Abstract. Left-censored data with one or more detection limits occur frequently in many application areas. In this paper, the computational procedure for calculation of maximum likelihood estimates of the parameters for type I multiply left-censored data from underlying Weibull distribution is suggested and used considering various numbers of detection limits. The expected Fisher information matrix is analytically determined and its performance is compared with sample (observed) Fisher information matrix using simulations. Simulations are focused primarily on the properties of estimators for small sample sizes. Real data illustration is included.

Keywords: Fisher information matrix, maximum likelihood estimator, multiply left-censored sample, type I censoring, Weibull distribution.

INTRODUCTION

Left-censored data occur frequently in analyses of environmental or chemical data when for a given experimental unit the attribute being measured is not present above the detection limit (DL) \( d \). In case the DL is fixed, we talk about type I (time) censoring. The number of experimental units under DL \( d \), i.e. the number of left-censored experimental units, is a random variable. In case the number of censored units is fixed, we talk about type II (failure) censoring. This paper will be focused on the type I left censoring which can be described as follows. Assume that \( n \) experimental units are observed, and \( X_1, \ldots, X_n \) are independent and identically distributed random variables. As a result of the experiment we get only measurements \( X_i \) and the number of observations under the detection limit \( d \). Most authors usually focus on right censoring. However, type I left-censored data are very frequent in real applications, and a brief overview of application areas can be found in [1]. In addition to that, multiply left-censored data often occur, mostly in environmental applications. They arise in case there are more, say \( k \), detection limits \( d_1 < d_2 \ldots < d_k \), \( k > 1 \), and only observations above the highest detection limit \( d_k \) and the numbers of observations between detection limits \( d_{i-1} \) and \( d_i \), \( i = 1, \ldots, k, d_0 = 0 \), are available [2–6]. In case \( k = 2 \), we talk about doubly left-censored data [7].

Various censoring techniques and statistical analyses of censored data are described in more details in many monographs [4, 8]. In order to estimate unknown parameters of particular distributions, the maximum likelihood (ML) approach is usually applied [9–11]. The ML method for estimating unknown parameters of the left-censored normal and log-normal distributions is well developed [5, 12, 13]. However, the interest in statistical analyses of censored data with asymmetric distributions which cannot be easily transformed into symmetric ones (e.g. normal distribution) has increased in many application areas recently. An overview of such distributions together with description of their properties can be found, for example, in [14]. Due to the monotonicity property of the hazard function and various shapes of the probability density function, Weibull distribution is very popular. Various approaches to inference on the parameters of the Weibull distribution can be found, for example, in [15–18]. A demonstration of parameters estimates properties based on numerical study for censored and uncensored Weibull...
distribution can be found in [19]. Moreover, the generalized exponential (GE) distribution [20], which is similar to Weibull distribution, has been of great interest lately. ML estimates of unknown parameters in case of censored GE distribution [1, 21, 22] were studied as well. In addition to that, some authors deal with comparison of censored Weibull and GE distributions with each other ([23] and references inside) and also with gamma and log-normal distributions [1, 24].

An advantage of the ML approach lies (under certain regularity conditions) in good asymptotic properties of the obtained estimates. The speed of convergence of estimators distribution to normal distribution and the asymptotic bias can be quite easily assessed in practical situations using simulations. The properties of ML estimates in case of singly, doubly and triply left-censored exponential distributions (a special case of the GE distribution) based on simulations can be found in [6]. On the other hand, ML approach often requires special numerical algorithms for solving likelihood equations [25]. In case of Weibull distribution, a simple fixed point type algorithm for calculating the ML estimates of unknown parameters was proposed in [19]. A proper optimization procedures often based on EM algorithm [26, 27] can also be used. This approach is rather beneficial when calculating ML estimates of unknown parameters. However, estimation of parameters variances using the Fisher information matrix (FIM) is computationally demanding. This inconvenience can be overcome using the exact formulas for calculating the (expected) FIM proposed in this paper.

This contribution is organized as follows: in Section 1, a derivation of a computational procedure for determination of ML estimates of parameters of type I multiply left-censored Weibull distribution is described. In Section 2, the expected FIM is analytically determined. In Section 3, bias of the estimates of Weibull distribution parameters considering various DLs (censoring schemes) and censoring multiplicity \( k = 1, 2, 3 \) (single, double and triple censoring) is described. Furthermore, the sample (observed) FIM and the expected FIM are compared and the bias of the FIM estimates is analyzed using simulations. Simulations focus primarily on estimator properties for small sample sizes and also on comparing the estimates from singly \((k = 1)\), doubly \((k = 2)\) and triply \((k = 3)\) left-censored samples. The paper was motivated by the need to process real environmental data described in [28]. The application of the derived method is presented in the last section.

1. ML ESTIMATION

Let \( X_1, \ldots, X_n \) be a random sample from Weibull distribution with scale parameter \( \lambda > 0 \), shape parameter \( \tau > 0 \), cumulative distribution function (cdf)

\[
F(x, \lambda, \tau) = \begin{cases} 
1 - \exp \left( -\left( \frac{x}{\lambda} \right)^\tau \right) & \text{for } x \geq 0, \\
0 & \text{for } x < 0,
\end{cases}
\]

(1)

and skewness

\[
\gamma(\lambda, \tau) = \frac{2 \Gamma^3 \left( \frac{1 + \frac{1}{\tau}}{\tau} \right) - 3 \Gamma \left( \frac{1 + \frac{2}{\tau}}{\tau} \right) \Gamma \left( \frac{1 + \frac{2}{\tau}}{\tau} \right) + \Gamma \left( \frac{1 + \frac{3}{\tau}}{\tau} \right)}{\left[ \Gamma \left( \frac{1 + \frac{2}{\tau}}{\tau} \right) - \Gamma^2 \left( \frac{1 + \frac{1}{\tau}}{\tau} \right) \right]^{3/2}}.
\]

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Furthermore, let \( X_{(1)}, \ldots, X_{(n)} \) be the ordered sample of \( X_1, \ldots, X_n \) which is type I multiply left-censored with detection limits \( d_1, \ldots, d_k \) and we put \( d_0 = 0 \). Moreover, \( N_i \) is the number of observations in the interval \((d_{i-1}, d_i)\) and \( N_0 \) is the number of uncensored observations \( X_{(n-N_0+1)}, \ldots, X_{(n)} \). Considering the type I censoring, the random vector \( (N_0, N_1, \ldots, N_k) \) has multinomial distribution \( \text{Mu}_{k+1}(n, \theta_0, \theta_1, \ldots, \theta_k) \), where \( \theta_i = F(d_i, \lambda, \tau) - F(d_{i-1}, \lambda, \tau), i = 1, \ldots, k \), \( \theta_0 = 1 - F(d_k, \lambda, \tau) \) and \( n = N_0 + N_1 + \ldots + N_k \). Particular marginal frequencies \( N_i \) have binomial distribution \( \text{Bi}(n, \theta_i) \), \( i = 0, 1, \ldots, k \).

