STRAIGHT LINES IN THREE-DIMENSIONAL SPACE AND THE ULTRAHYPERBOLIC EQUATION

© **Chrastinová V.**

Brno University of Technology, Veveří 331/95, 602 00 Brno, Czech Republic.
Department of Mathematics, Faculty of Civil Engineering,
e-mail: *chrastinova.v@fce.vutbr.cz*

Abstract. The straight lines in three-dimensional vector space realize the shortest distance for various metrics. This property is reformulated in terms of the inverse problem of the calculus of variations and closely related to the ultrahyperbolic equation with four independent variables. The interrelation is useful in both directions. For instance, polynomial solutions of the ultrahyperbolic equation provide all polynomial metrics with extremals the straight lines and conversely, a slight generalization of the Hilbert metrics leads to rather nontrivial (multi-valued or focusing) solutions of the ultrahyperbolic equation. In general, the article clarifies some well-known achievements concerning the 4th Hilbert Problem.

Introduction

The history of our topic goes back to the famous Hilbert Problems [1], namely to the 4th Problem concerning the determination of all metrics in the open subsets of $\mathbb{P}^n$ that have the straight lines as the shortest curves and the study of the relevant geometries. In this strong version, it is still far from a complete solution [2]. With additional smoothness assumptions, a close connection to the *inverse problem of the calculus of variations* ($\mathscr{I}\mathscr{P}$) and the prominent role of the *ultrahyperbolic equation* ($\mathscr{U}\mathscr{H}$) was soon indicated [3].

Let us recall that $\mathscr{I}\mathscr{P}$ consists in determination of the variational integral if the extremals are given in advance. In two dimensions, for the integral $\int f(x,y,v)\mathrm{d}x$ $(v=y')$, the solution is rather easy [4]. Especially in the particular case of extremal straight lines the formula $f_{vv} = U(v, y-vx)$ with arbitrary U resolves the problem. This result was adapted to three dimensions [3] with the following result. *The variational integral $\int f(x,y,z,v,w)\mathrm{d}x$ $(v=y', w=z')$ has the straight lines for extremals if and only if $f_{vv}=U, f_{vw}=V, f_{ww}=W$ are functions of the variables*

$$\alpha = v,\ \beta = y - vx,\ \gamma = w,\ \delta = z - wx.$$

One can check the compatibility conditions

$$U_\gamma = V_\alpha,\ V_\gamma = W_\alpha,\ U_\delta = V_\beta,\ U_\delta = W_\beta$$

and they imply the $\mathscr{U}\mathscr{H}$ equation $\partial^2(\cdot)/\partial_\beta\partial_\gamma = \partial^2(\cdot)/\partial_\alpha\partial_\delta$ for all functions $(\cdot) = U, V$ and W. *Then the function f can be reconstructed from U, V, W by double quadrature.*

Subsequently other solutions of $\mathscr{I}\mathscr{P}$ were discussed. In the ingenious article [5], the three-dimensional subcase was thoroughly analysed in full generality. However, in the particular case of extremal straight lines the path from $\mathscr{U}\mathscr{H}$ (formula (8.22)) to the kernel function f (pages 82-84) is not quite easy. The recent general solution [6] of $\mathscr{I}\mathscr{P}$ rests on non-elementary tools, the variational bicomplex, and the straight lines are not separately mentioned.

In this article we follow the geometrical approach [7] based on the systematical application of the Poincaré-Cartan ($\mathscr{P}\mathscr{C}$) forms [8] with intentional use of quite elementary methods of algorithmical nature. Our result is as follows: *for the variational integral* $\int f(x,y,z,v,w)\mathrm{d}x$ $(v = y', w = z')$ *with extremals the straight lines, every function* $\bar{f} = f(c,y,z,v,w)$ $(c = \text{const})$ *satisfies* $\mathscr{U}\mathscr{H}$ $\partial^2\bar{f}/\partial y\partial w = \partial^2\bar{f}/\partial z\partial v$ *and conversely, every solution* $\bar{f} = \bar{f}(y,z,v,w)$ of this $\mathscr{U}\mathscr{H}$ *permits to reconstruct the kernel function* f *of the variational integral* (which reduces to $\bar{f}$ if $x = c$ is kept fixed for a given constant c) *by a quadrature.*

On this occasion, a few examples are presented. The polynomial case related to the Bessel functions, a far going generalizations of the Hilbert projective metrics [9] on Riemannian surfaces with the multivalued and focusing solutions of $\mathscr{U}\mathscr{H}$, and finally the proof of analyticity of the elliptical Hilbert metrics employing very advanced results [10, 11].

We will establish a close relationship between the familiar property of the straight lines $y = Ax + B, z = Cx + D$ in the space $\mathbb{R}^3$ with coordinates x, y, z, i.e., that they represent the shortest curves for certain metrics, and the solutions $\bar{f} = \bar{f}(y,z,v,w)$ of the ultrahyperbolic equation $\partial^2\bar{f}/\partial y\partial w = \partial^2\bar{f}/\partial z\partial v$.

In order to employ the common tools of differential calculus, we shall deal with metrics ρ such that the limit

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\rho((x,y,z),(x+u\varepsilon,y+v\varepsilon,z+w\varepsilon)) = F(x,y,z,u,v,w) \tag{1}$$

is a smooth (infinitely-differentiable) function whenever $|u|+|v|+|w| \neq 0$. In geometrical terms, F is the rate of change of the distance at the point $(x,y,z) \in \mathbb{R}^3$ as one moves in the direction (u,v,w). Equation (1) reads

$$\rho((x,y,z),(x+u\varepsilon,y+v\varepsilon,z+w\varepsilon)) = (F(x,y,z,u,v,w) + o(\varepsilon))\varepsilon$$

and it follows that the length of a smooth curve

$$(x(t),y(t),z(t)) \in \mathbb{R}^3 \quad (a \le t \le b, |x'(t)|+|y'(t)|+|z'(t)| \neq 0) \tag{2}$$

is represented by a Riemannian integral as follows. The sum of the distances between the neighbouring points of a partition

$$x_i = x(t_i), y_i = y(t_i), z_i = z(t_i); \quad a < \cdots < t_i < t_{i+1} < \cdots < b$$

of the curve approximates the Riemannian integral sum and has a limit as the norm of the partition tends to zero:

$$\lim\sum\rho((x_i,y_i,z_i),\,(x_{i+1},y_{i+1},z_{i+1})) = \int_a^b F(x(t),y(t),z(t),x'(t),y'(t),z'(t))\mathrm{d}t.$$

We speak of a *generalized length* $L = \int_a^b F\mathrm{d}t$ of the curve (2) and our aim is to deal with metrics ρ such that the straight lines realize the shortest curves connecting two given (sufficiently close) points.

