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A SECOND ORDER MULTIVALUED BOUNDARY-VALUE PROBLEM AND IMPULSIVE NEUTRAL FUNCTIONAL DIFFERENTIAL INCLUSIONS IN BANACH SPACES*

БАГАТОЗНАЧНА ГРАНИЧНА ЗАДАЧА ДРУГОГО ПОРЯДКУ ТА ДИФЕРЕНЦІАЛЬНІ ВКЛЮЧЕННЯ З ІМПУЛЬСНОЮ ДІЄЮ ТА НЕЙТРАЛЬНИМ ФУНКЦІОНАЛОМ

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In this paper, by using the new fixed point theorem of O'Regan and Precup and noncompact measure, the existence of solutions of second order multivalued boundary-value problem in Banach spaces and the existence of a mild solution for impulsive neutral functional differential inclusions in Banach spaces are studied. The compactuess conditions and the upper semicontinuity conditions of multivalued integral operators are weaken in this paper.

З використанням нової теореми О'Регана та Прекупа, а також некомпактної міри доведено існування розв'язків багатозначної граничної задачі другого порядку в банахових просторах. Вивчається існування помірного розв'язку диференціальних включень з імпульсною дією та нейтральним функціоналом у банахових просторах. У роботі послаблено умови компактності та верхньої напівнеперервності на багатозначні інтегральні оператори.

1. Introduction. Differential inclusions is an important branch of the general theory of differential equations and has numerous applications. The problem of existence of solutions of differential inclusions has been studied by many authors, see [1–8]. The main tool used by these authors is the Leray–Schauder alternative theorem for set-valued mapping. However, in the Leray–Schauder alternative theorem, the multivalued operator must be upper semicontinuous and compact. In this paper, we will use the new fixed point theorem obtained by O'Regan and Precup [9] and a noncompact measure to study the existence of solutions of a second order multivalued boundary-value problem in Banach spaces and the existence of a mild solution for impulsive neutral functional differential inclusions in Banach spaces. The compactness conditions and upper semicontinuity conditions on multivalued integral operators are weaken in this paper.

In this paper, we denote by E a real Banach space, $\|\cdot\|$ is the norm in E. In the following, we denote

$$K(E) = \{A \subset E : \ A \text{ is nonempty and compact}\},$$

$$P(E) = \{A \subset E : \ A \text{ is nonempty and closed}\},$$

$$C(E) = \{A \subset E : \ A \text{ is nonempty and convex}\},$$

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$\|A\| = \sup\{\|x\| : x \in A\}$. Let $F : D \subset E \to 2^E \backslash \varnothing$ be a set-value mapping; $\forall A \subset E$ $F^{-1}(A) = \{x \in D : F(x) \bigcap A \neq \varnothing\}$, $\text{graph}\,(F) = \{(x, y) : x \in E, y \in F(x)\}$ is said to the graph of F.

Definition 1.1 [5]. *Let X, Y be metric spaces. $F : D \subset X \to 2^Y \backslash \varnothing$ is said to be upper semicontinuous (short as u.s.c.) if $F^{-1}(A)$ is closed in X wherever $A \subset Y$ is closed.*

Definition 1.2 [5]. *Let $(\Omega, \mathbf{A})$ be a measurable space, $F : \Omega \to 2^X \backslash \varnothing$ is said to be measurable if $F^{-1}(B) \in \mathbf{A}$ for every open subset $B \subset E$.*

Lemma 1.1 [5]. *Let $J = [0, a] \subset R$ and $F : J \times E \to 2^E \backslash \{\varnothing\}$ be compact values. If $F(t, \cdot)$ is u.s.c. and $F(\cdot, x)$ has a strongly measurable selection, then there exists $w(\cdot) \in F(\cdot, v(\cdot))$ for any $v \in C[J, E]$.*

Lemma 1.2 [10]. *Let $C \subset L^1([a, b], E)$ be separable. If there exists $h \in L^1[a, b]$ such that $\|u(t)\| \le h(t)$ for a.e. $t \in [a, b]$ and every $u \in C$, then*

$$\alpha\left(\left\{\int_a^b u(t)dt : u \in C\right\}\right) \le 2\int_a^b \alpha(C(t))\,dt.$$

Lemma 1.3 [5]. *Assume that $F : E \to K(E)$ is u.s.c., if $A \subset E$ is compact, then $F(A)$ is compact.*

Lemma 1.4 [11] (Ascoli – Arzela). *$H \in C[T, E]$ is a relatively compact set if and only if H is equicontinuous and for any $t \in T$, $H(t)$ is relatively compact in E.*

Lemma 1.5 [11] (Mazur). *Let $(E, \|\cdot\|)$ be a normed space, $\{x_n\}_{n \in N} \subset E$, $x_0 \in E$ and $w - \lim\limits_{n\to\infty} x_n = x_0$. Then for any $\epsilon > 0$ there exist $n \in N$, $\alpha_i \ge 0$, $i = 1, 2, \ldots, n$, $\sum\limits_{i=1}^{n} \alpha_i = 1$, such that $\|x_0 - \sum\limits_{i=1}^{n} \alpha_i x_i\| < \epsilon$.*

Lemma 1.6 [9]. *Let D be a closed, convex subset of a Banach space E and $N : D \to 2^D$. Assume* graph (N) *is closed, and for any compact set A, $N(A)$ is relatively compact. If there exists x_0 such that*

$$M \subset D, M = \text{co}\,(\{x_0\} \cup N(M)) \ \ and\ \overline{M} = \overline{C} \ \ with\ \ C \subset M \ \ countable \Rightarrow \overline{M} \ \ is\ \ compact,$$

then N has a fixed point in D.

