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STATE SPACE APPROACH TO THERMOELASTIC PROBLEM WITH THREE-PHASE-LAG MODEL

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Abstract. A two-dimensional problem of generalized thermoelasticity is formulated using state space approach. In this formulation, the governing equations are transformed into a matrix differential equation whose solution enables to write the solution of two-dimensional problem in terms of the boundary conditions. The resulting formulation is applied to an isotropic half-space problem within three-phase-lag model of thermoelasticity. The bounding surface is traction free and subjected to a time dependent thermal shock. The solution for temperature distribution, displacements and stress components are obtained and presented graphically as well as a comparison with other thermoelastic models is made.

Keywords: state space approach, normal mode analysis, thermal shock, three-phase-lag model.

1. Introduction.

The classical theory of thermoelasticity [1] suffers from so called 'paradox of heat conduction' i.e. the heat equations for both theories of a mixed parabolic-hyperbolic type, predicting infinite speeds of propagation for heat waves contrary to physical observations. To remove this paradox, the conventional theories of thermoelasticity has been generalized, where the generalization is in the sense that these theories involve a hyperbolic type heat transport equation supported by experiments, which exhibit the actual occurrence of wave type heat transport in solids, called second sound effect. To eliminate the second sound paradox of classical thermoelasticity theory, Lord and Shulman [2] established a generalized thermoelasticity theory which is often referred to as LS model and widely used in the case of heat flux and low temperature. Green and Lindsay [3] introduced one more theory, called GL theory, which involves two relaxation times. Later Green and Naghdi [4, 5, 6] developed three models for generalized thermoelasticity of homogeneous isotropic materials, which are labeled as G-N models I, II, III. Detailed information regarding these theories can be found in [7, 8].

The next generalization to the thermoelasticity is known as the dual-phase-lag model (DPL) developed by Tzou [9]. Tzou [9] considered micro-structural effects into the delayed response in time in macroscopic formulation by taking into account that increase of the lattice temperature is delayed due to phonon-electron interactions on the macroscopic level. Tzou [9] introduced two-phase lags to both the heat flux vector and the temperature gradient and considered as constitutive equation to describe the lagging behavior in the heat conduction in solids. Recently, Roychoudhuri [10] has established a generalized mathematical model of a coupled thermoelasticity theory that includes three-phase-lags in the heat flux vector, the temperature gradient and in the thermal displacement gradient. The more general model (TPL) established reduces to the previous models as special cases.

In three-phase-lag heat conduction equation the Fourier law of heat conduction is replaced by an approximation of three phase lags for the heat flux vector (τ_q), the temperature gradient (τ_T) and the thermal displacement gradient (τ_v). The previous established models can be obtained as special cases from this more general model. Three-phase-lag (TPL) model is very much useful in the problems of nuclear boiling, exothermic catalytic reactions, phonon-electron interactions, phonon scattering etc., where the delay time τ_q captures the thermal wave behavior (a small scale response in time), the phase lag τ_T captures the effect of phonon-electron interactions (a microscopic response in space), the other delay time τ_v is effective since in the three-phase-lag model , the thermal displacement gradient is considered as a constitutive variable.

State space methods are the cornerstone of modern control theory. The necessity of state space method is the characterization of the processes of interest by differential equations instead of transfer functions. In the earlier period the processes were simple enough to be characterized by a single differential equation of fairly low order. In the modern approach the processes are characterized by systems of coupled, first order differential equations. In principle, there is no limit to the order (i.e. the number of independent first order differential equations) and in practice the only limit to the order is the availability of computer software capable of performing the required calculations reliably.

A method for solving coupled thermoelastic problems by state space approach has been developed by Bahar and Hetnarski[11] . The state space formulation for the problems that do not contain heat sources have been done by Anwar and Sherief [12]. Ezzat et al. [13] considered state space approach to two-dimensional generalized thermoviscoelasticity with one relaxation time. Youssef and Al-Lehaibi [14] discussed two temperature generalized thermoelasticity using state space approach. El-Karamany and Ezzat [15] considered thermal shock problem in generalized thermoelasticity under four theories. Sherief et al. [16] discussed stochastic thermal shock in generalized thermoelasticity. Ezzat and Youssef [17] investigated three dimensional thermal shock problem of generalized thermoelastic medium. Wang et al. [18] considered thermoelastic behavior of elastic media with temperature-dependent properties under thermal shock. Baksi et al. [19] studied magneto-thermoelastic problem with thermal relaxation and heat sources in a three dimensional infinite rotating elastic medium. Biswas and Mukhopadhyay[20] proposed eigenfunction expansion method to analyze thermal shock behavior in magneto-thermoelastic orthotropic medium with three-phase-lag model. Biswas and S. M. Abo-Dahab [21] considered the effect of phase-lags on Rayleigh wave propagation in initially stressed magneto-thermoelastic orthotropic medium. Biswas [22] proposed modeling of memory-dependent derivatives in orthotropic medium with three phase lag model under the effect of magnetic field. Said [23] investigated the influence of gravity on generalized magneto-thermoelastic medium for three-phase -lag model. Kalkal and Deswal [24] examined the effects of phase lags on three dimensional wave propagation with temperature dependent properties.

El-Karamany and Ezzat [25] discussed linear micropolar thermoelasticity theory with three-phase-lag model. Sherief and Al-Sayed [26] proposed state space approach to two-dimensional generalized micropolar thermoelasticity. Ezzat et al. [27] employed state space approach to study two-dimensional electro-magnetic thermoelastic problem with two relaxation times.

In this problem we have considered two dimensional generalized thermoelasticity with three-phase-lag model of thermoelasticity. Normal mode analysis is employed to the governing equations and then the problem is solved using state space approach. Stress, displacements and temperature with respect to time and distance for different thermoelastic models are presented graphically.