The generalized length L is independent of the parametrization of the curve (2) which may cause some technical difficulties, however, on every sufficiently short segment of the curve one can choose one of the coordinates x, y, z for a new parameter. We shall mostly use the parameter x assuming $x'(t) \neq 0$ in (2). Then the curves under consideration are given by the equations $y = y(x), z = z(x)$ and the generalized length is represented by the integral

$$L = \int_a^b f(x, y(x), z(x), y'(x), z'(x))\mathrm{d}x, \ f(x, y, z, v, w) = F(x, y, z, 1, v, w). \qquad (3)$$

With this adaptation, the methods of the classical calculus of variations can be comfortably applied.

It is well-known that the curves of the minimal length connecting two given points satisfy the *Euler-Lagrange* ($\mathscr{EL}$) *system*

$$f_y(\cdots) = \frac{\mathrm{d}}{\mathrm{d}x} f_v(\cdots), \ f_z(\cdots) = \frac{\mathrm{d}}{\mathrm{d}x} f_w(\cdots), \text{ where } (\cdots) = (x, y, z, y', z').$$

Recall that the solutions of $\mathscr{EL}$ system are called *extremals*. We wish to determine functions f such that the straight lines are just the extremals. Recall that $\mathscr{EL}$ system represents only the necessary conditions and the local minimum property of the straight lines is ensured if moreover the familiar *Legendre condition* holds true, we shall however focus our interest just on the $\mathscr{EL}$ system.

With these preparations, our task can be explained in quite simple terms.

The $\mathscr{EL}$ system reads

$$f_y = f_{vx} + f_{vy}y' + f_{vz}z' + f_{vv}y'' + f_{vw}z'',$$
$$f_z = f_{wx} + f_{wy}y' + f_{wz}z' + f_{wv}y'' + f_{ww}z''.$$

It follows that all straight lines $y = vx + B$, $z = wx + D$ with the variable parameters $A = v$, $C = w$ are extremals (solutions of the $\mathscr{EL}$ system) if and only if the identities

$$f_y(\cdot) = f_{vx}(\cdot) + f_{vy}(\cdot)v + f_{vz}(\cdot)w,$$
$$f_z(\cdot) = f_{wx}(\cdot) + f_{wy}(\cdot)v + f_{wz}(\cdot)w,$$

where $(\cdot) = (x, y, z, v, w)$ hold true. They provide a system of the second order partial differential equations for the function $f = f(x, y, z, v, w)$ and we will also use the alternative transcription

$$f_y = Xf_v, \ f_z = Xf_w \qquad \left(X = \frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \right) \qquad (4)$$

of these equations in future. One can then observe that

$$f_{yw} = f_{vxw} + f_{vyw}v + f_{vzw}w + f_{vz},$$
$$f_{zv} = f_{wxv} + f_{wyv}v + f_{wy} + f_{wzv}w,$$

whence the *ultrahyperbolic equation* $f_{yw} = f_{zv}$ follows. The coordinate x appears as a mere parameter, so it is of interest to consider the equation

$$\bar{f}_{yw} = \bar{f}_{zv} \qquad (\bar{f} = \bar{f}(y, z, v, w) = f(c, y, z, v, w)) \qquad (5)$$

with c an arbitrary constant. We shall see that *the equation* (5) *for the function* $\bar{f} = \bar{f}(y, z, v, w)$ *of four variables is in certain sense equivalent to the system* (4). More precisely: *a solution* $\bar{f}$ *of* (5) *together with the choice of a constant* c *permits us to reconstruct the original function* $f = f(x, y, z, v, w)$ *satisfying* (4).

In this article a function $f = f(x, y, z, v, w)$ is called *resolving* if (4) is safisfied, i.e., if *all straight lines are extremals*. We will determine all resolving functions f by using the solutions $\bar{f}$ of (5). The converse setting will also be quite interesting; certain resolving functions f will be obtained by direct geometrical construction which provides rather nontrivial solutions $\bar{f}$ of the ultrahyperbolic equation (5).

Prerequisities

Our reasonings will be carried out in an open subset of the space $\mathbb{R}^5$ with coordinates denoted x, y, z, v, w. We also use the alternative coordinates

$$x,\ \alpha = v,\ \beta = y - vx,\ \gamma = w,\ \delta = z - wx.$$

The coordinates v, w correspond to the derivatives, therefore we consider straight lines given by the equations

$$y = Ax + B,\ z = Cx + D,\ v = A,\ w = C \qquad (A, B, C, D \text{ are constants}) \tag{6}$$

and, in terms of the alternative coordinates the equations (6) read:

$$\alpha = A,\ \beta = B,\ \gamma = C,\ \delta = D \quad (A, B, C, D\,\text{are constants}). \tag{7}$$

For a given function $g = g(x, y, z, v, w)$ clearly

$$\begin{aligned} \mathrm{d}g &= Xg\mathrm{d}x + g_y(\mathrm{d}y - v\mathrm{d}x) + g_z(\mathrm{d}z - w\mathrm{d}x) + g_v\mathrm{d}v + \mathrm{d}g_w\mathrm{d}w = \\ &= Xg\mathrm{d}x + g_y\mathrm{d}\beta + g_z\mathrm{d}\delta + (g_v + xg_y)\mathrm{d}\alpha + (g_w + xg_z)\mathrm{d}\gamma. \end{aligned} \tag{8}$$

The functions $\alpha,\ \beta,\ \gamma,\ \delta$ are *reduced* to $\bar{\alpha} = v,\ \bar{\beta} = y - cv,\ \bar{\gamma} = w,\ \bar{\gamma} = z - cw$ if they are considered on the hyperplane $x = c$ (a constant). In general, a function $g = g(x, y, z, v, w)$ is *reduced* to $\bar{g} = g(c, y, z, v, w) = \bar{g}(y, z, v, w)$. In the alternative coordinates, a function $h = h(x, \alpha, \beta, \gamma, \delta)$ is reduced to

$$\bar{h} = h(c, \bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}) = h(c, v, y - cv, w, z - cw)$$

and this reduction will be again denoted $\bar{h} = \bar{h}(y, z, v, w)$ when regarded as a function of the original coordinates.

Conversely, every function $h = h(\alpha, \beta, \gamma, \delta)$ *independent of* x (better: *expressible in terms of* $\alpha, \beta, \gamma, \delta$) can be *restored* from its restriction $\bar{h}(y, z, v, w)$ expressed in terms of the original coordinates since

$$h(\alpha, \beta, \gamma, \delta) = \bar{h}(\beta + c\alpha,\ \delta + c\gamma,\ \alpha,\ \gamma).$$

Indeed, the restriction of the function is $\bar{h}(\bar{\beta} + c\bar{\alpha},\ \bar{\delta} + c\bar{\gamma},\ \bar{\alpha},\ \bar{\gamma}) = \bar{h}(y, z, v, w)$; use the formulae for $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}$ given above.