Using the results from [4], the log-likelihood function of the censored sample can be written in the form of

\[
\ell(\lambda, \tau, N_0, \ldots, N_k, X_{(n-N_0+1)}, \ldots, X_{(n)}) = \log \left( \frac{n!}{N_1! \ldots N_k!} \right) + \sum_{i=n-N_0+1}^{n} \log \left( f(X_{(i)}) \right) + \sum_{i=1}^{k} N_i \log \left[ F(d_i, \lambda, \tau) - F(d_{i-1}, \lambda, \tau) \right],
\]

and we put \( \sum_{i=n-N_0+1}^{n} \log \left( f(X_{(i)}) \right) = 0 \) for \( N_0 = 0 \).

The usual approach to estimation of parameters \( \lambda, \tau \) is to derive likelihood equations and solve them numerically using, for example, the Newton–Raphson method or other methods for searching of extremes (see e.g. [29]) can be used. However, it was found out that in case the shape parameter value is low, there are numerical difficulties with obtaining the solution. For that very reason, ML estimates \( \hat{\lambda} \) and \( \hat{\tau} \) were obtained by maximizing the log-likelihood function (3) using the Nelder–Mead simplex algorithm [30]. When using this type of algorithm, it is necessary to select initial values of the parameters that need to be estimated. Starting values of the algorithm were selected using the moment estimator of parameters of the Weibull distribution based on samples in which the censored observations were replaced by a constant lying between the detection limits.

2. FISHER INFORMATION MATRIX

In this section, the expected FIM for type I multiply left-censored samples will be derived. According to [9], the sample FIM can be defined (under the regularity conditions) using formula

\[
\tilde{J}_n = \begin{bmatrix}
J_{11} & \hat{J}_{12} \\
\hat{J}_{21} & \hat{J}_{22}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial^2 \ell}{\partial \lambda^2} & \frac{\partial^2 \ell}{\partial \lambda \partial \tau} \\
\frac{\partial^2 \ell}{\partial \tau \partial \lambda} & \frac{\partial^2 \ell}{\partial \tau^2}
\end{bmatrix}.
\]

The sample FIM \( \tilde{J}_n \) is an unbiased estimator of the expected FIM \( J_n \) and \( \tilde{J}_n(\lambda, \tau) \rightarrow J_n(\lambda, \tau) \) in probability for \( n \rightarrow \infty \). On that account, in many applications when the exact determination of the expected FIM is complicated, the sample FIM is used instead [3, 31]. One major disadvantage of this approach is the rather extensive variability of the sample FIM. Many authors prefer another approach like bootstrap or Bayesian methods [19] for description of variability of parameters estimates. The expected FIM can be used for statistical inference and more precise description of the asymptotic variability of obtained estimates. On that account, our attention will now be paid to determination of the expected FIM. Similar problems were solved, for example, in [6] for exponential distribution, in [1] for GE distribution or in [22] and [23] using the hazard function according to [32].
The expected FIM can be calculated using formula

\[
J_n = \begin{bmatrix}
J_{11} & J_{12} \\
J_{21} & J_{22}
\end{bmatrix} = EJ_n = \begin{bmatrix}
EJ_{11} & EJ_{12} \\
EJ_{21} & EJ_{22}
\end{bmatrix},
\]

where

\[
J_{11} = \frac{\tau^2 \left\{ \sum_{i=1}^{k} d_i^\tau \exp \left[ -\frac{(d_{i-1})^\tau}{\alpha} \right] - d_i^\tau \exp \left[ -\frac{(d_i)^\tau}{\alpha} \right] \right\}^2}{\chi_{2r+2}}
\]

\[
J_{12} = J_{21} = -n \sum_{i=1}^{k} \left\{ \frac{d_i^\tau \ln \left( \frac{d_{i-1}}{\alpha} \right) \exp \left[ -\frac{(d_{i-1})^\tau}{\alpha} \right] - d_i^\tau \ln \left( \frac{d_i}{\alpha} \right) \exp \left[ -\frac{(d_i)^\tau}{\alpha} \right] \right\}
\]

\[
+ \frac{(\ln \lambda)^2}{\lambda^\tau} E_1 - \frac{2 \ln \lambda}{\lambda^\tau} E_2 + \frac{1}{\lambda^\tau} E_3,
\]

\[
J_{22} = \frac{n^2 \left\{ \sum_{i=1}^{k} d_i^\tau \exp \left[ -\frac{(d_i)^\tau}{\alpha} \right] - d_i^\tau \exp \left[ -\frac{(d_i)^\tau}{\alpha} \right] \right\}^2}{\chi_{2r+2}}
\]

\[
- n \frac{\sum_{i=1}^{k} \left\{ \frac{d_i^\tau \ln \left( \frac{d_{i-1}}{\alpha} \right) \exp \left[ -\frac{(d_{i-1})^\tau}{\alpha} \right] - d_i^\tau \ln \left( \frac{d_i}{\alpha} \right) \exp \left[ -\frac{(d_i)^\tau}{\alpha} \right] \right\}}{\chi_{2r+2}}
\]

\[
+ \frac{1}{\lambda^\tau} \sum_{i=1}^{k} \left\{ \frac{d_i^\tau \ln \left( \frac{d_{i-1}}{\alpha} \right) \exp \left[ -\frac{(d_{i-1})^\tau}{\alpha} \right] - d_i^\tau \ln \left( \frac{d_i}{\alpha} \right) \exp \left[ -\frac{(d_i)^\tau}{\alpha} \right] \right\}
\]

\[
+ \frac{1}{\lambda^\tau} \sum_{i=1}^{k} \left\{ d_i^\tau \ln \left( \frac{d_{i-1}}{\alpha} \right) \exp \left[ -\frac{(d_{i-1})^\tau}{\alpha} \right] - d_i^\tau \ln \left( \frac{d_i}{\alpha} \right) \exp \left[ -\frac{(d_i)^\tau}{\alpha} \right] \right\}
\]

\[
+ \frac{n}{\lambda^2} \exp \left[ -\left( \frac{d_k}{\alpha} \right)^\tau \right] + \frac{n}{\lambda^2} \exp \left[ -\left( \frac{d_k}{\alpha} \right)^\tau \right]
\]