2. Formulation of the problem.

We consider a two-dimensional problem and assume that the thermoelastic medium is governed by the equations of generalized thermoelasticity whose state depends on the space variables x', y' and the time variable t'.

The displacement vector has components $[u(x, z, t), 0, w(x, z, t)]$.

The displacement equations are given by

$$(\lambda+\mu)u_{j,ij}+\mu u_{i,jj}-\gamma T_{,i}=\rho \ddot{u}_i\,. \tag{1}$$

The constitutive equation is

$$\tau_{ij}=\lambda e_{kk}\delta_{ij}+2\mu e_{ij}-\gamma T\delta_{ij} \tag{2}$$

and the strain-displacement relations are as follows:

$$e_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i})\,. \tag{3}$$

In the above equations, a superposed dot denotes differentiation with respect to time, while a comma denotes spatial derivatives.

The heat conduction equation of three-phase-lag model (Roychoudhuri [10]) is given by

$$K\left(1+\tau_T\frac{\partial}{\partial t}\right)\nabla^2\dot{T}+K^*\left(1+\tau_\nu\frac{\partial}{\partial t}\right)\nabla^2 T=\left(1+\tau_q\frac{\partial}{\partial t}+\frac{\tau_q^2}{2}\frac{\partial^2}{\partial t^2}\right)\left(\rho C_e\ddot{T}+\gamma T_0\ddot{e}\right), \tag{4}$$

where τ_q, τ_T and τ_ν are the phase lags for heat flux, temperature gradient and thermal displacement gradient respectively.

We use the following non-dimensional variables:

$$\left(x',z',u',w'\right)=c_1\eta_0\left(x,z,u,w\right);\ \left(t',\tau_q',\tau_T',\tau_\nu'\right)=c_1^2\eta_0\left(t,\tau_q,\tau_T,\tau_\nu\right);\ T'=\frac{\gamma T}{\rho c_1^2};\ \tau_{ij}'=\frac{\tau_{ij}}{\mu},$$

where the dashed quantities denote non-dimensional variables.

In terms of these non-dimensional variables, the equations of motion has the form (dropping primes).

$$\beta^2 u_{,xx}+u_{,zz}+(\beta^2-1)w_{,xz}-\beta^2 T_{,x}=\beta^2\ddot{u}\,; \tag{5}$$

$$(\beta^2-1)u_{,xz}+\beta^2 w_{,zz}+w_{,xx}-\beta^2 T_{,z}=\beta^2\ddot{w} \tag{6}$$

and the components of the stress are:

$$\tau_{xx}=c_1^2 u_{,x}+(c_1^2-2)w_{,z}-\beta^2 T\,; \tag{7a}$$

$$\tau_{xz}=u_{,z}+w_{,x}\,; \tag{7b}$$

$$\tau_{zz}=(c_1^2-2)u_{,x}+c_1^2 w_{,z}-\beta^2 T\,. \tag{7c}$$

The equation (4) in non-dimensional form is obtained as

$$\left(1+\tau_T\frac{\partial}{\partial t}\right)\left(\dot{T}_{,xx}+\dot{T}_{,zz}\right)+\bar{K}\left(1+\tau_\nu\frac{\partial}{\partial t}\right)\left(T_{,xx}+T_{,zz}\right)=\left(1+\tau_q\frac{\partial}{\partial t}+\frac{\tau_q^2}{2}\frac{\partial^2}{\partial t^2}\right)(\ddot{T}+\varepsilon\ddot{e})\,, \tag{8}$$

where $\bar{K}=\dfrac{K^*}{Kc_1^2\eta_0}$.

These equations will be supplemented with appropriate boundary conditions.

3. Normal mode analysis.

Generally, Laplace transformation and Fourier transformation are employed to solve a two- dimensional generalized thermoelastic problem. In the application of this method, the partial differential equations can be converted into ordinary differential equations. By solving differential equations in the transformed domain and adopting inverse Fourier transformation and inverse Laplace transformation in the time domain, the solutions of the problem can be attained. But this method entails a tiresome process. The key problem is that it introduces a discrete error and truncation error in the process of numerical inverse integrated transformation, so the second sound of heat conduction cannot be fully demonstrated. To compensate for the defects of the above mentioned method, we solve the problem of generalized thermoelasticity by employing normal mode analysis to the considered equations.

Now we seek the solution of equations (5), (6) and (8) in the following form:

$$(u, w, e, T)(x, z, t) = (\overline{u}, \overline{w}, \overline{e}, \overline{T})(z)\exp\left[ik(x - ct)\right] , \tag{9}$$

where k is wave number and c is the phase velocity.

Applying normal mode analysis to both sides of the equations (5), (6) and (8), we obtain

$$-\beta^2 k^2 \overline{u} + D^2\overline{u} + ik(\beta^2 - 1)D\overline{w} - ik\beta^2\overline{T} = -\beta^2 k^2 c^2 \overline{u}; \tag{10}$$

$$ik(\beta^2 - 1)D\overline{u} + \beta^2 D^2\overline{w} - k^2\overline{w} - \beta^2 D\overline{T} = -\beta^2 k^2 c^2 \overline{w}; \tag{11}$$

$$\left(D^2 - k^2\right)\overline{T} = P(\overline{T} + \varepsilon\overline{e}), \tag{12}$$

where $D \equiv \dfrac{d}{dz}$; $P = \dfrac{k^2c^2}{\left(ikc\tau_1 - \overline{K}\tau_2\right)}$ with $\tau_1 = \dfrac{1 - ikc\tau_T}{1 - ikc\tau_q - \dfrac{k^2c^2}{2}\tau_q^2}$; $\tau_2 = \dfrac{1 - ikc\tau_\nu}{1 - ikc\tau_q - \dfrac{k^2c^2}{2}\tau_q^2}$.