2. Boundary-value problem for differential inclusions. In this section, we prove the existence of a C^1-solution of the following second order multivalued boundary-value problem in Banach spaces:

$$\begin{aligned} x''(t) \in F(t, x(t), x'(t)) \quad \text{a.e.} \quad t \in [0, 1],\\ ax(0) - bx'(0) = x_0, \\ cx(1) + dx'(1) = x_1, \end{aligned} \tag{2.1}$$

where $a \geq 0$, $b \geq 0$, $c \geq 0$, $d \geq 0$, $ad + bc > 0$, $x_0, x_1 \in E$. Let

$$Gx = \left\{ w \in L^1([0,1], E) : w(\cdot) \in F(\cdot, x(\cdot), x'(\cdot)), x \in C^1([0,1], E) \right\},$$

where $w(\cdot)$ is a strongly measurable selection of $F(\cdot, x(\cdot), x'(\cdot))$. Let

$$Jw(t) = h(t) + \int_0^1 g(t,s) w(s)\, ds, \quad w \in Gx, \quad N = J \circ G,$$

where g is the Green function with respect to inclusions (2.1), and $h(t)$ is a solution of

$$x''(t) = 0 \quad \text{a.e.} \quad t \in [0,1],$$
$$ax(0) - bx'(0) = x_0,$$
$$cx(1) + dx'(1) = x_1.$$

Let $a_0 = \max\{|g(t,s)|, |g_t(t,s)| : t, s \in [0,1]\}$, and for any $R > 0$, $U_R = \{x \in C^1[T, E] : \|x\|_1 \leq R\}$, where $\|x\|_1 = \max\{\|x\|, \|x'\|\}$.

We first make some assumptions about the multivalued map $F : T \times E \to 2^E$.

(C_1) $F(\cdot, x, y)$ has a strongly measurable selection for any $x, y \in E$;

(C_2) $F(t, \cdot, \cdot)$ is u.s.c., for a.e. $t \in [0,1]$;

(C_3) for any $r > 0$ there exists $l_r \in L^1[T, R_+]$ such that $\|F(t, x, y)\| \leq l_r(t)$ if $\|x\| \leq r$ and $\|y\| \leq r$;

(C_4) $\limsup_{\rho \to \infty} \dfrac{a_0}{\rho} \displaystyle\int_0^1 l_\rho(t) dt < 1$;

(C_5) for any $R > 0$ there exists $w : T \times [0, 2R] \to R_+$ such that for any bounded sets $A, B \subset U_R$, the inequality $\alpha(F(s, A, B)) \leq w(s \max\{\alpha(A)), \alpha(B)\})$ holds, and

$$\varphi(t) \leq 2 \int_0^1 (|g(t,s)| + |g_t(t,s)|) w(s, \varphi(s))\, ds$$

has a unique nonnegative continuous zero solution.

Lemma 2.1. *If $F : [0,1] \times E \times E \to CK(E)$ satisfies (C_1), (C_2) and (C_3), then $N : U_R \to C(C^1[T, E])$ has closed graph, and $N(B)$ is relatively compact for any compact set B.*

Proof. (a) We prove $Nx \neq \varnothing$ for any $x \in U_R$. By Lemma 1.1 we know that $F(\cdot, x(\cdot), x'(\cdot))$ has a strongly measurable selection. From (C_3), $F(\cdot, x(\cdot), x'(\cdot))$ has a Bochner selection, i.e., $Nx \neq \varnothing$.

(b) Since F has convex values, clearly N has convex values.

(c) Suppose $x_n \to x$, $v_n \to v$, $n \to \infty$, and

$$v_n(\cdot) = h(\cdot) + \int_0^1 g(\cdot, s) w_n(s)\, ds,$$

where

$$v_n \in N(x_n), \quad w_n(\cdot) \in F(\cdot, x_n(\cdot), x_n'(\cdot)).$$

It follows from Exercise 9.6 in [5] that $\{w_n : n \geq 1\}$ has a weakly convergent subsequence in $L^1[T, E]$. Assume $w_{n_k} \rightharpoonup w$, by $Mazur$ Theorem we have

$$w \in \overline{\text{co}}\left(\bigcap_{m=1}^{\infty}\bigcup_{k=m}^{\infty}\{w_{n_k}\}\right) \subset \overline{\text{co}}\left(\bigcap_{m=1}^{\infty}\bigcup_{k=m}^{\infty}\{F(\cdot, x_{n_k}(\cdot), x_{n_k}'(\cdot))\}\right) \subset F(\cdot, x(\cdot), x'(\cdot)).$$

Since

$$v_{n_k}(\cdot) = h(\cdot) + \int_0^1 g(\cdot, s) w_{n_k}(s)\, ds \to v, \quad n_k \to \infty,$$

we have

$$v(\cdot) = h(\cdot) + \int_0^1 g(\cdot, s) w(s)\, ds.$$

Thus N is a closed graph operator.