The explicit expressions for \( E_1, E_2, E_3 \) and details of the derivation can be found in the Appendix 1.
The expected FIM can be used for statistical inference of parameters \( \lambda \) and \( \tau \). Considering the asymptotic properties of the ML estimator \( \hat{\lambda} \) (\( \hat{\tau} \) respectively), according to \cite{10}, \( \sqrt{n}(\hat{\lambda} - \lambda) \) (\( \sqrt{n}(\hat{\tau} - \tau) \) respectively) has asymptotically normal distribution \( \mathcal{N}(0, \sigma_\lambda^2) \) (\( \mathcal{N}(0, \sigma_\tau^2) \) respectively), where \( \sigma_\lambda^2 = J_{11}^{-1} \) (\( \sigma_\tau^2 = J_{22}^{-1} \) respectively). The properties of estimators \( \hat{\lambda} \), \( \hat{\tau} \) considering various sample sizes \( n \), various number of detection limits \( k \) and various censoring schemes will be analyzed using simulations in the next section.

3. ESTIMATORS BEHAVIOR BASED ON SIMULATIONS

The estimates of parameters and parametric functions derived in the previous section will be analyzed and compared using simulations. Firstly, 20,000 Type I singly, doubly and triply left-censored random samples with size \( n = 10, 20, 30, 50, 100 \) from Weibull distribution were generated. Since \( \lambda \) is the scale parameter, and the maximum likelihood estimators are scale invariant, we take \( \lambda = 1 \) without loss of generality. In order to describe various shapes of Weibull distribution, we take \( \tau = 0.5, 1.5, 3 \), which corresponds to skewnesses \( \gamma = 6.62, 1.07, 0.17 \). Limits of detection \( d_i \), \( i = 1, \ldots, k \), \( k = 1, 2, 3 \), were chosen as quantiles of the Weibull distribution using equations \( q_i = F(d_i, \lambda, \tau) \), where \( q_i \) are given in Table 1. Particular censoring schemes are denoted as \( c_1, \ldots, c_9 \), and correspond to various quantiles which determine detection limits for \( k = 1, 2, 3 \). Thus, for example, the \( q_1 \) given in column “Double” in Table 1 denotes the proportion of doubly censored observations, and describes the given censoring scheme. The censoring scheme \( c_1 \) represents the smallest proportion (10%) of censored data, and the censoring scheme \( c_9 \) represents the largest proportion (90%) of censored data in case of singly, doubly and triply censored samples. The large proportion of censored data corresponds to real data which will be analyzed later on.

Next, ML estimates of parameters \( \lambda, \tau \) from 20,000 samples were calculated by maximization of \( (3) \), and their mean values \( \tilde{\lambda}, \tilde{\tau} \) were determined. It was found out (see Fig. 1) that the estimate of parameter \( \lambda \) has lower bias for higher values of parameter \( \tau \) (i.e. for lower skewness) considering various censoring schemes from Table 1. In all the following tables, only results for censoring schemes \( c_1, c_3, c_5, c_7, c_9 \) are presented. Moreover, estimates \( \tilde{\tau} \) are similar bias-wise for various values of \( \tau \) (not shown in figures). It can be seen from Table 2 (only results for single and double censoring

### Table 1

<table>
<thead>
<tr>
<th>Censoring</th>
<th>Quantiles for determination of DL values considering single, double and triple censoring and various censoring schemes</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Single</td>
</tr>
<tr>
<td></td>
<td>( q_1 )</td>
</tr>
<tr>
<td>( c_1 )</td>
<td>0.10</td>
</tr>
<tr>
<td>( c_2 )</td>
<td>0.20</td>
</tr>
<tr>
<td>( c_3 )</td>
<td>0.30</td>
</tr>
<tr>
<td>( c_4 )</td>
<td>0.40</td>
</tr>
<tr>
<td>( c_5 )</td>
<td>0.50</td>
</tr>
<tr>
<td>( c_6 )</td>
<td>0.60</td>
</tr>
<tr>
<td>( c_7 )</td>
<td>0.70</td>
</tr>
<tr>
<td>( c_8 )</td>
<td>0.80</td>
</tr>
<tr>
<td>( c_9 )</td>
<td>0.90</td>
</tr>
</tbody>
</table>
are shown), where the average ML estimates $\tilde{\lambda}$, $\tilde{\tau}$ and their average mean square errors (MSE) can be found, that the ML estimate $\hat{\lambda}$ is rather satisfying until the censoring scheme $c_7$ even when the sample size is small ($n = 10$). For $n > 10$, the bias of $\hat{\lambda}$ is very small, and from the practical point of view negligible for the censoring scheme $c_7$ and lower. The effect of multiplicity of the censoring on the estimation of parameter $\lambda$ is noticeable only for higher detection limits depending on the sample size. For $n \geq 30$, the differences among the single, double and triple censoring are almost negligible until scheme $c_8$ when, in accordance with expectations, the highest bias of estimate is present in case of single censoring. The ML estimate $\tilde{\tau}$ is significantly biased even when the censoring is low, and sample size $n = 100$ (see Table 2).

Fig. 1. The average ML estimates of parameter $\hat{\lambda}$ considering various values of $\tau$ and single (index S), double (index D), triple (index T) censoring; $\tau = 0.5$ (a), $\tau = 1.5$ (b), $\tau = 3$ (c); sample size $n = 30$ estimates averaged over 20,000 repetitions. These estimates together with the corresponding empirical sample variances $S^2(\hat{\lambda}) = ns^2(\hat{\lambda})$, $S^2(\hat{\tau}) = ns^2(\hat{\tau})$ will be further compared with the asymptotic variances considering various sample sizes $n$. Due to a rather large number of samples, the estimators $S^2(\hat{\lambda})$, $S^2(\hat{\tau})$ allow us to assess the bias of estimators $\sigma^2(\hat{\lambda})$, $\sigma^2(\hat{\tau})$, $\sigma^2(\hat{\lambda})$, $\sigma^2(\hat{\tau})$, and the bias of asymptotic variances $\sigma^2(\hat{\lambda})$, $\sigma^2(\hat{\tau})$ from the true (simulation-based) variances $S^2(\hat{\lambda})$, $S^2(\hat{\tau})$ of the estimates.
It can be seen from Fig. 2 that behavior of estimates $\hat{\sigma}^2(\lambda)$, $\tilde{\sigma}^2(\lambda)$ is significantly influenced by the value of parameter $\tau$. In case $\tau < 1$, i.e. the skewness of the sample is high (> 2), both estimates are higher than the asymptotic variance $\sigma^2(\lambda)$. When $\tau > 1$, i.e. the skewness of the sample is low (< 2), both estimates are lower than the asymptotic variance $\sigma^2(\lambda)$.