4. State-space formulation.

We take the quantities e, T, De, DT as state variables. Now

$$\overline{e} = ik\overline{u} + D\overline{w}. \tag{13}$$

Eliminating $\overline{u}$ and $\overline{w}$ between equations (10), (11) and (12) with the help of equation (13), we obtain the following equations:

$$D^2\overline{e} = (-k^2c^2 + k^2 + P\varepsilon)\overline{e} + P\overline{T}; \tag{14}$$

$$D^2\overline{T} = P\varepsilon\overline{e} + (k^2 + P)\overline{T}. \tag{15}$$

Equations (14) and (15) can be written in matrix differential equation form as follows:

$$\frac{d\tilde{V}(z)}{dz} = \tilde{A}\tilde{V}(z), \tag{16}$$

where

$$\tilde{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k^2c^2 + k^2 + P\varepsilon & P & 0 & 0 \\ P\varepsilon & k^2 + P & 0 & 0 \end{bmatrix}; \quad \tilde{V} = \begin{bmatrix} \overline{e} \\ \overline{T} \\ D\overline{e} \\ D\overline{T} \end{bmatrix}.$$

The formal solution of system (16) can be written in the form

$$\tilde{V}(z) = \exp(\tilde{A}z)\tilde{V}(z_0), \tag{17}$$

where z_0 denotes any arbitrarily chosen initial value for z.

The characteristic equation for the matrix $\tilde{A}$ is

$$\lambda^4 - (-k^2c^2 + 2k^2 + P\varepsilon + P)\lambda^2 + k^4 + k^2(-k^2c^2 + \varepsilon P + P) - Pk^2c^2 = 0. \tag{18}$$

The roots of the equation (18) satisfy the relations

$$k_1^2 + k_2^2 = -k^2c^2 + 2k^2 + P\varepsilon + P; \tag{19a}$$

$$k_1^2 k_2^2 = k^4 + k^2(-k^2c^2 + P\varepsilon + P) - Pk^2c^2. \tag{19b}$$

The Mclaurin series expansion of $\exp\left(\tilde{A}z\right)$ is given by

$$\exp\left(\tilde{A}z\right) = \sum_{n=0}^{\infty} \frac{\left[\tilde{A}.z\right]^n}{n!}.$$

Using Cayley – Hamilton theorem, the infinite series representing $\exp\left(\tilde{A}z\right)$ can be truncated to the following form:

$$\exp(\tilde{A}z) = \tilde{L} = b_0\tilde{I} + b_1\tilde{A} + b_2\tilde{A}^2 + b_3\tilde{A}^3, \tag{20}$$

where $\tilde{I}$ is the unit matrix of order 4 and b_0, ..., b_3 are some parameters depending on z, k and t.

We shall stress here that the above expressions for the matrix exponential is a formal one. In the actual physical problem, the space is divided into two regions accordingly as $z \geq 0$ or $z \leq 0$.

By Cayley – Hamilton theorem, the characteristic roots $\pm k_1$ and $\pm k_2$ of the matrix $\tilde{A}$ must satisfy the equations

$$\exp(k_1z) = b_0 + b_1k_1 + b_2k_1^2 + b_3k_1^3; \quad \exp(-k_1z) = b_0 - b_1k_1 + b_2k_1^2 - b_3k_1^3;$$

$$\exp(k_2z) = b_0 + b_1k_2 + b_2k_2^2 + b_3k_2^3; \quad \exp(-k_2z) = b_0 - b_1k_2 + b_2k_2^2 - b_3k_2^3.$$

The solution of the above system is given by

$$b_0 = \frac{1}{k_1^2 - k_2^2}[k_1^2\cosh(k_2z) - k_2^2\cosh(k_1z)];$$

$$b_1 = \frac{1}{k_1^2 - k_2^2}\left[\frac{k_1^2}{k_2}\sinh(k_2z) - \frac{k_2^2}{k_1}\sinh(k_1z)\right]; \tag{21}$$

$$b_2 = \frac{1}{k_1^2 - k_2^2}[\cosh(k_1z) - \cosh(k_2z)]; \quad b_3 = \frac{1}{k_1^2 - k_2^2}\left[\frac{1}{k_1}\sinh(k_1z) - \frac{1}{k_2}\sinh(k_2z)\right].$$

Substituting the expressions (21) into (20) and computing $\tilde{A}^2$ and $\tilde{A}^3$, we obtain after repeated use of equations (19a) and (19b), the elements $l_{ij}\,(i, j = 1,2,3,4)$ of the matrix $\tilde{L}$ as

$$l_{11} = \frac{1}{k_1^2 - k_2^2}\left[(k_1^2 - k^2 - P)\cosh(k_1z) - (k_2^2 - k^2 - P)\cosh(k_2z)\right];$$

$$l_{12} = \frac{P}{k_1^2 - k_2^2}\left[\cosh(k_1z) - \cosh(k_2z)\right];$$

$$l_{13} = \frac{1}{k_1^2 - k_2^2}\left[\frac{(k_1^2 - k^2 - P)}{k_1}\sinh(k_1z) - \frac{(k_2^2 - k^2 - P)}{k_2}\sinh(k_2z)\right];$$

$$l_{14} = \frac{P}{k_1^2 - k_2^2}\left[\frac{1}{k_1}\sinh(k_1 z) - \frac{1}{k_2}\sinh(k_2 z)\right]; \quad l_{21} = \frac{P\varepsilon}{k_1^2 - k_2^2}\left[\cosh(k_1 z) - \cosh(k_2 z)\right];$$

$$l_{22} = \frac{1}{k_1^2 - k_2^2}\left[(k_1^2 - k^2 - P)\cosh(k_2 z) - (k_2^2 - k^2 - P)\cosh(k_1 z)\right];$$