Analogous procedure can be applied to differential forms. A differential form ψ can be *reduced* to the form denoted $\bar{\psi}$ and conversely, every differential form ψ *expressible only in terms the functions* $\alpha, \beta, \gamma, \delta$ (*without the use of* x) can be *restored* from the reduction $\bar{\psi}$ expressed in terms of y, z, v, w if the functions $\beta + c\alpha,\ \delta + c\gamma,\ \alpha,\ \gamma$ are substituted for y, z, v, w, respectively. One can verify that the differential is preserved: $\mathrm{d}\bar{h} = \overline{\mathrm{d}h}$, $\mathrm{d}\bar{\psi} = \overline{\mathrm{d}\psi}$. We omit the simple direct proof (alternatively: reductions and restorations are special pull-backs).

Definition. For every function $f = f(x, y, z, v, w)$ we introduce the Poincaré-Cartan ($\mathscr{PC}$) form

$$\breve{\varphi} = f\mathrm{d}x + f_v(\mathrm{d}y - v\mathrm{d}x) + f_w(\mathrm{d}z - w\mathrm{d}x). \tag{9}$$

The exterior differential $\Phi = \mathrm{d}\breve{\varphi}$ has the obvious properties $\mathrm{d}\Phi = 0$ (Φ is a *closed form*) and $\Phi \cong 0 \pmod{\mathrm{d}x, \mathrm{d}y, \mathrm{d}z}$). If f is a resolving function, we speak of a *resolving $\mathscr{PC}$ form.* These resolving $\mathscr{PC}$ forms will be alternatively characterized in the following lemmas.

Direct Lemma. *If $\breve{\varphi}$ is a resolving $\mathscr{PC}$ form, then $\Phi = \mathrm{d}\breve{\varphi}$ can be expressed in terms of the functions α, β, γ, δ* (without the use of the coordinate x and the differential $\mathrm{d}x$).

Proof. Obviously $\mathrm{d}y - v\mathrm{d}x = \mathrm{d}\beta + x\mathrm{d}\alpha$, $\mathrm{d}z - w\mathrm{d}x = \mathrm{d}\delta + x\mathrm{d}\gamma$, $\mathrm{d}v = \mathrm{d}\alpha$, $\mathrm{d}w = \mathrm{d}\gamma$, therefore

$$\Phi = \mathrm{d}f \wedge \mathrm{d}x + \mathrm{d}f_v \wedge (\mathrm{d}\beta + x\mathrm{d}\alpha) + \mathrm{d}f_w \wedge (\mathrm{d}\delta + x\mathrm{d}\gamma) - (f_v\mathrm{d}\alpha + f_w\mathrm{d}\gamma) \wedge \mathrm{d}x.$$

Using (8) for $g = f,\ f_v,\ f_w,$ if follows that

$$\Phi = ((f_y - Xf_v)(\mathrm{d}\beta + x\mathrm{d}\alpha) + (f_z - Xf_w)(\mathrm{d}\delta + x\mathrm{d}\gamma)) \wedge \mathrm{d}x + f^{\alpha\beta}\mathrm{d}\alpha \wedge \mathrm{d}\beta + \cdots + f^{\gamma\delta}\mathrm{d}\gamma \wedge \mathrm{d}\delta$$

with certain coefficients $f^{\alpha\beta}, \cdots, f^{\gamma\delta}$ (they need not be explicitly stated). By virtue of (4), we obtain

$$\Phi = f^{\alpha\beta}\mathrm{d}\alpha \wedge \mathrm{d}\beta + \cdots + f^{\gamma\delta}\mathrm{d}\gamma \wedge \mathrm{d}\delta,$$

where the coefficients can be expressed in terms of the alternative coordinates x, α, β, γ, δ. However, they are in fact independent of x as follows from the identity $\mathrm{d}\Phi = 0$.

$\square$

Converse Lemma. *Let Ψ be a closed 2-form satisfying $\Psi \cong 0$* (mod $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}z$). *If Ψ is expressible only in terms of α, β, γ, δ then there (locally) exists a resolving $\mathscr{PC}$ form $\breve{\varphi}$ such that $\mathrm{d}\breve{\varphi} = \Psi$.*

Proof. By virtue of the Poincaré lemma, $\Psi = \mathrm{d}\psi$ for an appropriate form

$$\psi = K\mathrm{d}x + L\mathrm{d}y + R\mathrm{d}z + M\mathrm{d}v + N\mathrm{d}w.$$

The congruence $\Psi = \mathrm{d}\psi \cong 0$ implies $M_w = N_v$ and there exists $g = g(x, y, z, v, w)$ such that $g_v = M$, $g_w = N$. Clearly $\mathrm{d}(\psi - \mathrm{d}g) = \Psi$ where the corrected form

$$\psi - \mathrm{d}g = U\mathrm{d}x + V\mathrm{d}y + W\mathrm{d}z \qquad (U = K - g_x,\ V = L - g_y,\ W = R - g_z)$$

has no terms in $\mathrm{d}v$, $\mathrm{d}w$. We will see that $\psi - \mathrm{d}g$ is identical to the sought resolving $\mathscr{PC}$ form $\breve{\varphi}$. Obviously

$$\psi - \mathrm{d}g = f\mathrm{d}x + V(\mathrm{d}y - V\mathrm{d}x) + W(\mathrm{d}z - w\mathrm{d}x)$$

where $f = U + vV + wW$, whence

$$\Psi = \mathrm{d}(\psi - \mathrm{d}g) = \mathrm{d}f \wedge \mathrm{d}x + \mathrm{d}V \wedge (\mathrm{d}\beta + x\mathrm{d}\alpha) + \mathrm{d}W \wedge (\mathrm{d}\delta + x\mathrm{d}\gamma) - (V\mathrm{d}\alpha + W\mathrm{d}\gamma) \wedge \mathrm{d}x.$$

Using (8), one can verify that

$$\Psi = ((f_y - XV)(\mathrm{d}\beta + x\mathrm{d}\alpha) + (f_z - XW)(\mathrm{d}\delta + x\mathrm{d}\gamma) +$$
$$+ (f_v - V)\mathrm{d}\alpha + (f_w - W)\mathrm{d}\gamma) \wedge \mathrm{d}x + \cdots$$

where all the products $\mathrm{d}\alpha \wedge \mathrm{d}\beta, \cdots, \mathrm{d}\gamma \wedge \mathrm{d}\delta$ are neglected. Since Ψ is expressible in terms of α, β, γ, δ, we conclude that $f_v = V$, $f_w = W$, hence Ψ is indeed a $\mathscr{PC}$ form. Moreover $f_y = XV = Xf_v$, $f_z = XW = Xf_w$, therefore $\breve{\varphi}$ is resolving. The proof is done.

□

Crucial Lemma. *Let Ψ be a 2-form expressible only in terms of α, β, γ, δ. Then the congruence $\Psi \cong 0 \pmod{\mathrm{d}x, \mathrm{d}y, \mathrm{d}z}$ holds true if and only if the reduction $\bar{\Psi}$ to some* (equivalently: *to any*) *hyperplane $x = c$ is of the special kind*

$$\bar{\Psi} = (M\mathrm{d}v + L\mathrm{d}w) \wedge \mathrm{d}y + (L\mathrm{d}v + N\mathrm{d}w) \wedge \mathrm{d}z. \tag{10}$$

Proof. Assuming (10), we obtain

$$\Psi = (M\mathrm{d}\alpha + L\mathrm{d}\gamma) \wedge \mathrm{d}(\beta + c\alpha) + (L\mathrm{d}\alpha + N\mathrm{d}\gamma) \wedge (\mathrm{d}\delta + c\gamma)$$

by restoration. Then the desired congruence can be directly verified.