The comparison of above mentioned characteristics of variance (see Table 3 in case of double censoring) shows that the anticipated bias of estimator $\sigma^2(\lambda)$ is substantial for small sample sizes. Furthermore, the estimator $S^2(\lambda)$ is of lower (respectively higher) values than asymptotic variance $\sigma^2(\lambda)$ for $\tau > 1$ ($\tau < 1$ respectively). All of the estimators of variance almost coincide for $\tau > 1$ and the sample size $n \geq 100$. Furthermore, the asymptotic variance $\sigma^2(\lambda)$ (obtained from the expected FIM) was analyzed considering various sample sizes and censoring schemes. With the exception of schemes $c_1 - c_3$, variability of the estimators is, as expected, the lowest for triple censoring for an arbitrary sample size.

### Table 2

<table>
<thead>
<tr>
<th>Censoring</th>
<th>$\lambda = 1$, $n = 10$</th>
<th>$\lambda = 1$, $n = 30$</th>
<th>$\lambda = 1$, $n = 100$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>S</td>
<td>D</td>
<td>S</td>
</tr>
<tr>
<td>$c_1$</td>
<td>1.00 (0.050)</td>
<td>1.00 (0.050)</td>
<td>1.00 (0.017)</td>
</tr>
<tr>
<td>$c_2$</td>
<td>0.99 (0.052)</td>
<td>1.00 (0.050)</td>
<td>1.00 (0.017)</td>
</tr>
<tr>
<td>$c_3$</td>
<td>0.99 (0.058)</td>
<td>0.99 (0.051)</td>
<td>1.00 (0.020)</td>
</tr>
<tr>
<td>$c_4$</td>
<td>1.03 (0.071)</td>
<td>1.00 (0.053)</td>
<td>1.00 (0.029)</td>
</tr>
<tr>
<td>$c_5$</td>
<td>1.33 (0.214)</td>
<td>1.07 (0.066)</td>
<td>1.10 (0.112)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Censoring</th>
<th>$\tau = 1.5$, $n = 10$</th>
<th>$\tau = 1.5$, $n = 30$</th>
<th>$\tau = 1.5$, $n = 100$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>S</td>
<td>D</td>
<td>S</td>
</tr>
<tr>
<td>$c_1$</td>
<td>1.75 (0.355)</td>
<td>1.75 (0.350)</td>
<td>1.57 (0.067)</td>
</tr>
<tr>
<td>$c_2$</td>
<td>1.78 (0.446)</td>
<td>1.76 (0.382)</td>
<td>1.58 (0.081)</td>
</tr>
<tr>
<td>$c_3$</td>
<td>1.96 (0.588)</td>
<td>1.77 (0.408)</td>
<td>1.60 (0.119)</td>
</tr>
<tr>
<td>$c_4$</td>
<td>2.76 (1.169)</td>
<td>1.77 (0.442)</td>
<td>1.67 (0.273)</td>
</tr>
<tr>
<td>$c_5$</td>
<td>5.69 (5.589)</td>
<td>1.62 (0.418)</td>
<td>2.84 (8.011)</td>
</tr>
</tbody>
</table>

Fig. 2. Comparison of the estimates of variance $S^2(\hat{\lambda})$, $\sigma^2(\hat{\lambda})$, $\tilde{\sigma}^2(\hat{\lambda})$ and the asymptotic variance $\sigma^2(\lambda)$ considering double censoring and sample size $n = 30$; $\tau = 0.5$ (a), $\tau = 1.5$ (b); logarithmic scale on the y-axis.
The behavior of \( \hat{\tau} \) variance estimators is similar for various values of \( \tau \). The comparison of the characteristics of variance (see Table 4 in case of double censoring) shows that the anticipated bias of estimator \( \hat{\sigma}^2(\hat{\tau}) \) is substantial for small sample sizes. Furthermore, the estimator \( \hat{S}^2(\hat{\tau}) \) is of higher values than asymptotic variance \( \sigma^2(\tau) \) for all sample sizes and censoring schemes. Furthermore, the asymptotic variance \( \sigma^2(\tau) \) (obtained from the expected FIM) was analyzed considering various sample sizes and censoring schemes. With the exception of schemes \( c_1-c_2 \), the variability of the estimators is, as expected, the lowest for triple censoring for arbitrary sample size.

Finally, using the variance estimators \( \hat{\sigma}^2(\hat{\lambda}), \sigma^2(\hat{\tau}), \hat{\sigma}^2(\hat{\lambda}), \hat{\sigma}^2(\hat{\tau}) \), the lower and the upper confidence limits of the estimate of parameters \( \lambda \) and \( \tau \) can be obtained. The coverage probability of 95% confidence interval, computed as the proportion of the number of times, out of 20,000 replications, the estimated 95% confidence interval contains the true parameter value, is calculated. In general, the coverage probability of \( \lambda \) is better with higher values of \( \tau \), because the estimator of parameter \( \lambda \) performs better for higher values of parameter \( \tau \). When \( \tau < 1 \), i.e., the skewness of the sample is high (>2), the estimator based on expected FIM (5) performs better than the estimator based on sample FIM (4) for all censoring schemes, especially for small sample sizes.