$$l_{23} = \frac{P\varepsilon}{k_1^2 - k_2^2}\left[\frac{1}{k_1}\sinh(k_1 z) - \frac{1}{k_2}\sinh(k_2 z)\right];$$

$$l_{24} = \frac{1}{k_1^2 - k_2^2}\left[\frac{(k_1^2 - k^2 - P)}{k_2}\sinh(k_2 z) - \frac{(k_2^2 - k^2 - P)}{k_1}\sinh(k_1 z)\right];$$

$$l_{31} = \frac{1}{k_1^2 - k_2^2}\left\{\left[k_1(k_1^2 - k^2 - P)\right]\sinh(k_1 z) - \left[k_2(k_2^2 - k^2 - P)\right]\sinh(k_2 z)\right\};$$

$$l_{32} = \frac{P}{k_1^2 - k_2^2}\left[k_1 \sinh(k_1 z) - k_2 \sinh(k_2 z)\right];$$

$$l_{33} = \frac{1}{k_1^2 - k_2^2}\left[(k_1^2 - k^2 - P)\cosh(k_1 z) - (k_2^2 - k^2 - P)\cosh(k_2 z)\right];$$

$$l_{34} = \frac{P}{k_1^2 - k_2^2}\left[\cosh(k_1 z) - \cosh(k_2 z)\right];$$

$$l_{41} = \frac{P\varepsilon}{k_1^2 - k_2^2}\left[k_1 \sinh(k_1 z) - k_2 \sinh(k_2 z)\right];$$

$$l_{42} = \frac{1}{k_1^2 - k_2^2}\left\{\left[k_2(k_1^2 - k^2 - P)\right]\sinh(k_2 z) - \left[k_1(k_2^2 - k^2 - P)\right]\sinh(k_1 z)\right\}; \tag{22}$$

$$l_{43} = \frac{\varepsilon P}{k_1^2 - k_2^2}[\cosh(k_1 z) - \cosh(k_2 z)];$$

$$l_{44} = \frac{1}{k_1^2 - k_2^2}[(k_1^2 - k^2 - P)\cosh(k_1 z) - (k_2^2 - k^2 - P)\cosh(k_2 z)].$$

It should be noted that we have repeatedly used equations (19a) and (19b) in order to write (29) in the simplest possible form.

Using equation (24), upon equating Matrices we obtain

$$\bar{e}(z) = l_{11}e_0 + l_{12}\theta_0 + l_{13}e_0' + l_{14}\theta_0'; \tag{23}$$

$$\bar{T}(z) = l_{21}e_0 + l_{22}\theta_0 + l_{23}e_0' + l_{24}\theta_0', \tag{24}$$

where $e_0 = \bar{e}(z_0)$, $\theta_0 = \bar{T}(z_0)$, $e_0' = D\bar{e}(z_0)$, $\theta_0' = D\bar{T}(z_0)$

Using equation (22) into equation (23) and (24) we obtain

$$\bar{T} = \sum_{i=1}^{2}\left[M_i \cosh(k_i z) + \frac{M_i'}{k_i}\sinh(k_i z)\right]; \tag{25}$$

$$\bar{e}=\sum_{i=1}^{2}\left[N_i\cosh\left(k_i z\right)+\frac{N_i'}{k_i}\sinh\left(k_i z\right)\right], \tag{26}$$

where

$$M_i=\frac{(-1)^{i-1}}{k_1^2-k_2^2}\left\{\varepsilon Pe_0+\left(k_i^2-k^2-P\right)\theta_0\right\};\quad M_i'=\frac{(-1)^{i-1}}{k_1^2-k_2^2}\left\{\varepsilon Pe_0'+\left(k_i^2-k^2-P\right)\theta_0'\right\};$$

$$N_i=\frac{(-1)^{i-1}}{k_1^2-k_2^2}\left\{\left(k_i^2-k^2-P\right)e_0+P\theta_0\right\};\quad N_i'=\frac{(-1)^{i-1}}{k_1^2-k_2^2}\left\{\left(k_i^2-k^2-P\right)e_0'+P\theta_0'\right\}. \tag{27}$$

Substituting from equation (13) in equation (10), we obtain

$$\left(D^2-k_3^2\right)\bar{u}=\sum_{i=1}^{2}ik\left\{\left[\left(1-\beta^2\right)N_i+\beta^2M_i\right]\cosh\left(k_i z\right)+\frac{\left[\left(1-\beta^2\right)N_i'+\beta^2M_i'\right]}{k_i}\sinh\left(k_i z\right)\right\},$$

where $k_3^2=-\beta^2k^2c^2+k^2$.

Now solving the above equation, we get

$$\bar{u}=C\cos\left(k_3 z\right)+\frac{ik}{k_1^2-k_2^2}\sum_{i=1}^{2}\left\{\frac{\left(1-\beta^2\right)N_i+\beta^2M_i}{\left(k_i^2-k_3^2\right)}\cosh\left(k_i z\right)+\frac{\left(1-\beta^2\right)N_i'+\beta^2M_i'}{k_i\left(k_i^2-k_3^2\right)}\sinh\left(k_i z\right)\right\}. \tag{28}$$

Substituting (28) into (13) and integrating the resulting equation, we get

$$\bar{w}=\frac{-ikC}{k_3}\sinh\left(k_3 z\right)+\frac{1}{k_1^2-k_2^2}\sum_{i=1}^{2}\left\{\left[\frac{N_i}{k_i}+\frac{k^2\left(\left(1-\beta^2\right)N_i+\beta^2M_i\right)}{k_i\left(k_i^2-k_3^2\right)}\right]\sinh\left(k_i z\right)+\right.$$
$$\left.+\left[\frac{N_i'}{k_i^2}+\frac{k^2\left(\left(1-\beta^2\right)N_i'+\beta^2M_i'\right)}{k_i^2\left(k_i^2-k_3^2\right)}\right]\cosh\left(k_i z\right)\right\}. \tag{29}$$

5. Boundary Conditions.

We consider the case where the surface of the half space is subjected to a time dependent thermal shock and the surface is traction free.

