbers $\{a_k\}_k$ is *log-concave* (LC for short) if $a_{k-1}a_{k+1} \leqslant a_k^2$ for all $k > 0$. This condition is equivalent to $a_{i-1}a_{j+1} \leqslant a_i a_j$ for all $j \geqslant i \geqslant 1$ (see [7], for instance). It is easy to see that if a sequence is log-concave, then it is unimodal [7]. A sufficient condition for log-concavity of a polynomial can be given by the location of its zeros: if all the zeros of a polynomial are real and negative, then

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it is log-concave and therefore it is unimodal [16]. A second criterion for the log-concavity of a polynomial was established by Brenti [8]. A sequence of real numbers is said to have no internal zeros if whenever a_i, $a_k \neq 0$ and $i < j < k$, then $a_j \neq 0$. Such a sequence of positive numbers whose generating function has only real zeros is called a Pólya frequency in the theory of total positivity. See Karlin [12] for a standard reference on total positivity and Brenti [8–10] for applications of total positivity to unimodality and log-concavity problems. It often occurs that unimodality of a sequence is known, yet to determine the exact number and location of modes is a much more difficult problem. The case for Pólya frequency sequences is somewhat different. Darroch [11] showed that if the polynomial $P(x) = \sum_{k=0}^{m} a_k x^k$ with positive coefficients has only real zeros, then the unimodal sequence $a_0, \ldots, a_m$ has at most two modes and each mode m^* satisfies

$$\left\lfloor \frac{P'(1)}{P(1)} \right\rfloor \leqslant m^* \leqslant \left\lceil \frac{P'(1)}{P(1)} \right\rceil .$$

Brenti's criterion states that if $P(x)$ is a log-concave polynomial with nonnegative coefficients and with no internal zeros, then $P(x+1)$ is log-concave. Boros and Moll in [6] showed that if $P(x) = \sum_{k=0}^{m} a_k x^k$ is a polynomial with $0 \leqslant a_0 \leqslant a_1 \cdots \leqslant a_m$, then $P(x+1)$ is also unimodal.

In this paper, we generalize the work of Boros and Moll showing that $P(x^s + \cdots + x + 1)$ is unimodal. It is easy to see that all the coefficients of this polynomial are associated with the bi^{s}nomial coefficients which are also called ordinary multinomial numbers $\binom{m}{k}_s$ (see for instance [4]).

The ordinary multinomial number is defined as the kth coefficient in the expansion

$$(1 + x + x^2 + \cdots + x^s)^m = \sum_{k \geqslant 0} \binom{m}{k}_s x^k. \tag{1}$$

Using the classical binomial coefficients, one has

$$\binom{m}{k}_s = \sum_{j_1 + j_2 + \cdots + j_s = k} \binom{m}{j_1} \binom{j_1}{j_2} \cdots \binom{j_{s-1}}{j_s}. \tag{2}$$

The following properties of multinomial numbers are well known.

- The symmetry relation

$$\binom{m}{k}_s = \binom{m}{sm-k}_s. \tag{3}$$

- The longitudinal recurrence relation

$$\binom{m}{k}_s = \sum_{j=0}^{s} \binom{m-1}{k-j}_s. \tag{4}$$

- The diagonal recurrence relation

$$\binom{m}{k}_s = \sum_{j=0}^{m} \binom{m}{j} \binom{j}{k-j}_{s-1}. \tag{5}$$

These coefficients can be found using the "s-Pascal triangle", a generalized version of the Pascal triangle for ordinary binomial coefficients. One can find the first values of the s-Pascal triangle in Sloane [14] as A027907 for $s = 2$, as A008287 for $s = 3$ and as A035343 for $s = 4$.

$n \setminus k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	1													
1	1	1	1	1	1									
2	1	2	3	4	5	4	3	2	1					
3	1	3	6	10	15	18	19	18	15	10	6	3	1	
4	1	4	10	20	35	52	68	80	85	80	68	52	35	...
5	1	5	15	35	70	121	185	255	320	365	379	365	320	...

Table 1. Triangle of quintinomial coefficients: $s = 4$.

For other properties of ordinary multinomial coefficients, see, for instance, [3, 4].