In order to prove the converse, we use the formula

$$\Psi = f^{\alpha\beta}\mathrm{d}\alpha \wedge \mathrm{d}\beta + \cdots + f^{\gamma\delta}\mathrm{d}\gamma \wedge \mathrm{d}\delta =$$
$$= f^{\alpha\beta}\mathrm{d}v \wedge \mathrm{d}(y - vx) + \cdots + f^{\gamma\delta}\mathrm{d}w \wedge \mathrm{d}(z - wx) \cong$$
$$\cong (f^{\alpha\gamma} - x\,(f^{\alpha\delta} + f^{\beta\gamma}) + x^2 f^{\beta\delta})\mathrm{d}u \wedge \mathrm{d}v \qquad (\mathrm{mod}\ \mathrm{d}x, \mathrm{d}y, \mathrm{d}z).$$

Assuming the congruence $\Psi \cong 0$, it follows that $f^{\alpha\gamma} = f^{\alpha\delta} + f^{\beta\gamma} = f^{\beta\delta} = 0$, hence the reduction is

$$\bar{\Psi} = \bar{f}^{\alpha\beta}\mathrm{d}\bar{\alpha} \wedge \mathrm{d}\bar{\beta} + \cdots + \bar{f}^{\gamma\delta}\mathrm{d}\bar{\gamma} \wedge \mathrm{d}\bar{\delta} =$$
$$= (\bar{f}^{\alpha\beta}\mathrm{d}v + \bar{f}^{\alpha\delta}\mathrm{d}w) \wedge \mathrm{d}y + (\bar{f}^{\alpha\delta}\mathrm{d}v + \bar{f}^{\gamma\delta}\mathrm{d}w) \wedge \mathrm{d}z$$

after some calculations. This is exactly (10).

(Alternative proof. Clearly $X\alpha = \cdots = X\delta = 0$, hence $\mathscr{L}_X\mathrm{d}\alpha = \cdots = \mathscr{L}_X\mathrm{d}\delta = 0$ and therefore $\mathscr{L}_X\Psi = 0$ where $\mathscr{L}_X$ denotes the Lie derivative. Let us denote $\Theta = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \wedge \Psi$. Then

$$\mathscr{L}_X\Theta = \mathrm{d}x \wedge (\mathrm{d}v \wedge \mathrm{d}z + \mathrm{d}y \wedge \mathrm{d}w) \wedge \Psi,$$
$$\mathscr{L}_X^2\Theta = 2\mathrm{d}x \wedge \mathrm{d}v \wedge \mathrm{d}w \wedge \Psi,$$
$$\mathscr{L}_X^3\Theta = 0.$$

The form Θ satisfies a third order linear differential equation, therefore Θ is determined if the initial values

$$\Theta_{x=c} = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \wedge \bar{\Psi},$$
$$\mathscr{L}_X \Theta_{x=c} = \mathrm{d}x \wedge (\mathrm{d}v \wedge \mathrm{d}z + \mathrm{d}y \wedge \mathrm{d}w) \wedge \bar{\Psi},$$
$$\mathscr{L}_X^2 \Theta_{x=c} = \mathrm{d}x \wedge \mathrm{d}v \wedge \mathrm{d}w \wedge \bar{\Psi},$$

at the point $x = c$ are given. In particular $\Theta = 0$ identically if and only if the initial values are all vanishing, however, this is true just for the forms $\bar{\Psi}$ of the special kind (10). Finally, the identity $\Theta = 0$ is a mere transcription of the congruence $\Psi \cong 0 \pmod{\mathrm{d}x, \mathrm{d}y, \mathrm{d}z}$ and therefore the last congruence is equivalent to (10). The alternative proof is not elementary. However, it may be applied even to the general $\mathscr{I}\mathscr{P}$ where the given extremals need not be the straight lines [7]. As for the other Lemmas, there need not be any change.) □

Final results

Let us overview our achievements and add some remarks.

By virtue of the Converse Lemma (better: the proof), if ψ is such a 1-form that the differential $\Psi = \mathrm{d}\psi$ is expressible in terms of $\alpha, \beta, \gamma, \delta$ and moreover satisfies $\Psi \cong 0 \pmod{\mathrm{d}x, \mathrm{d}y, \mathrm{d}z}$, then $\psi - \mathrm{d}g \cong 0 \pmod{\mathrm{d}x, \mathrm{d}y, \mathrm{d}z}$ for an appropriate function g and for every correction $\mathrm{d}g$, the result $\breve{\varphi} = \psi - \mathrm{d}g$ is a resolving $\mathscr{P}\mathscr{C}$ form. The Direct Lemma ensures that every resolving $\mathscr{P}\mathscr{C}$ form can be obtained in this manner (triviality: put $\psi = \breve{\varphi}$, $\mathrm{d}g = 0$).

In order to determine all resolving $\mathscr{P}\mathscr{C}$ forms $\breve{\varphi}$, we have to search for all forms $\psi = K\mathrm{d}x + L\mathrm{d}y + R\mathrm{d}z + M\mathrm{d}v + N\mathrm{d}w$ with the above two properties of the differential $\Psi = \mathrm{d}\psi$. The first property will be guaranteed if ψ is chosen as a restoration of a form $\bar{\psi} = \bar{L}\mathrm{d}y + \bar{R}\mathrm{d}z + \bar{M}\mathrm{d}v + \bar{N}\mathrm{d}w$, the second property by the use of the Crucial Lemma (namely $\bar{\Psi} = \mathrm{d}\bar{\psi}$ must be of the form (10)). In fact it is sufficient to deal with the reductions of the special kind

$$\bar{\psi} = \bar{L}\mathrm{d}y + \bar{R}\mathrm{d}z = V(y, z, v, w)\mathrm{d}y + W(y, z, v, w)\mathrm{d}z. \tag{11}$$

Indeed, let us suppose that we search for a resolving $\mathscr{P}\mathscr{C}$ form $\breve{\varphi}$ (not yet explicitly known). Clearly $\bar{\breve{\varphi}} \cong 0 \pmod{\mathrm{d}y, \mathrm{d}z}$ for the reduction and so we may put $\bar{\psi} = \bar{\breve{\varphi}}$ which is just of the form (11) (with V, W as yet unknown). Let ψ be the restoration of the form (11). Then $\Psi = \mathrm{d}\psi$ is the restoration of

$$\bar{\Psi} = \mathrm{d}\bar{\psi} = \mathrm{d}\bar{\breve{\varphi}} = \bar{\Phi} \quad (\Phi = \mathrm{d}\breve{\varphi}),$$

hence $\Psi = \Phi$ (restorations are uniquely determined) which means $\mathrm{d}\psi = \mathrm{d}\breve{\varphi}$ and $\breve{\varphi} = \psi - \mathrm{d}g$ for appropriate correction $\mathrm{d}g$ therefore we do not lose any possible resolving $\mathscr{P}\mathscr{C}$ form $\breve{\varphi}$.