(a) Thermal boundary condition that the surface of the half-space is subjected to a time dependent thermal shock

$$T(x,\,0,\,t)=F\left(t\right)H\left(a-|x|\right); \tag{30}$$

(b) Mechanical boundary condition that the surface to the half-space is traction free

$$\tau_{zz}(x,\,0,\,t)=0;\ \tau_{xz}(x,\,0,\,t)=0, \tag{31}$$

where H denotes Heaviside function and a is constant.

6. Application.

We shall apply our results to solve a problem for a half-space $\left(z\geq 0\right)$. Inside the region $0\leq z\leq\infty$, the positive exponential terms, not bounded at infinity, must be suppressed. Thus, for $z\geq 0$ we should replace each $\sinh\left(kz\right)$ by $-0,5\exp\left(-kz\right)$ and each $\cosh\left(kz\right)$ by $0,5\exp\left(-kz\right)$.

The solution of the problem is given by equation (24) with z_0 chosen as zero for convenience. Thus, the two components of the initial vectors $V_0=V\left(0\right)$ are known, i.e.,

$$e_0' = \theta_0' = 0\,.$$

Now replacing each $\sinh(kz)$ by $-\,0,5\exp(-kz)$ and each $\cosh(kz)$ by $0,5\exp(-kz)$, we obtain

$$\bar{T} = \frac{1}{2}\sum_{i=1}^{2} M_i \exp(-k_i z); \quad \bar{e} = \frac{1}{2}\sum_{i=1}^{2} N_i \exp(-k_i z);$$

$$\bar{u} = \frac{C}{2}\exp(-k_3 z) + \frac{ik}{2}\sum_{i=1}^{2}\frac{(1-\beta^2)N_i + \beta^2 M_i}{(k_i^2 - k_3^2)}\exp(-k_i z); \tag{32}$$

$$\bar{w} = \frac{ikC}{2k_3}\exp(-k_3 z) - \frac{1}{2}\sum_{i=1}^{2}\left\{\frac{N_i}{k_i} + \frac{k^2\left[(1-\beta^2)N_i + \beta^2 M_i\right]}{k_i(k_i^2 - k_3^2)}\right\}\exp(-k_i z).$$

Using (32) in equations (7a – 7c), the stress components are obtained as

$$\tau_{xx} = \left[\frac{-c_1^2 k^2}{2}\sum_{i=1}^{2}\frac{(1-\beta^2)N_i + \beta^2 M_i}{(k_i^2 - k_3^2)}\exp(-k_i z) + ikC\exp(-k_3 z) + \right.$$
$$+\frac{(c_1^2 - 2)}{2}\sum_{i=1}^{2}\left\{N_i + \frac{k^2\left[(1-\beta^2)N_i + \beta^2 M_i\right]}{(k_i^2 - k_3^2)}\right\}\exp(-k_3 z) - \tag{33}$$
$$\left. -\frac{\beta^2}{2}\sum_{i=1}^{2} M_i \exp(-k_i z)\right]\exp\left[ik(x-ct)\right];$$

$$\tau_{zz} = \left[\frac{-k^2(c_1^2 - 2)}{2}\sum_{i=1}^{2}\frac{(1-\beta^2)N_i + \beta^2 M_i}{(k_i^2 - k_3^2)}\exp(-k_i z) + \right.$$
$$+\frac{c_1^2}{2}\sum_{i=1}^{2}\left\{N_i + \frac{k^2\left[(1-\beta^2)N_i + \beta^2 M_i\right]}{(k_i^2 - k_3^2)}\right\}\exp(-k_i z) - \tag{34}$$
$$\left. -ikC\exp(-k_3 z) - \frac{\beta^2}{2}\sum_{i=1}^{2} M_i \exp(-k_i z)\right]\exp\left[ik(x-ct)\right];$$

$$\tau_{xz} = \left[\frac{-ik}{2}\sum_{i=1}^{2}\frac{\left[(1-\beta^2)N_i + \beta^2 M_i\right]k_i}{(k_i^2 - k_3^2)}\exp(-k_i z) - \right.$$
$$-\frac{ik}{2}\sum_{i=1}^{2}\left\{\frac{N_i}{k_i} + \frac{k^2\left[(1-\beta^2)N_i + \beta^2 M_i\right]}{k_i(k_i^2 - k_3^2)}\right\}\exp(-k_i z) - \tag{35}$$
$$\left. -\frac{C}{2k_3}(k^2 + k_3^2)\exp(-k_3 z)\right]\exp\left[ik(x-ct)\right].$$

Now applying boundary conditions (37) and (38), we obtain

$$\frac{1}{2}\sum_{i=1}^{2} M_i = F(t)H(a-|x|)\exp\left[-ik(x-ct)\right]; \tag{36}$$

$$\frac{-k^2\left(c_1^2-2\right)}{2}\sum_{i=1}^{2}\frac{\left(1-\beta^2\right)N_i+\beta^2 M_i}{\left(k_i^2-k_3^2\right)}+$$
$$+\frac{c_1^2}{2}\sum_{i=1}^{2}\left\{N_i+\frac{k^2\left[\left(1-\beta^2\right)N_i+\beta^2 M_i\right]}{\left(k_i^2-k_3^2\right)}\right\}-ikC-\frac{\beta^2}{2}\sum_{i=1}^{2}M_i=0; \tag{37}$$

$$\frac{ik}{2}\sum_{i=1}^{2}\frac{\left[\left(1-\beta^2\right)N_i+\beta^2 M_i\right]k_i}{\left(k_i^2-k_3^2\right)}+\frac{ik}{2}\sum_{i=1}^{2}\left\{\frac{N_i}{k_i}+\frac{k^2\left[\left(1-\beta^2\right)N_i+\beta^2 M_i\right]}{k_i\left(k_i^2-k_3^2\right)}\right\}+\frac{C}{2k_3}\left(k^2+k_3^2\right)=0. \tag{38}$$

From (27), we see that M_i and N_i are expressed in terms of e_0 and θ_0. By solving the equations (36), (37) and (38), e_0, θ_0 and C can be obtained.