The ordinary multinomial coefficients were studied regarding to their unimodality, log-concavity and log-convexity. In [1] and [2] we respectively established preserving log-concavity and log-convexity properties for these coefficients.

Our main result is the following.

Theorem 1. *If $P(x) = \sum_{k=0}^{m} a_k x^k$ is a polynomial with positive non-decreasing coefficients, then $P(x^s + \cdots + x + 1)$ is unimodal for each integer $s \geqslant 1$.*

1. Proof of the main result

To prove the theorem, we need two basic lemmas.

Lemma 1 ([12, 13])**.** *If the sequences $\{x_n\}$ and $\{y_n\}$ are log-concave, then so is their ordinary convolution $z_n = \sum_{k=0}^{n} x_k y_{n-k}$, $n = 0, 1, \ldots$*

Lemma 2 ([8]). *A log-concave sequence $\{x_k\}_k$ with no internal zeros is unimodal.*

The following proposition plays a key role in the proof of our theorem.

Proposition 1. *The polynomial*

$$P_{m,r}(x) = (x^s + \cdots + x + 1)^m - (x^s + \cdots + x + 1)^r \quad \textit{with } 0 \leqslant r \leqslant m-1,$$

is unimodal.

Proof. Observe that

$$\begin{aligned} P_{m,r}(x) &= \sum_{k=0}^{sm} \binom{m}{k}_s x^k - \sum_{k=0}^{sr} \binom{r}{k}_s x^k = \sum_{k=0}^{sm} \left[\binom{m}{k}_s - \binom{r}{k}_s \right] x^k \\ &= \sum_{k=0}^{sm} \left[\sum_{j_1,\ldots,j_{m-r} \leqslant s / j_1+\cdots+j_{m-r} \geqslant 0} \binom{r}{k-j_1-\cdots-j_{m-r}}_s - \binom{r}{k}_s \right] x^k \\ &\quad \text{(using relation (4))} \\ &= \sum_{k=0}^{sm} \sum_{j_1,\ldots,j_{m-r} \leqslant s / j_1+\cdots+j_{m-r} \geqslant 1} \binom{r}{k-j_1-\cdots-j_{m-r}}_s x^k = \sum_{k=0}^{sm} b_k x^k, \end{aligned}$$

where $b_k = \sum_{j_1,\ldots,j_{m-r} \leqslant s / j_1+\cdots+j_{m-r} \geqslant 1} \binom{r}{k-j_1-\cdots-j_{m-r}}_s$.

Further, we can rewrite the b_k as

$$b_k = \sum_{j_1 \leqslant s} \cdots \sum_{j_{m-r} \leqslant s} \binom{r}{k-j_1-\cdots-j_{m-r}}_s \quad \text{with } j_1 + \cdots + j_{m-r} \geqslant 1.$$

As the ordinary multinomial coefficients are log-concave (see [5], for instance), then by Lemma 1 the linear transformation $\sum_{j_{m-r} \leqslant s} \binom{r}{k-j_1-\cdots-j_{m-r}}_s$ is also log-concave. Hence, by the induction hypothesis, the sequence $\{b_k\}_k$ is log-concave and therefore unimodal by Lemma 2. □

Remark 1. The polynomial $P_{m,r}(x)$ is unimodal with the following smallest mode:

$$m^* := \begin{cases} \left\lfloor \frac{sm}{2} \right\rfloor, & \text{or} \\ \left\lfloor \frac{sm}{2} \right\rfloor + 1 \end{cases} \qquad \text{for all } 0 \leqslant r \leqslant m,$$

since

$$m^* = \left\lfloor \frac{P'_{m,r}(1)}{P_{m,r}(1)} \right\rfloor = \left\lfloor \frac{\frac{sm}{2}(s+1)^m - \frac{sr}{2}(s+1)^r}{(s+1)^m - (s+1)^r} \right\rfloor \quad \text{by Darroch [11]}$$

$$
\begin{aligned}
&= \left\lfloor \frac{sm}{2} \frac{(s+1)^{m-r} - \frac{r}{m}}{(s+1)^{m-r} - 1} \right\rfloor \\
&= \left\lfloor \frac{sm}{2} \left[1 + \frac{\frac{l}{m}}{(s+1)^{l} - 1} \right] \right\rfloor \qquad \text{by setting } l = m - r \\
&= \begin{cases} \left\lfloor \frac{sm}{2} \right\rfloor, & \text{or} \\ \left\lfloor \frac{sm}{2} \right\rfloor + 1. \end{cases}
\end{aligned}
$$