Altogether, in order to obtain an arbitrary resolving $\mathscr{P}\mathscr{C}$ form $\breve{\varphi}$, it is sufficient to start with the restorations ψ of the special differential form (11). So let us choose a differential form (11). Then the restoration ψ, hence the differential $\Psi = \mathrm{d}\psi$ can be expressed in terms of $\alpha, \beta, \gamma, \delta$; the first requirement is satisfied. The congruence $\Psi \cong 0 \pmod{\mathrm{d}x, \mathrm{d}y, \mathrm{d}z}$, that is, the second requirement is ensured if the reduction is of the form (10). However clearly

$$\bar{\Psi} = \mathrm{d}\bar{\psi} = (W_y - V_z)\mathrm{d}y \wedge \mathrm{d}z + (V_v\mathrm{d}v + V_w\mathrm{d}w) \wedge \mathrm{d}y + (W_v\mathrm{d}v + W_w\mathrm{d}w) \wedge \mathrm{d}z$$

and we have the conditions $W_y = V_z$, $W_v = V_w$ for the coefficients in (11). Now we recall the $\mathscr{U}\mathscr{H}$ equation (5). If $\bar{f}$ is a solution, the functions $V = \bar{f}_v, W = \bar{f}_w$ clearly satisfy the conditions and the converse also holds true. We are done.

Our achievements can be summarized as follows.

Main Theorem. *We start with an arbitrary form*

$$\bar{\psi} = \bar{f}_v(y,z,v,w)\mathrm{d}y + \bar{f}_w(y,z,v,w)\mathrm{d}z \tag{12}$$

where $\bar{f} = \bar{f}(y,z,v,w)$ *is a solution of* (5). *Then the restoration*

$$\psi = \bar{f}_v(\cdots)\mathrm{d}(\beta+\alpha c) + \bar{f}_w(\cdots)\mathrm{d}(\delta+\gamma c) =$$

$$= \bar{f}_v(\cdots)(\mathrm{d}y - v\mathrm{d}x) + \bar{f}_w(\cdots)(\mathrm{d}z - w\mathrm{d}x) + (c-x)(\bar{f}_v(\cdots)\mathrm{d}v + \bar{f}_w(\cdots)\mathrm{d}w) \tag{13}$$

where

$$(\cdots) = (\beta+\alpha c, \delta+\gamma c, \alpha, \gamma) = (y+(c-x)v, z+(c-x)w, v, w) \tag{14}$$

needs the correction $-\mathrm{d}g(x,y,z,v,w)$ *such that*

$$g_v = (c-x)\bar{f}_v(\cdots),\ g_w = (c-x)\bar{f}_w(\cdots) \tag{15}$$

in order to obtain the resolving $\mathscr{P}\mathscr{C}$ *form*

$$\breve{\varphi} = \psi - \mathrm{d}g = -Xg\mathrm{d}x + (\bar{f}_v(\cdots) - g_y)(\mathrm{d}y - v\mathrm{d}x) + (\bar{f}_w(\cdots) - g_z)(\mathrm{d}z - w\mathrm{d}x) \tag{16}$$

whence $f = -Xg$ *is the desired resolving function.*

Equations (15) for the unknown function g can be resolved by the line integrals in the two-dimensional subspaces $x = x_0, y = y_0, z = z_0$ (with coordinates v, w) of the total five-dimensional space with the coordinates x, y, z, v, w. For instance

$$g(x_0,y_0,z_0,v,w) = g_0 + (c-x)\int\limits_0^1 (\bar{f}_v(\cdots)(v-v_0) + \bar{f}_w(\cdots)(w-w_0))\mathrm{d}t \tag{17}$$

where $x_0, y_0, z_0, v_0 + t(v-v_0), w_0 + t(w-w_0)$ is substituted for x, y, z, v, w (respectively) into the arguments (14). The integral is taken along the straight line segment with the endpoints (v_0, w_0) and (v, w). Arbitrary smooth functions of the parameters x_0, y_0, z_0 can be in principle chosen for g_0, v_0, w_0, however, one can also assume v_0, w_0 are constants without any loss of generality.

If we are not interested in $\mathscr{P}\mathscr{C}$ forms but only in the corresponding resolving function $f = -Xg$, the formula

$$f(x_0,y_0,z_0,v,w) = -(\frac{\partial}{\partial x_0} + v\frac{\partial}{\partial y_0} + w\frac{\partial}{\partial z_0})g_0(x_0,y_0,z_0) +$$

$$+(v-v_0)\int\limits_0^1 \bar{f}_v(\cdots)\mathrm{d}t + (w-w_0)\int\limits_0^1 \bar{f}_w(\cdots)\mathrm{d}t -$$

$$-(c-v_0)((v-v_0)^2\int\limits_0^1 \bar{f}_{vy}(\cdots)(1-t)\mathrm{d}t + 2(v-v_0)(w-w_0)\int\limits_0^1 \bar{f}_{vz}(\cdots)(1-t)\mathrm{d}t +$$

$$+(w-w_0)^2\int\limits_0^1 \bar f_{wz}(\cdots)(1-t)\mathrm{d}t \tag{18}$$

directly follows from (17). For convenience, we recall the arguments $(\cdots)=$ $(y_0+(c-x_0)(v_0+t(v-v_0)), z_0+(c-x_0)(w_0+t(w-w_0)), v_0+t(v-v_0), w_0+t(w-w_0))$ appearing in the integrands (17) and (18).

A few applications

First Example. We shall deal with the resolving functions f which are polynomials. For this aim, it is sufficient to determine all solutions $\bar f$ of the equation (5) that are homogeneous polynomials

$$\mathbb{P}_n=\sum P_n^{a_ib_jc_kd_l}y^{a_i}z^{b_j}v^{c_k}w^{d_l}\quad (a_i+b_j+c_k+d_l=n) \tag{19}$$

of given degree n. The equation (5) is satisfied if and only if all the equalities

$$\frac{\partial^n}{\partial y^{a_i}\partial z^{b_j}\partial v^{c_k}\partial w^{d_l}}\mathbb{P}_n=\frac{\partial^n}{\partial y^{a_i-1}\partial z^{b_j+1}\partial v^{c_k+1}\partial w^{d_l-1}}\mathbb{P}_n$$

between n-th order derivatives are valid whenever $a_i, d_l\geq 1$. However

$$\frac{\partial^n}{\partial y^{a_i}\partial z^{b_j}\partial v^{c_k}\partial w^{d_l}}\mathbb{P}_n=a_i!\,b_j!\,c_k!\,d_l!\,P_n^{a_ib_jc_kd_l}$$