This completes the solution of the problem.

7. Special cases.

Now we discuss some special cases as follows:

(a) If we take $K^* = 0$ then equation (4) becomes the heat conduction equation for dual-phase lag (DPL) model.

(b) If we take $\tau_q = \tau_T = \tau_\nu = 0$ then equation (4) will reduce to Green-Naghdi-III (GN-III) model.

(c) If we take $K^* = \tau_T = \tau_q^2 = 0$ and $\tau_q \neq 0$ then equation (4) will reduce to Lord-Shulman (LS) model.

(d) If we take $\tau_q = \tau_T = K^* = 0$ then equation (4) will reduce to classical thermoelasticity (CT).

8. Numerical discussion.

In order to illustrate the above results graphically the time dependent thermal shock $F(t)$ is taken in the following form:

$$F(t) = \theta_0 \exp(-bt).$$

The copper material is chosen for purposes of numerical evaluations.

$$\lambda = 7.76\cdot 10^{10}\,\text{N/m}^2;\ \mu = 3.86\cdot 10^{10}\,\text{N/m}^2;\ \alpha_t = 1.78\cdot 10^{-5}\,\text{K}^{-1};$$

$$\rho = 8954\,\text{Kg/m}^3;\ C_e = 383.1\,\text{m}^2/\text{K},\ K = 386\,\text{W/mK};\ K^* = 124\,\text{W/mKs};\ T_0 = 293\,\text{K};$$

$$\tau_q = 2\cdot 10^{-7}\,\text{s};\ \tau_T = 1\cdot 10^{-7}\,\text{s};\ \tau_\nu = 1\cdot 10^{-8}\,\text{s}.$$

Further for numerical purpose we take $\theta_0 = 10,\ b = 0.1,\ k = 1.2.$

The numerical applications will be carried out for the temperature T, displacements u, w and stress τ_{xx} at $x = 0$. Fig. 1 – 4 represent the graphs for τ_{zz}, u, w and T versus t for fixed values of z for stress free boundary. Keeping t fixed, the graphs for τ_{zz}, u, w and T versus z are presented in fig. 5 – 9.

In fig. 1, a significant difference is noticed for the stress field predicted by different models. It is observed that the value of stress for LS model is greater than TPL model and less than CT model.

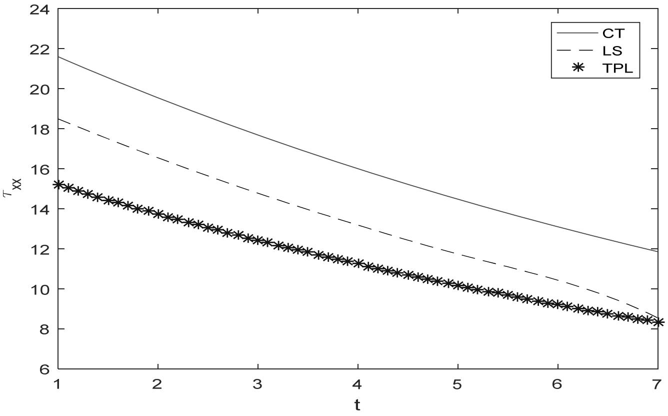

Fig. 1. Comparison of stress $\left(\tau_{xx}\right)$ with respect to time.

In fig. 2 it is found that the value of u for three phase lag model is greater than the value of u for GN-III model but less than the value of u for LS model. It finally converges to zero with the increase of time.

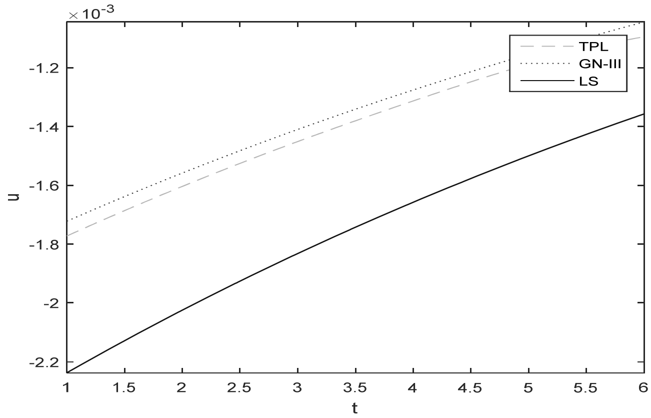

Fig. 2. Comparison of u with respect to time.

In fig. 3 it is found that the value of w for three phase lag model is greater than the value of u for GN-III model but less than the value of w for LS model. It finally converges to zero with the increase of time.

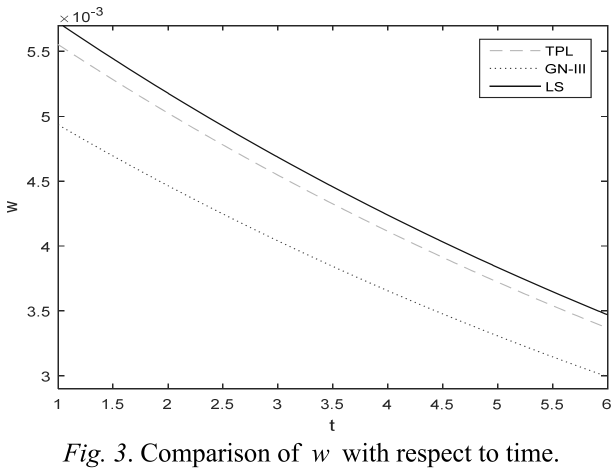

Fig. 3. Comparison of w with respect to time.