Proof of Theorem 1. Now, we have

$$
\begin{aligned}
P(x^s + \cdots + x + 1) &= \sum_{k=0}^{m} a_k (x^s + \cdots + x + 1)^k \\
&= a_0 + a_1 (x^s + \cdots + x + 1) + \cdots + a_m (x^s + \cdots + x + 1)^m \\
&= \frac{1}{(x^s + \cdots + x)} \left[a_0 P_{m,0}(x) + (a_1 - a_0) P_{m,1}(x) + \cdots \right. \\
&\quad \left. + (a_m - a_{m-1}) P_{m,m}(x) \right].
\end{aligned}
$$

By Proposition 1, the polynomial $P(x^s + \cdots + x + 1)$ is a sum of unimodal polynomials with the same mode, and hence it is unimodal.

Also we can rewrite the polynomial as

$$
\begin{aligned}
&P(x^s + \cdots + x + 1) \\
&\quad = \sum_{k=0}^{m} a_k \sum_{j=0}^{sk} \binom{k}{j}_s x^j = \sum_{j=0}^{sm} x^j \sum_{k=\lceil j/s \rceil}^{m} a_k \binom{k}{j}_s = \sum_{j=0}^{sm} b_j x^j,
\end{aligned}
$$

where $b_j = \sum_{k=\lceil j/s \rceil}^{m} a_k \binom{k}{j}_s$. Thus, the sequence of coefficients $\{b_j\}_{0 \leqslant j \leqslant sm}$ is also unimodal. □

Setting $s = 1$, we immediately obtain the results of Boros and Moll [6].

Corollary 1. *If $P(x) = \sum_{k=0}^{m} a_k x^k$ is a polynomial with positive nondecreasing coefficients, then $P(x+1)$ is unimodal.*

2. Examples

Example 1. Let $n, m \in \mathbb{N}$ be fixed. Then the sequences

$$
A_j := \sum_{k=\lceil \frac{j}{s} \rceil}^{m} n^k \binom{k}{j}_s, \quad B_j := \sum_{k=\lceil \frac{j}{s} \rceil}^{m} k^n \binom{k}{j}_s \quad \text{and} \quad C_j := \sum_{k=\lceil \frac{j}{s} \rceil}^{m} k^k \binom{k}{j}_s
$$

are unimodal for $0 \leqslant j \leqslant sm$.

Example 2. Let $2 < \alpha_1 < \cdots < \alpha_l$ and $n_1, \ldots, n_l$ be two sequences of l positive integers. For $0 \leqslant j \leqslant sm$, define

$$\alpha_j := \sum_{k=\lceil \frac{j}{s} \rceil}^{m} \binom{\alpha_1 m}{k}^{n_1} \cdots \binom{\alpha_l m}{k}^{n_l} \binom{k}{j}_s. \tag{6}$$

Then α_j is unimodal.

3. The remark and the question about the mode

We have established the unimodality property of fundamental polynomials $P(x^s + \cdots + x + 1)$. However, the number and location of the modes of these polynomials remains a question to be answered.

Generaly, it is not easy to find the number and location of modes. In our case, it is suffice to find the modes of the unimodal sequence $\{b_j\}_{0 \leqslant j \leqslant sm}$ defined by:

$$b_j = \sum_{k=\lceil \frac{j}{s} \rceil}^{m} a_k \binom{k}{j}_s, \qquad j = 0, 1, \ldots, sm. \tag{7}$$

This lets us to finish this paper by the following question.

Question. Find the number and location of modes of the unimodal sequence $\{b_j\}_{0 \leqslant j \leqslant sm}$ defined by:

$$b_j = \sum_{k=\lceil \frac{j}{s} \rceil}^{m} a_k \binom{k}{j}_s, \qquad j = 0, 1, \ldots, sm,$$

where $\{a_k\}_k$ is sequence of positive nondecreasing numbers.

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