and so we have the conditions

$$a_i!\,b_j!\,c_k!\,d_l!\,P_n^{a_ib_jc_kd_l}=(a_i-1)!\,(b_j+1)!\,(c_k+1)!\,(d_l-1)!\,P_n^{a_i-1,\,b_j+1,\,c_k+1,\,d_l-1}$$

for the coefficients. The products yw and zv play a significant role here which is not clearly expressed. Let us therefore rearrange (19) as follows

$$\mathbb{P}_n=\sum y^{A_i}z^{B_j}v^{C_k}w^{D_l}\mathbb{P}_{2m}^{A_iB_jC_kD_l}\quad (A_i+B_j+C_k+D_l+2m=n) \tag{20}$$

where the homogeneous polynomials

$$\mathbb{P}_{2m}^{A_iB_jC_kD_l}=\sum P_{2m}^{A_iB_jC_kD_l\,rs}(yw)^r(zv)^s\quad (r+s=m)$$

involve all factors yw and zv, i.e, we suppose $A_iD_l=B_jC_k=0$ in the sum (20). After this arrangement, we have the conditions

$$(A_i+r)!(B_j+s)!(C_k+s)!(D_l+r)!P_{2m}^{A_iB_jC_kD_l\,rs}=$$
$$=(A_i+r-1)!(B_j+s+1)!(C_k+s+1)!(D_l+r-1)!P_{2m}^{A_iB_jC_kD_l,r-1,s+1} \tag{21}$$

whenever $r\geq 1$. They can be explicitly resolved:

$$\mathbb{P}_{2m}^{A_iB_jC_kD_l}=\ \mathrm{const}\sum_{r,s}\frac{(yw)^r(zv)^s}{(A_i+r)!(D_l+r)!(B_j+s)!(C_k+s)!}\quad (r+s=m) \tag{22}$$

where const $=$ $C^{A_iB_jC_kD_l}$. *Substituting* (22) *into* (20), *we obtain all polynomial solutions* $\bar f=\mathbb{P}_n$ *of the ultrahyperbolic equation* (5).

A certain relationship to Bessel functions is worth mentioning. For the particular case $A_i = \cdots = D_l = 0$ this is simple since

$$\sum_{r=0}^{\infty} \frac{(yw\lambda)^r}{r!^2} \sum_{s=0}^{\infty} \frac{(zv\lambda)^s}{s!^2} = \sum_{r=0}^{\infty} \mathbb{P}_{2m}^{0\ldots0}\lambda^m \tag{23}$$

which reads $I_0(2(yw\lambda)^{1/2})I_0(2(zv\lambda)^{1/2}) = \sum \mathbb{P}_{2m}^{0\ldots0}\lambda^m$. As the general case is concerned, recall that A_i, B_j, C_k, D_l are natural numbers such that either A_i, D_l or B_j, C_k are vanishing. Assume $A_i = C_k = 0$. Then the obvious identity

$$\sum_{r=0}^{\infty} \frac{(yw\lambda)^r}{r!(D_l+r)!} \sum_{s=0}^{\infty} \frac{(zv\lambda)^s}{(B_j+s)!s!} = \sum \mathbb{P}_{2m}^{0Bj0D_l}\lambda^m$$

reads

$$I_{D_l}(2(yw\lambda)^{1/2})I_{B_j}(2(zv\lambda)^{1/2}) = (2(yw\lambda))^{D_l}(2(zv\lambda))^{B_j}\sum \mathbb{P}_{2m}^{0B_j0D_l}\lambda^m$$

where $I_\nu(z) = (z/2)^\nu \sum (z/2)^{2k}/(k!\Gamma(\nu+k+1))$ are the Bessel functions. We have obtained the *generating functions* for the ultrahyperbolic polynomials.

The generating functions are useful if one calculates the resolving functions f. We mention only the particular case when $A_i = B_j = C_k = D_l = 0$ here. Then, applying (17) with the functions $\bar{f} = \sum \mathbb{P}_{2m}^{0\ldots0}\lambda^m$ and choosing $g_0 = v_0 = w_0 = c = 0$ for simplicity, we obtain the result

$$f(x, y, z, v, w; \lambda) =$$

$$= (\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}) \sum_{i,j,r,s}^{\infty} (-x)^{i+j} \frac{y^{r-i}z^{s-j}v^{s+i}w^{r+j}}{i!(r-i)!j!(s-j)!r!s!} \frac{r+s}{r+s+i+j} \lambda^{r+s}$$

(sum over $i, j, s, r \geq 0$ and moreover $r \geq i, s \geq j, r+s+i+j > 0$). The coefficient of λ^m on the right-hand side is the resolving function which corresponds to the particular solution $\bar{f} = \mathbb{P}_{2m}^{0\ldots0}$ of the ultrahyperbolic equation (5).

By using the "general solution" $\bar{f} = \int\limits_{-\infty}^{\infty} \xi(t, y - vt, z - wt)\mathrm{d}t$ of $\mathscr{U}\mathscr{H}$ with the choice

$$\xi = g(t)(y - tv)^p(z - tw)^q \qquad (p, q, = 0, 1, \cdots)$$

where $g(t)$ is an arbitrary function with a compact support, we obtain the above polynomial solutions as well. However, this is in fact a misleading strategy: the proofs of the most interesting geometrical results to follow rest on quite other principles.

Second Example. Together with the primary coordinates x, y, z, v, w we shall use the alternative coordinates $x, \alpha, \beta, \gamma, \delta$. Then the straight lines (6) have the alternative equations (7). Let $\xi = \xi(\alpha, \beta, \gamma, \delta)$be a given function. In virtue of (7), ξ may be regarded as a function on the family of straight lines. We will determine all resolving functions of the form $f = F(x, \alpha, \beta, \gamma, \delta) = h(x, \xi)$. One can verify that conditions (4) for our resolving function read

$$2F_\beta = F_{\alpha x} - xF_{\beta x},\ 2F_\delta = F_{\gamma x} - xF_{\delta x} \tag{24}$$

in the alternative coordinates. Continuing in this way we obtain the system of differential equations

$$2\xi_\beta h_\xi = (\xi_\alpha - x\xi_\beta)h_{\xi x}, 2\xi_\delta = (\xi_\gamma - x\xi_\delta)h_{\xi x}$$

for the functions h, ξ where the variables can be separated as

$$\frac{\xi_\alpha}{\xi_\beta} = \frac{\xi_\gamma}{\xi_\delta} = \frac{2}{H} + x \quad \left(H = \frac{h_{\xi x}}{h_\xi}\right). \tag{25}$$

We have omitted the exceptional (and quite simple) cases when some of the functions $\xi_\beta, \xi_\delta, h_\xi, H$ vanish. It follows that $\partial(2/H + x)/\partial x = 0$ whence

$$H = -\frac{2}{x - p(\xi)},\ h_\xi = \frac{q(\xi)}{(x - p(\xi))^2},\ h = \int \frac{q(\xi)}{(x - p(\xi))^2}\mathrm{d}\xi + r(x)$$

where $p(\xi), q(\xi), r(x)$ are arbitrary functions. With this function $h,$ conditions (25) reduce to the system

$$\xi_\alpha = p(\xi)\xi_\beta,\ \xi_\gamma = p(\xi)\xi_\delta. \tag{26}$$

The solution is given by the implicit equation

$$\xi = M(\beta + p(\xi)\alpha, \delta + p(\xi)\gamma) \tag{27}$$

and we have the result.