It is shown in fig. 4 that temperature for three phase lag model is greater than GN-III model and less than LS model. The region of influence is much larger in case of LS model as compared to other models. Temperature is decreasing with the increasing of time and converging towards zero.

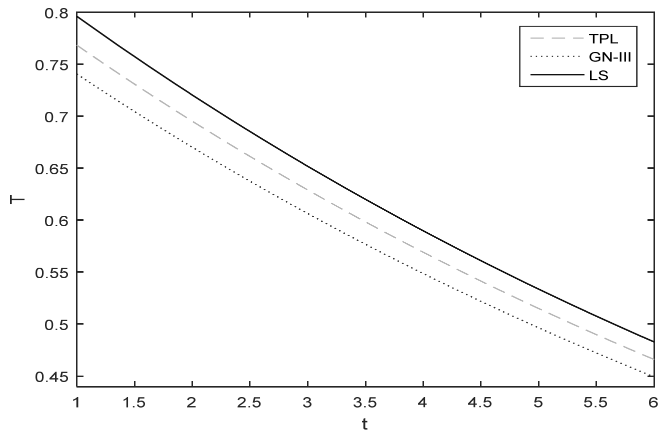

Fig. 4. Comparison of temperature with respect to time.

In fig. 5, a significant difference is observed for the stress field predicted by different models. The value of stress for three-phase-lag model with and without magnetic field is greater than GN-III model and lower than LS model. Initially stress has the maximum value and then with the increase of distance it decreases.

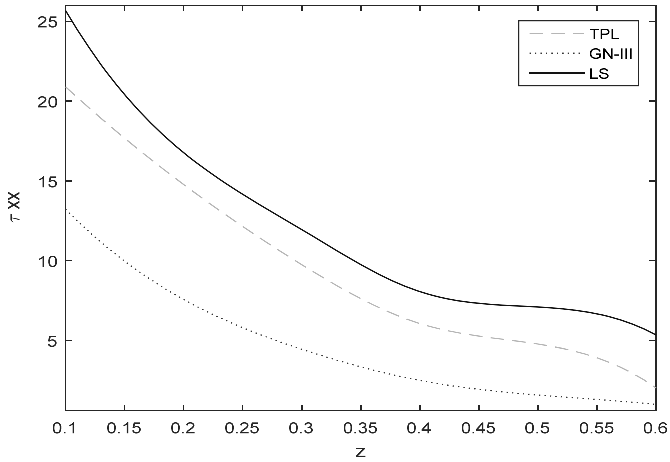

Fig. 5. Comparison of stress $\left(\tau_{xx}\right)$ with respect to distance.

In fig. 6, it is observed that the value of u for three phase lag model is greater than the value of u for GN-III model but less than the value of u for LS model. The figure indicates that u has maximum value initially and it decreases with the increase of distance.

In fig. 7, it is observed that the value of w for three phase lag model is greater than the value of w for GN-III model but less than the value of w for LS model. The figure indicates that w has maximum value initially and it decreases with the increase of distance.

It is shown in fig. 8 that temperature for three phase lag model is greater than GN-III model and less than LS model. The region of influence is much larger in case of LS model as compared to other models. Temperature is decreasing with the increasing of distance and converging towards zero.

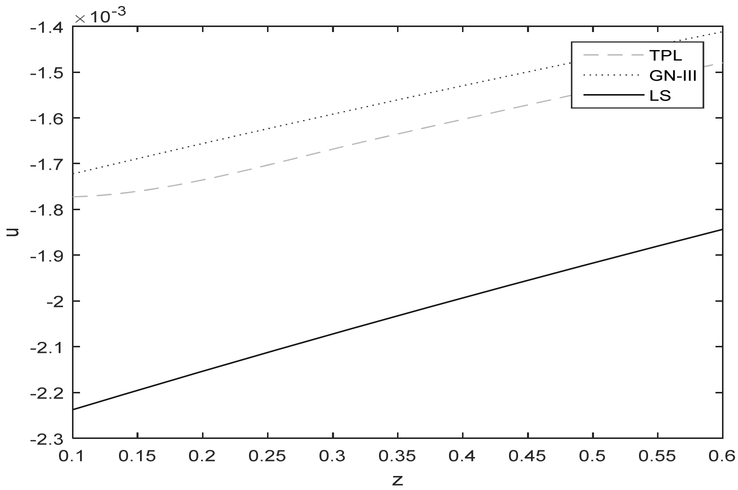

Fig. 6. Comparison of u with respect to distance.

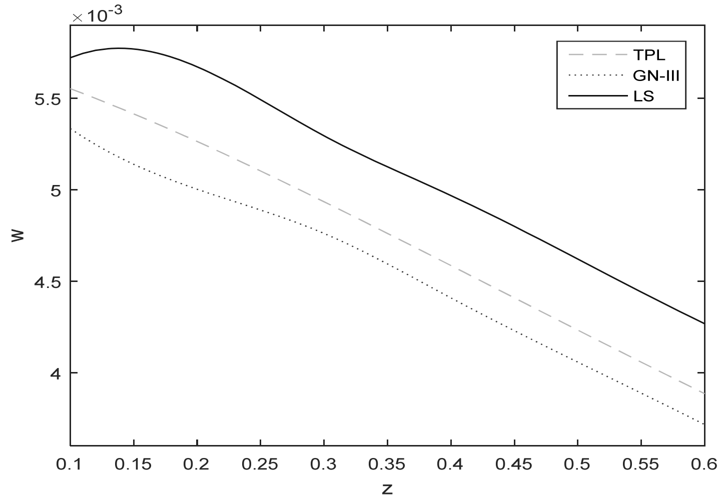

Fig. 7. Comparison of w with respect to distance.