### Table 3

<table>
<thead>
<tr>
<th>Censoring</th>
<th>Comparison of the estimates of variance ( S^2(\hat{\lambda}), \sigma^2(\hat{\lambda}), \hat{\sigma}^2(\hat{\lambda}) ) and the asymptotic variance ( \sigma^2(\hat{\lambda}) ) considering double censoring and sample size ( n )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \tau = 0.5, ; n = 10 )</td>
</tr>
<tr>
<td></td>
<td>( S^2(\hat{\lambda}) )</td>
</tr>
<tr>
<td>( c_1 )</td>
<td>6.2696</td>
</tr>
<tr>
<td>( c_3 )</td>
<td>6.0369</td>
</tr>
<tr>
<td>( c_7 )</td>
<td>6.3032</td>
</tr>
<tr>
<td>( c_9 )</td>
<td>8.7636</td>
</tr>
</tbody>
</table>

### Table 4

<table>
<thead>
<tr>
<th>Censoring</th>
<th>Comparison of the estimates of variance ( S^2(\hat{\tau}), \sigma^2(\hat{\tau}), \hat{\sigma}^2(\hat{\tau}) ) and the asymptotic variance ( \sigma^2(\hat{\tau}) ) considering double censoring and sample size ( n; \tau = 1.5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( n = 10 )</td>
</tr>
<tr>
<td></td>
<td>( S^2(\hat{\tau}) )</td>
</tr>
<tr>
<td>( c_1 )</td>
<td>2.8844</td>
</tr>
<tr>
<td>( c_3 )</td>
<td>3.1440</td>
</tr>
<tr>
<td>( c_5 )</td>
<td>3.3591</td>
</tr>
<tr>
<td>( c_7 )</td>
<td>3.6670</td>
</tr>
<tr>
<td>( c_9 )</td>
<td>4.0300</td>
</tr>
</tbody>
</table>
**Fig. 3.** Coverage probabilities for parameter $\hat{\lambda}$ considering various estimates of variance and double censoring: $n = 10$ (a), $n = 30$ (b), $n = 100$ (c); $\tau = 0.5$

**Table 5**

<table>
<thead>
<tr>
<th>Censoring</th>
<th>Coverage probabilities (CP) for parameter $\hat{\lambda}$ based on expected FIM (5) considering single (S), double (D) and triple (T) censoring and sample size $n$; $\tau = 1.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$n = 10$, S</td>
</tr>
<tr>
<td>c1</td>
<td>LCL</td>
</tr>
<tr>
<td>c3</td>
<td>0.593</td>
</tr>
<tr>
<td>c5</td>
<td>0.580</td>
</tr>
<tr>
<td>c7</td>
<td>0.562</td>
</tr>
<tr>
<td>c9</td>
<td>0.729</td>
</tr>
<tr>
<td></td>
<td>$n = 30$, S</td>
</tr>
<tr>
<td>c1</td>
<td>LCL</td>
</tr>
<tr>
<td>c3</td>
<td>0.754</td>
</tr>
<tr>
<td>c5</td>
<td>0.745</td>
</tr>
<tr>
<td>c7</td>
<td>0.730</td>
</tr>
<tr>
<td>c9</td>
<td>0.570</td>
</tr>
<tr>
<td></td>
<td>$n = 100$, S</td>
</tr>
<tr>
<td>c1</td>
<td>LCL</td>
</tr>
<tr>
<td>c3</td>
<td>0.863</td>
</tr>
<tr>
<td>c5</td>
<td>0.859</td>
</tr>
<tr>
<td>c7</td>
<td>0.850</td>
</tr>
<tr>
<td>c9</td>
<td>0.819</td>
</tr>
<tr>
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</tr>
</tbody>
</table>
For $c_1$, coverage probabilities of both estimators almost coincide. The results showed that the coverage probability is very similar considering double and triple censoring for all sample sizes and various values of $c_1$ (see Table 5 for $c_1 = 1.5$; the average lower (LCL) and upper (UCL) confidence limits are included). The coverage probability gets higher with a higher censoring scheme for small sample sizes ($n < 50$). In case of single censoring, the behavior is similar until censoring scheme $c_7$.

The coverage probability of $c_7$ is similar considering double and triple censoring for all values of $c_7$ (see Table 6 for $c_7 = 1.5$; the average lower (LCL) and upper (UCL) confidence limits are included), because all the estimates of $c_7$ are similar bias-wise. The coverage probabilities are quite close to the prescribed significance level for both estimators (based on the expected and the sample FIM) and practically coincide for $n > 50$.

4. REAL DATA EXAMPLE

Statistical methods derived in previous sections were used in the analysis of the worldwide commonly used synthetic musk compounds (e.g. galaxolide, musk ketone, musk xylene, etc.) present in fish muscle (see [28] for more details). Here we show one example, specifically modeling of galaxolide concentration using doubly left-censored Weibull distribution (see Fig. 4). The real sample consists of 30 fish from the carp family (*Leuciscus cephalus*). Fish tissue samples were analyzed, and, among others, polycyclic musk compound called galaxolide was explored with $N_0 = 4$, $N_1 = 3$, $N_2 = 23$, $d_1 = 8.9488 \mu g / kg$, $d_2 = 29.8294 \mu g / kg$ and $X_{(27)} = 30.3630 \mu g / kg$, $X_{(28)} = 39.2597 \mu g / kg$, $X_{(29)} = 48.9161 \mu g / kg$, $X_{(30)} = 79.7756 \mu g / kg$. The level of censoring is high, and approximately corre-
sponds to censoring scheme $c_9$. The unknown parameters of the Weibull distribution were estimated, and particular 95% confidence intervals were calculated using estimators of variance based on the expected and the sample FIM. Specifically, the ML estimate $\hat{\lambda} = 23.98$ ($\hat{\tau} = 1.61$ respectively) with confidence interval $\lambda \in (17.91, 30.05)$ ($\tau \in (1.10, 2.12)$ respectively) with estimate of variance based on the expected FIM, and $\lambda \in (17.82, 30.13)$ ($\tau \in (1.17, 2.06)$ respectively) with estimate of variance based on the sample FIM.

From the practical point of view, it would be more interesting to have confidence limits for the mean concentration. In order to obtain them, results from the previous section can be utilized, and the delta method can be used [28, 33].

CONCLUSIONS

This paper dealt with type I multiply left-censored Weibull distribution. It was described how to estimate parameters of censored Weibull distribution using the method of maximum likelihood, and the expected FIM was analytically determined. Moreover, simulation results showed what bias of estimators $\hat{\lambda}$, $\hat{\tau}$ and $\sigma^2(\hat{\lambda})$, $\sigma^2(\hat{\tau})$ can be expected considering various sample sizes, various censoring schemes, and degree of censoring. Furthermore, it is shown what change in the variance estimate can be expected when the sample FIM is used instead of the expected FIM. Using different variance estimators, the lower and the upper confidence limits of the parameters estimates were obtained and the coverage probabilities of 95% confidence intervals were calculated. It was shown that the coverage probability of $\hat{\lambda}$ is better with higher values of $\tau$, and when $\tau < 1$, i.e. the skewness of the sample is high ($>2$), the estimator based on the expected FIM performs better than the estimator based on the sample FIM for all censoring schemes, especially for small sample sizes.