If a function $\xi = \xi(\alpha, \beta, \gamma, \delta)$ *satisfies the implicit equation* (27) *and* $p(\xi), q(\xi),\ r(x)$ *are arbitrary, then*

$$h(x, \xi) = \int \frac{q(\xi)}{(x - p(\xi))^2}\mathrm{d}\xi + r(x) \tag{28}$$

is a resolving function. The function $r(x)$ is immaterial here and may be omitted.

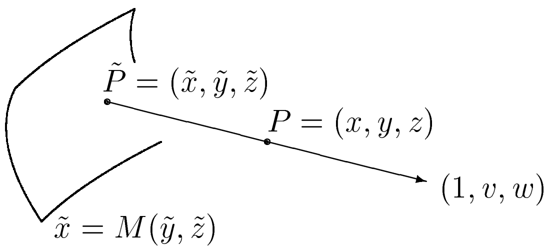
Without much loss of generality, we may assume $p(\xi) = \xi$. Then (27) simplifies as

$$\xi = M(\beta + \xi\alpha, \delta + \xi\beta). \tag{29}$$

In terms of the primary coordinates, we have the equation

$$\tilde{x} = M(y - (x - \tilde{x})v,\ z - (x - \tilde{x})w) \quad (\tilde{x} = \xi(v, y - vx, w, z - wx)) \tag{30}$$

and a nice geometrical interpretation is as follows.



$$y - \tilde{y} = (x - \tilde{x})v$$
$$z - \tilde{z} = (x - \tilde{x})w$$

Let a smooth surface $x = M(y, z)$ in the space $\mathbb{R}^3$ with the coordinates x, y, z be given, the domain of M being a certain open subset of $\mathbb{R}^2$. Given $x, y, z, v, w,$ we consider the point $\tilde{P} = (\tilde{x}, \tilde{y}, \tilde{z})$ of intersection of the straight line $y-\tilde{y} = (x-\tilde{x})v,\ z-\tilde{z} = (x-\tilde{x})w$ with this surface; then the intersection point $\tilde{P}$ is determined just by the implicit equation (30).

Smooth dependence of the intersection $\tilde{P}$ on the variables x, y, z, v, w is ensured if the inequality

$$\frac{\partial(\tilde{x} - M)}{\partial\tilde{x}} = 1 - M_y(\tilde{y}, \tilde{z})v - M_z(\tilde{y}, \tilde{z})w \neq 0 \tag{31}$$

holds true (i.e., if the line transversally intersects the surface).

Assuming $p(\xi) = \xi$, the formula (28) also greatly simplifies and leads to a huge family of resolving functions. We shall however deal only with the simplest possible case $q(\xi) = 1$, hence

$$h(x, \xi) = \int \frac{\mathrm{d}\xi}{(x-\xi)^2} = \frac{1}{x - \xi(\alpha, \beta, \gamma, \delta)} = \frac{1}{x - \tilde{x}(x, y, z, v, w)} \tag{32}$$

from now on.

Let us recall the $\mathscr{U}\mathscr{H}$ equation (5). In general, if $f = \bar{f}(y, z, v, w)$ is a solution then

$$\left(\frac{\partial^2}{\partial y \partial w} - \frac{\partial^2}{\partial z \partial v}\right) k(\bar{f}) = k''(\bar{f})(\bar{f}_y \bar{f}_w - \bar{f}_z \bar{f}_v)$$

for any function k. In particular, let us choose the function (32), hence $\bar{f} = 1/(x - \tilde{x})$ where x is regarded as a mere parameter. Then $\bar{f}_y \bar{f}_w = \bar{f}_z \bar{f}_v$ identically. (Direct verification: the formulae

$$\tilde{x}_y = \frac{M_y}{\Delta},\ \tilde{x}_z = \frac{M_z}{\Delta},\ \tilde{x}_v = -(x - \tilde{x})\frac{M_y}{\Delta},\ \tilde{x}_w = -(x - \tilde{x})\frac{M_z}{\Delta}$$

with $\Delta = 1 - vM_y - wM_z$ easily follow from (30).) So we have the result: every composed function $k(1/(x - \tilde{x}))$ is a solution of $\mathscr{U}\mathscr{H}$ (if $x = c$ is kept fixed) and therefore *every composed function of the form* $K(\tilde{x})$ *and in particular* $\tilde{x}$ *itself is a solution of* $\mathscr{U}\mathscr{H}$. According to the geometrical meaning of the function $\tilde{x} = \tilde{x}(x, y, z, v, w)$ which is the x-coordinate of the intersection point $\tilde{P}$, we have very clear insight into a huge the family of *(generalized) solutions of* $\mathscr{U}\mathscr{H}$ *with the singularities at the exceptional "focusing points"where the inequality* (31) *is not satisfied.*

By employing the latter result, one can obtain a certain counterpart of the formula (18) by applying (17) with the choice $\bar{f} = K(\tilde{x}(c, y, z, v, w))$. Clearly

$$g = (c - x) \int_0^1 K'(\tilde{x}(c, \ldots))(\tilde{x}_v(c, \ldots)(v - v_0) + \tilde{x}_w(c, \ldots)(w - w_0))\mathrm{d}t.$$

In the case $v_0 = w_0 = 0$ we may substitute

$$\tilde{x}_v v + \tilde{x}_w w = (c - \tilde{x})\frac{-M_y v - M_z w}{\Delta} = (c - \tilde{x})\frac{\Delta - 1}{\Delta}$$

and therefore the final result

$$g = (c - x) \int_0^1 K'(\tilde{x}(c, \ldots))(c - \tilde{x}(c, \ldots))\frac{\Delta - 1}{\Delta}\mathrm{d}t$$

with the arguments $(\cdots) = (y_0 + (c - x_0)tv, z + (c - x_0)tw, tv, tw)$ looks fairly good.

Third Example. We shall deal with the *complex–valued* functions $\mathbf{f} = f_1 + if_2$ of the common real variables (either of the original ones x, y, z, v, w or the alternative $x, \alpha, \beta, \gamma, \delta$). Such a function is called *resolving* if (4) is satisfied (with $\mathbf{f}$ substituted for f) and this is true if and only if the components f_1, f_2 are resolving functions in the common sense. Nontrivial results can be nevertheless obtained if one deals with the generalized length $\int \mathbf{f}\mathrm{d}x$. We wish to obtain *real values* after appropriate adaptations.