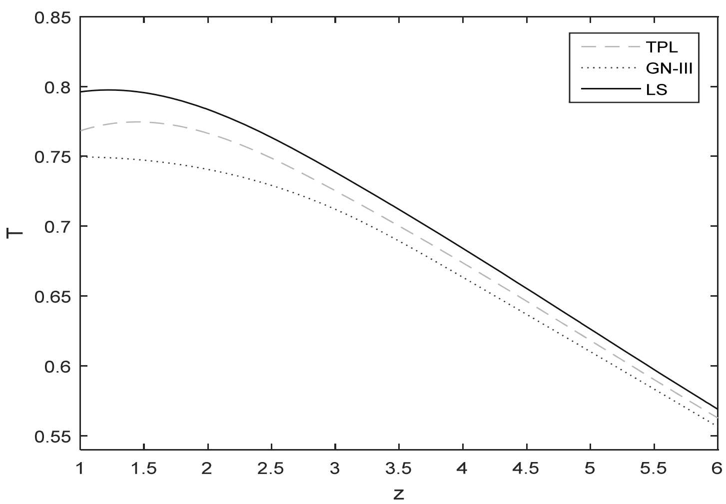

Fig. 8. Comparison of temperature with respect to distance.

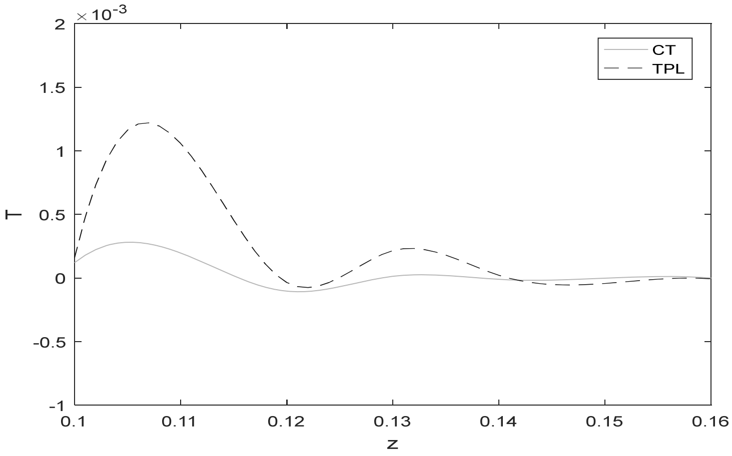

Fig. 9. Comparison of temperature with respect to distance.

It is shown in fig. 9 that temperature for three phase lag model is greater than CT model. Temperature is decreasing with the increase of distance and converging towards zero. Temperature for TPL model is showing oscillatory behavior like wave with the increase of distance.

9. Conclusion.

The present article provides a detail analysis of propagation of thermoelastic disturbances in presence of a time dependent thermal shock on traction free half-space. With the view of theoretical analysis and numerical computation, we can conclude the following phenomena:

(a) The importance of state space analysis is recognized in fields where the time behavior of any physical process is of interest. The state space approach is more general than the classical Laplace and Fourier transform theory. State space theory is applicable to all systems that can be analyzed by integral transforms in time and is applicable to many systems for which transform theory breaks down. Furthermore, state space theory gives a somewhat different insight into the time behavior of linear systems.

(b) The problem with three-phase-lag model is a more general one as the other thermoelastic models can be obtained as special cases from it.

(c) The thermoelastic disturbances converge towards zero with the distances from the application of the thermal shock.

(d) The influence of various thermal relaxation times is also observed from this investigation. The Lord-Shulman model shows maximum thermoelastic deformation and GN-III model of thermoelasticity experiences least amount of disturbances.

The results presented in this article may be useful for researchers who are working on material science, mathematical physics and thermodynamics with low temperatures as well as on the development of the hyperbolic thermoelasticity theory.

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РЕЗЮМЕ. Двовимірна задача узагальненої термопружності сформульована з використанням підходу простору станів. У цій постановці основні рівняння перетворюються на матричне диференціальне рівняння, розв’язування якого дає змогу записати розв’язок двовимірної задачі через граничні умови. Остаточне формулювання застосовується до задачі про ізотропний напівпростір в рамках моделі запізнювання трьох фаз термопружності. Гранична поверхня є вільною від розтягу і піддається залежному від часу тепловому удару. Отримані та представлені графічно розв’язки для розподілу температури, переміщень та напружень. Також проведено порівняння з іншими термопружними моделями.

Nomenclature:

$\lambda,\ \mu$	Lame′ constants;
ρ	density
C_e	specific heat at constant strain
t	time
T	temperature above reference temperature
T_0	reference temperature chosen so that $\lvert T/T_0 \rvert << 1$
τ_{ij}	components of stress tensor
e_{ij}	components of strain tensor
$u,\ w$	components of displacement vector
K	thermal conductivity
K^*	material constant characteristic of the theory
c_1^2	$\dfrac{\lambda+2\mu}{\rho}$
c_2^2	$\dfrac{\mu}{\rho}$
β^2	$\dfrac{c_1^2}{c_2^2}$
e	$u_{,x}+w_{,z}$
η_0	$\dfrac{\rho C_e}{K}$
α_t	coefficient of linear thermal expansion
γ	$(3\lambda+2\mu)\alpha_t$
ε	$\dfrac{\gamma^2 T_0}{\rho C_e(\lambda+2\mu)}$.

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