The problem of measured values found below the detection limits is common in many application areas. The methods derived in this paper can be favorably used instead of various ad hoc methods (e.g. replacing censored values with a constant) whenever dealing with type I multiply left-censored data from Weibull distribution. All the procedures used were implemented in the Matlab environment (version R2015a), and are available from the first author upon request.

APPENDIX 1. DERIVATION OF THE EXPECTED FIM

The expected FIM is calculated using formula

$$J_n = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \begin{bmatrix} -E \frac{\partial^2 l}{\partial \lambda^2} - E \frac{\partial^1 l}{\partial \lambda \partial \tau} \\ -E \frac{\partial^2 l}{\partial \tau^2} - E \frac{\partial^1 l}{\partial \tau^2} \end{bmatrix},$$

where

$$J_{11} = -\sum_{i=1}^{k} H_i^2(\lambda, \tau) E(N_i) - \frac{\sum_{i=1}^{n} N_i}{\lambda^2} + \frac{\tau^2 + \tau}{\lambda^2} E \left( \sum_{i=1}^{n} X_i^2 \right).$$

Fig. 4. Histogram of galaxolide concentration with Weibull density
\[
J_{22} = -\sum_{i=1}^{k} H_i^{\alpha \tau}(\lambda, \tau) E(N_{i}) + \frac{E(N_{0})}{\tau^2} + \frac{E(\ln^2 \lambda)}{\kappa^2} \left( \sum_{i=n-N_0+1}^{n} X_i^\tau \ln X_i \right) - \frac{2 \ln \lambda}{\kappa^2} \left( \sum_{i=n-N_0+1}^{n} X_i^\tau \ln X_i \right) + \frac{1}{\kappa^2} \left( \sum_{i=n-N_0+1}^{n} X_i^\tau \ln X_i \right)^2.
\]

\[
J_{12} = J_{21} = -\sum_{i=1}^{k} H_i^{\alpha \tau}(\lambda, \tau) E(N_{i}) + \frac{E(N_{0})}{\lambda} \ln \lambda - 1 \frac{\kappa}{\kappa^2} \left( \sum_{i=n-N_0+1}^{n} X_i^\tau \ln X_i \right) - \frac{\tau}{\kappa^2} \left( \sum_{i=n-N_0+1}^{n} X_i^\tau \ln X_i \right),
\]

and

\[
H_i^{\alpha \beta}(\lambda, \tau) = \frac{\frac{1}{\kappa^2 + 2} \exp \left[-\left( \frac{d_{i-1}^\tau}{\lambda} \right)^2 - \exp \left[-\left( \frac{d_i^\tau}{\lambda} \right)^2 \right] \right]}{\kappa^2 + 2} \exp \left[-\left( \frac{d_{i-1}^\tau}{\lambda} \right)^2 - \exp \left[-\left( \frac{d_i^\tau}{\lambda} \right)^2 \right] \right] \exp \left[-\left( \frac{d_{i-1}^\tau}{\lambda} \right)^2 - \exp \left[-\left( \frac{d_i^\tau}{\lambda} \right)^2 \right] \right] \exp \left[-\left( \frac{d_{i-1}^\tau}{\lambda} \right)^2 - \exp \left[-\left( \frac{d_i^\tau}{\lambda} \right)^2 \right] \right]
\]

\[
H_i^{\alpha \tau}(\lambda, \tau) = \frac{\left( \frac{\ln \left( \frac{d_{i-1}^\tau}{\lambda} \right)^2 \exp \left[-\left( \frac{d_i^\tau}{\lambda} \right)^2 \right] - \left( \frac{d_{i-1}^\tau}{\lambda} \right)^2 \exp \left[-\left( \frac{d_i^\tau}{\lambda} \right)^2 \right] \right) \exp \left[-\left( \frac{d_{i-1}^\tau}{\lambda} \right)^2 - \exp \left[-\left( \frac{d_i^\tau}{\lambda} \right)^2 \right] \right]}{\kappa^2 + 2} \exp \left[-\left( \frac{d_{i-1}^\tau}{\lambda} \right)^2 - \exp \left[-\left( \frac{d_i^\tau}{\lambda} \right)^2 \right] \right] \exp \left[-\left( \frac{d_{i-1}^\tau}{\lambda} \right)^2 - \exp \left[-\left( \frac{d_i^\tau}{\lambda} \right)^2 \right] \right] \exp \left[-\left( \frac{d_{i-1}^\tau}{\lambda} \right)^2 - \exp \left[-\left( \frac{d_i^\tau}{\lambda} \right)^2 \right] \right]
\]

\[
H_i^{\alpha \tau}(\lambda, \tau) = \frac{\left[ \frac{\ln \left( \frac{d_{i-1}^\tau}{\lambda} \right)^2 \exp \left[-\left( \frac{d_i^\tau}{\lambda} \right)^2 \right] - \left( \frac{d_{i-1}^\tau}{\lambda} \right)^2 \exp \left[-\left( \frac{d_i^\tau}{\lambda} \right)^2 \right] \right] \exp \left[-\left( \frac{d_{i-1}^\tau}{\lambda} \right)^2 - \exp \left[-\left( \frac{d_i^\tau}{\lambda} \right)^2 \right] \right]}{\kappa^2 + 2} \exp \left[-\left( \frac{d_{i-1}^\tau}{\lambda} \right)^2 - \exp \left[-\left( \frac{d_i^\tau}{\lambda} \right)^2 \right] \right] \exp \left[-\left( \frac{d_{i-1}^\tau}{\lambda} \right)^2 - \exp \left[-\left( \frac{d_i^\tau}{\lambda} \right)^2 \right] \right] \exp \left[-\left( \frac{d_{i-1}^\tau}{\lambda} \right)^2 - \exp \left[-\left( \frac{d_i^\tau}{\lambda} \right)^2 \right] \right]
\]
\[
tau \left\{ d_{n-1} \exp \left[ -\left( \frac{d_{n-1}}{\lambda} \right)^\tau \right] - d_n \exp \left[ -\left( \frac{d_n}{\lambda} \right)^\tau \right] \right\} \times \\
\lambda^{2r+1} \left\{ \exp \left[ -\left( \frac{d_{n-1}}{\lambda} \right)^\tau \right] - \exp \left[ -\left( \frac{d_n}{\lambda} \right)^\tau \right] \right\} \times \\
d_n \ln \left( \frac{d_{n-1}}{\lambda} \right) \exp \left[ -\left( \frac{d_{n-1}}{\lambda} \right)^\tau \right] - d_n \ln \left( \frac{d_n}{\lambda} \right) \exp \left[ -\left( \frac{d_n}{\lambda} \right)^\tau \right],
\]