Recall that every (real) straight line is determined by the real constants $\alpha = A, \beta = B, \gamma = C, \delta = D$ in terms of the alternative coordinates $\alpha, \beta, \gamma, \delta$ and the equations of the straight line are as before

$$y = Ax + B,\ z = Cx + D$$

in terms of the original coordinates. However we will choose a *complex point* $\tilde{\mathbf{P}} = (\tilde{\mathbf{x}}, \tilde{\mathbf{y}}, \tilde{\mathbf{z}})$ on the real straight line (hence $\tilde{\mathbf{y}} = A\tilde{\mathbf{x}} + B, \tilde{\mathbf{z}} = C\tilde{\mathbf{x}} + D$). It is clearly determined by the (arbitrary) choice of a *complex-valued* function $\tilde{\mathbf{x}} = \xi(\alpha, \beta, \gamma, \delta)$ of real variables $\alpha, \beta, \gamma, \delta$.

Theorem. *Assuming Im* $\xi \neq 0$, *then the function* $\mathbf{f} = 1/(x - \xi)$ *is resolving if and only if* ξ *is a holomorphic function such that (except for some degenerate cases) a certain implicit equation* $\xi = \mathbf{M}(\beta + \xi\alpha, \delta + \xi\gamma)$ *is satisfied, where* $\mathbf{M} = \mathbf{M}(\mathbf{y}, \mathbf{z})$ *is a holomorphic function.*
One can observe that we again deal with the point of intersection $\tilde{\mathbf{P}} = (\tilde{\mathbf{x}}, \tilde{\mathbf{y}}, \tilde{\mathbf{z}})$ *where* $\tilde{\mathbf{x}} = \xi$, *with the complex surface* $\tilde{\mathbf{x}} = \mathbf{M}(\tilde{\mathbf{y}}, \tilde{\mathbf{z}})$.

Proof is a mere slight adaptation of the above reasonings. Let us denote $\mathbf{f} = \mathbf{F}(x, \alpha, \beta, \gamma, \delta)$ in terms of the alternative coordinates. Then $\mathbf{f}$ is resolving if (24) is satisfied for the (complex-valued) function $\mathbf{F}$ instead of the previous F. In particular, assuming $\mathbf{F} = 1/(x - \xi)$, we obtain the requirement (26) with $p(\xi) = \xi$ and the new, complex-valued function ξ (instead of the previous real ξ). However, $\xi = \xi_1 + i\xi_2$ and separation of the real and imaginary components provides the systems

$$\xi_1\frac{\partial\xi_1}{\partial\beta} - \xi_2\frac{\partial\xi_2}{\partial\beta} = \frac{\partial\xi_1}{\partial\alpha},\ \xi_1\frac{\partial\xi_2}{\partial\beta} + \xi_2\frac{\partial\xi_1}{\partial\beta} = \frac{\partial\xi_2}{\partial\alpha}, \tag{33}$$

$$\xi_1\frac{\partial\xi_1}{\partial\delta} - \xi_2\frac{\partial\xi_2}{\partial\delta} = \frac{\partial\xi_1}{\partial\gamma},\ \xi_1\frac{\partial\xi_2}{\partial\delta} + \xi_2\frac{\partial\xi_1}{\partial\delta} = \frac{\partial\xi_2}{\partial\gamma}. \tag{34}$$

Both (33) and (34) are elliptical systems (see below) if $\xi_2 \neq 0$, therefore any classical solution ξ_1, ξ_2 is in reality an analytical function of the variables $\alpha, \beta, \gamma, \delta$ (which may be therefore extended to complex variables). Then equations (26) ensure the existence of certain nontrivial identity $\mathbf{M}(\xi, \beta + \xi\alpha, \delta + \xi\gamma) = 0$ where $\mathbf{M}$ is appropriate analytic function. The proof is done.

Note to the ellipticity. We mention the particular case of the quasilinear system $\sum A^j_{ik}\partial\xi^i/\partial y^k = B^j \quad (i, j = 1, \ldots, m; k = 1, \ldots, n)$ with the holomorphic coefficients A^j_{ik}, B^j. The system is called *elliptical* if $\det(\sum A^j_{ik}t_k) \neq 0$ for all real nonvanishing vectors $(t_1, \dots, t_n)$. In our particular case, i.e., for the system (33), we have denoted $y^1 = \alpha, y^2 = \beta$ $(n = m = 2)$ and then

$$\sum A^1_{1k}t_k = \sum A^2_{2k}t_k = \xi_1 t_2 - t_1, \sum A^2_{1k}t_k = -\sum A^1_{2k}t_k = \xi_2 t_2$$

whence $\det(\sum A^j_{ik}t_k) = (\xi_1 t_2 - t_1)^2 + (\xi_2 t_2)^2 \neq 0$ if $\xi_2 \neq 0$ and $(t_1, t_2) \neq (0, 0)$.

Passing to the example proper, one can see that the distances (lengths of the segments along a fixed straight line where $\xi = \text{const}$) are given by the complex-valued and multi-valued logarithm

$$\int \mathbf{f}\mathrm{d}x = \int \frac{\mathrm{d}x}{x - \xi} = \mathrm{Ln}(x - \xi) = \ln|x - \xi| + i\mathrm{Arg}(x - \xi).$$

It is desirable to introduce the real-valued components

$$\mathrm{Re}\int \mathbf{f}\mathrm{d}x = \frac{1}{2}\int \left(\frac{1}{x - \xi_1 - i\xi_2} + \frac{1}{x - \xi_1 + i\xi_2}\right)\mathrm{d}x = \int \frac{x - \xi_1}{|x - \xi|^2}\mathrm{d}x = \ln|x - \xi|,$$

$$\mathrm{Im}\int \mathbf{f}\mathrm{d}x = \frac{1}{2i}\int \left(\frac{1}{x - \xi_1 - i\xi_2} - \frac{1}{x - \xi_1 + i\xi_2}\right)\mathrm{d}x = \int \frac{\xi_2}{|x - \xi|^2}\mathrm{d}x = \mathrm{Arg}(x - \xi).$$

Both are rather interesting. For instance, if one takes the imaginary sphere $x^2 + y^2 + z^2 = -1$ for the surface $\tilde{\mathbf{x}} = \mathbf{M}(\tilde{\mathbf{y}}, \tilde{\mathbf{z}})$, the resolving functions

$$\mathrm{Re}\,\mathbf{f} = ((y - vx)v + (z - wx)w)/(1 + x^2 + y^2 + z^2),$$

$$\mathrm{Im}\,\mathbf{f} = (1 + v^2 + w^2 + ((y - vz)^2 + (z - wx)^2 - (vz + wy)^2)/(1 + x^2 + y^2 + z^2)$$

appear. The latter one provides the distance in the non-Euclidean elliptical geometry.

One can observe that(in contrast to the hyperbolic case) the distances are defined on total lines and for the component $\mathrm{Im}\,\mathbf{f}$ also at the points of infinity.

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