\( i = 1, \ldots, k. \) Since the frequencies \( N_i, i = 0, \ldots, k, \) have binomial distribution, their expectations are

\[
E(N_i) = n \theta_i = \begin{cases} 
n \exp \left[ -\left( \frac{d_{n-1}}{\lambda} \right)^\tau \right] - \exp \left[ -\left( \frac{d_n}{\lambda} \right)^\tau \right] & \text{for } i = 1, \ldots, k, \\
n \exp \left[ -\left( \frac{d_n}{\lambda} \right)^\tau \right] & \text{for } i = 0.
\end{cases}
\]

Furthermore, it remains to derive the expectations

\[
E_1 = E \left( \sum_{i=n-N+1}^{n} X^\tau_{(i)} \right), \quad E_2 = E \left( \sum_{i=n-N+1}^{n} X^\tau_{(i)} \ln X_{(i)} \right), \quad E_3 = E \left( \sum_{i=n-N+1}^{n} X^\tau_{(i)} (\ln X_{(i)})^2 \right),
\]

where \( X_{(i)} \) is the \( i \)-th ordered statistic.

Using properties of the conditional expectation \( E(E(X \mid Y)) = E(X) \) \( (X,Y) \) are random variables), and due to the fact that \( N_0 \sim Bi(n, \theta_0) \), one gets

\[
E_1 = E \left( \sum_{i=n-N+1}^{n} X^\tau_{(i)} \right) = E \left[ E \left( \sum_{i=n-N+1}^{n} X^\tau_{(i)} \mid N_0 \right) \right] = \\
= \sum_{n=0}^{n} \left( \sum_{i=n-N+1}^{n} \frac{n\theta_i}{\theta_0} \right) \frac{n}{\theta_0} \exp \left[ -\frac{n\theta_i}{\theta_0} \left( \frac{d_n}{\lambda} \right)^\tau \right] \left[ 1 - \exp \left[ -\frac{n\theta_i}{\theta_0} \left( \frac{d_n}{\lambda} \right)^\tau \right] \right]^{n-n_0},
\]

and analogically

\[
E_2 = E \left( \sum_{i=n-N+1}^{n} X^\tau_{(i)} \ln X_{(i)} \right) = \sum_{n=0}^{n} \left[ \sum_{i=n-N+1}^{n} E(X^\tau_{(i)} \ln X_{(i)}) \right] \times \\
\times \left( \frac{n}{\theta_0} \right) \exp \left[ -\frac{n\theta_i}{\theta_0} \left( \frac{d_n}{\lambda} \right)^\tau \right] \left[ 1 - \exp \left[ -\frac{n\theta_i}{\theta_0} \left( \frac{d_n}{\lambda} \right)^\tau \right] \right]^{n-n_0},
\]

\[
E_3 = E \left( \sum_{i=n-N+1}^{n} X^\tau_{(i)} (\ln X_{(i)})^2 \right) = \sum_{n=0}^{n} \left[ \sum_{i=n-N+1}^{n} E(X^\tau_{(i)} (\ln X_{(i)})^2) \right] \times \\
\times \left( \frac{n}{\theta_0} \right) \exp \left[ -\frac{n\theta_i}{\theta_0} \left( \frac{d_n}{\lambda} \right)^\tau \right] \left[ 1 - \exp \left[ -\frac{n\theta_i}{\theta_0} \left( \frac{d_n}{\lambda} \right)^\tau \right] \right]^{n-n_0}.
\]
The pdf of the variable $X_i$ is in the form of (see [34])

$$f_{i}(x) = n \left( \frac{n-1}{i-1} \right) f(x)[F(x)]^{(i-1)}(1-F(x))^{n-i}, \quad i = 1, 2, \ldots, n,$$

where $F$ is cdf (1) and $f$ is pdf (2) of the uncensored Weibull distribution. Gradually, one gets pdf of $f_{i}(x)$ in the form of

$$f_{i}(x) = n \left( \frac{n-1}{i-1} \right) \frac{X^{i-1}}{\Gamma(i)} \sum_{j=0}^{i-1} \frac{(-1)^j}{j!} \left( \frac{x}{\lambda} \right)^j \exp \left[ -\left( \frac{x}{\lambda} \right)^i \right] (n-i+j+1).$$

Using $\int_0^{\infty} e^{-t} dt = \Gamma(2) = 1$, $\int_0^{\infty} t \ln e^{-t} dt = \Gamma'(2) = 1-\gamma_e$ and $\int_0^{\infty} (\ln t)^2 e^{-t} dt = \Gamma''(2) = \frac{\pi^2}{6} - 2\gamma_e^2 + \gamma_e^2$

(see [35]), the expectations (8) are obtained in the form of

$$E(X^r_{i}) = n \left( \frac{n-1}{i-1} \right) \frac{\lambda^i}{\Gamma(i)} \sum_{j=0}^{i-1} \frac{(-1)^j}{j!} \left( \frac{\lambda^i}{n-i+j+1} \right)^2 \ln \left( \frac{\lambda^i}{n-i+j+1} \right) + 1-\gamma_e \right) \frac{n^2}{i-1}.$$

$$E_X(ln X_{i}) = n \lambda^{i-1} \left( \frac{n-1}{i-1} \right) \frac{\lambda^i}{\Gamma(i)} \sum_{j=0}^{i-1} \frac{(-1)^j}{j!} \left( \frac{\lambda^i}{n-i+j+1} \right)^2 \ln \left( \frac{\lambda^i}{n-i+j+1} \right) + 1-\gamma_e \right) \frac{n^2}{i-1}.$$

$$E[\ln(X_{i}^r)(\ln X_{i})^2] = n \lambda^{i-1} \left( \frac{n-1}{i-1} \right) \frac{\lambda^i}{\Gamma(i)} \sum_{j=0}^{i-1} \frac{(-1)^j}{j!} \left( \frac{\lambda^i}{n-i+j+1} \right)^2 \ln \left( \frac{\lambda^i}{n-i+j+1} \right) + 1-\gamma_e \right) \frac{n^2}{i-1}.$$

where $\gamma_e \approx 0.57722$ is Euler's constant. Substituting (12)--(14) into (9)--(11), one gets

$$E_1 = n^2 \frac{\lambda}{\Gamma(i)} \sum_{n_0=0}^{n} \sum_{i=n-n_0+1}^{n} \left( \frac{n-1}{i-1} \right) \frac{\lambda^i}{\Gamma(i)} \sum_{j=0}^{i-1} \frac{(-1)^j}{j!} \left( \frac{\lambda^i}{n-i+j+1} \right)^2 \left( \frac{n}{n_0} \right)^2 \exp \left[ -\left( \frac{\lambda}{n_0} \right)^n \right] \left[ 1 - \exp \left[ -\left( \frac{\lambda}{n_0} \right)^n \right] \right]^{n-n_0},$$

$$E_2 = n^2 \frac{\lambda}{\Gamma(i)} \sum_{n_0=0}^{n} \sum_{i=n-n_0+1}^{n} \left( \frac{n-1}{i-1} \right) \frac{\lambda^i}{\Gamma(i)} \sum_{j=0}^{i-1} \frac{(-1)^j}{j!} \left( \frac{\lambda^i}{n-i+j+1} \right)^2 \left( \frac{n}{n_0} \right)^2 \exp \left[ -\left( \frac{\lambda}{n_0} \right)^n \right] \left[ 1 - \exp \left[ -\left( \frac{\lambda}{n_0} \right)^n \right] \right]^{n-n_0},$$

$$E_3 = n^2 \frac{\lambda}{\Gamma(i)} \sum_{n_0=0}^{n} \sum_{i=n-n_0+1}^{n} \left( \frac{n-1}{i-1} \right) \frac{\lambda^i}{\Gamma(i)} \sum_{j=0}^{i-1} \frac{(-1)^j}{j!} \left( \frac{\lambda^i}{n-i+j+1} \right)^2 \left( \frac{n}{n_0} \right)^2 \exp \left[ -\left( \frac{\lambda}{n_0} \right)^n \right] \left[ 1 - \exp \left[ -\left( \frac{\lambda}{n_0} \right)^n \right] \right]^{n-n_0},$$

Moreover, substituting (7) and (15)--(17) into (6), the elements of the FIM (5) are obtained.
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Íàä³éøëà äî ðåäàêö³¿ 08.11.2018

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ÑÒÀÒÈÑÒÈ×ÅÑÊÈÉ ÂÛÂÎÄ ÄËß ÁÀÃÀÒÎÐÀÇÎÂÎ ÖÅÍÇÓÐÎÂÀÍί Ç˲ÂÀ ÂÈÁ²ÐÊÈ ÒÈÏÓ I ÄËß ÐÎÇÏÎIJËÓ ÂÅÉÁÓËËÀ
Àíîòàö³ÿ.
Ó áàãàòüîõ ãàëóçÿõ íàóêè ÷àñòî çóñòð³÷àþòüñÿ çàäà÷³ ç öåíçóðî -âàíèìè çë³âà äàíèìè ç îäí³ºþ àáî ê³ëüêîìà ìåæàìè âèÿâëåííÿ. Ó ö³é ðî -áîò³ çàïðîïîíîâàíî ïðîöåäóðó äëÿ îá÷èñëåííÿ îö³íîê ìàêñèìàëüíî¿ ïðàâäîïîä³áíîñò³ ïàðàìåòð³â áàãàòîðàçîâîãî öåíçóðóâàííÿ çë³âà òèïó I ç ðîçïîä³ëó Âåéáóëëà ç óðàõóâàííÿì ð³çíî¿ ê³ëüêîñò³ ìåæ âèÿâëåííÿ. Î÷³êóâàíó ³íôîð-ìàö³éíó ìàòðèöþ Ô³øåðà âèçíà÷åíî àíàë³òè÷íî òà ¿¿ âèãëÿä ïîð³âíÿíî ç âèá³ðêîâîþ (ñïîñòåðåæóâàíîþ) ³íôîðìàö³éíîþ ìàòðèöåþ Ô³øåðà. Ìîäåëþ-âàííÿ çäåá³ëüøîãî ´ðóíòóºòüñÿ íà âëàñòèâîñòÿõ îö³íîê âèá³ðîê ìàëèõ ðîçì³ð³â. Ïðèêëàäè ïðî³ëþñòðîâàíî íà ðåàëüíèõ äàíèõ.

Këþ÷îâ³ ñëîâà:
áàãàòîðàçîâî öåíçóðîâàíà çë³âà âèá³ðêà, îö³íêà ìàêñèìàëüíîé ïðàâäîïîäîáíîñò³, ðîçïîä³ë Âåéáóëëà, ³íôîðìàö³éíà ìàòðèöÿ Ô³øåðà, öåíçóðóâàíèå òèïó ².

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Àííîòàöèÿ.
Âî ìíîãèõ îáëàñòÿõ íàóêè ÷àñòî â ñòðå÷àþòñÿ çàäà÷è ñ öåíçóðèðîâàííûìè ñëåâà äàííûìè ñ îäíîé èëè í åñêîëüêèìè ãðàíèöàìè îáíàðóæåíèÿ. Â äàííîé ðàáîòå ïðåäëîæåíà ïðîöåäóðà ä ëÿ âû÷èñëåíèÿ îöåíîê ìàêñèìàëüíîé ïðàâäîïîäîáíîñòè ïàðàìåòðîâ ìíîãîêðàòíîãî öåíçóðèðîâàíèÿ ñëåâà òèïà I äëÿ ðàñïðåäåëåíèÿ Âåéáóëëà ñ ó÷åòîì ðàçíîãî ÷èñëà ãðàíèö îáíàðóæåíèÿ. Îæèäàìè ñíîâàíî, ãëàâíûì îáðàçîì, íà ñâîéñòâàõ îöåíîê âûáîðîê ìàëûõ ðàçìåðîâ. Ïðèìåðû ïðîèëëþñòðèðîâàíû íà ðåàëüíûõ äàííûõ.

Këþ÷åâûå ñëîâà:
ìíîãîêðàòíî öåíçóðèðîâàííàÿ ñëåâà âûáîðêà, îöåíêà ìàêñèìàëüíîé ïðàâäîïîäîáíîñòè, ðàñïðåäåëåíèå Âåéáóëëà, èíôîðìàö³éíà ìàòðèöà Ôèøåðà, öåíçóðèðîâàíèå òèïó ².

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