(d) We prove that N maps any compact set $M \subset U_R$ into a relatively compact set. For this aim, it is enough to prove that $\{v_n\}_{n\geq 1} \subset N(M)$ has a convergent subsequence, that is, $\{v_n\}_{n\geq 1} \subset N(M)$ is relatively compact. Suppose

$$v_n(\cdot) = h(\cdot) + \int_0^1 g(\cdot, s) w_n(s)\, ds$$

where

$$v_n \in N(x_n), w_n(\cdot) \in F(\cdot, x_n(\cdot), x_n'(\cdot)), \quad x_n \in M.$$

Since M is a compact subset of $C^1[T, E]$, M is bounded, hence by (C_3), there exists $k \in \in L^1[0,1]$ such that $\|w_n(s)\| \leq k(s)$ for a.e. $s \in [0,1]$, and then $\{v_n : n \geq 1\}$ is bounded. Then we have

$$\alpha(\{v_n(t) : n \geq 1\}) = \alpha\left(h(t) + \left\{\int_0^1 g(t,s) w_n(s) ds : n \geq 1\right\}\right) =$$

$$= \alpha\left(\left\{\int_0^1 g(t,s) w_n(s) ds : n \geq 1\right\}\right) \leq$$

$$\leq 2\int_0^1 \alpha(\{g(t,s) w_n(s) : n \geq 1\}) ds \leq$$

$$\leq 2 \max\{|g(t,s)| : s \in [0,1]\} \int_0^1 \alpha(\{w_n(s) : n \geq 1\}) ds.$$

It follows from the compactness of M that $M(t), M'(t)$ are compact for any $t \in [0,1]$. Since $F(s,\cdot,\cdot)$ is u.s.c. and has compact values, we know that $F(s, M(s), M'(s))$ is relatively compact by Lemma 1.3. Thus, for any $s \in [0,1]$, $\alpha(\{w_n(s) : n \geq 1\}) = 0$. This shows that $\alpha(\{v_n(s) : n \geq 1\}) = 0$. Similarly, from

$$\alpha(\{v_n'(t) : n \geq 1\}) = \alpha\left(h'(t) + \left\{\int_0^1 g_t(t,s)w_n(s)ds : n \geq 1\right\}\right) =$$

$$= \alpha\left(\left\{\int_0^1 g_t(t,s)w_n(s)ds : n \geq 1\right\}\right) \leq$$

$$\leq 2\int_0^1 \alpha(\{g_t(t,s)w_n(s) : n \geq 1\})ds \leq$$

$$\leq 2\max\{|g_t(t,s)| : s \in [0,1]\}\int_0^1 \alpha(\{w_n(s) : n \geq 1\})ds,$$

we have that $\alpha(\{v_n'(s) : n \geq 1\}) = 0$. Thus for any $t \in [0,1]$, $\{v_n(t)\}_{n\geq 1}$ and $\{v_n'(t)\}_{n\geq 1}$ are relatively compact. From

$$\|v'_n(t_1) - v'_n(t_2)\| \leq \|h'(t_1) - h'(t_2)\| + \left\|\int_0^1 g_t(t_1,s)w_n(s)\,ds - \int_0^1 g_t(t_2,s)w_n(s)\,ds\right\| \leq$$

$$\leq \|h'(t_1) - h'(t_2)\| + \int_0^1 |g_t(t_1,s) - g_t(t_2,s)|\,\|w_n(s)\|\,ds \leq$$

$$\leq \|h'(t_1) - h'(t_2)\| + \int_0^1 |g_t(t_1,s) - g_t(t_2,s)|k(s)ds, \tag{2.2}$$

we know that $\{v_n'(\cdot)\}_{n\geq 1}$ is equicontinuous. By Theorem 1.2.7 in [10], $N(M)$ is relatively compact.

Lemma 2.2. *Assume that $F : T \times E \times E \to 2^E$ satisfies (C_3) and (C_5), then N satisfies*

$$M \subset U, M = \text{co}\,(\{x_0\} \cup N(M)) \ \text{ and } \ \overline{M} = \overline{C} \ \text{ with } \ C \subset M \ \text{ countable} \Rightarrow \overline{M} \ \text{ is compact.}$$

Proof. Suppose $M \subset U$, $M = \text{co}\,(\{x_0\} \cup N(M))$ and $\overline{M} = \overline{C}$ with $C \subset M$ countable. Let

$x \in M$ and $v \in N(x)$, $v = h + \int_0^1 g(\cdot, s)w(s)ds$, where $w(\cdot) \in F(\cdot, x(\cdot), x'(\cdot))$. Thus

$$\|v(t_1) - v(t_2)\| \le \|h(t_1) - h(t_2)\| + \left\| \int\limits_0^1 g(t_1, s)w(s)\, ds - \int\limits_0^1 g(t_2, s)w(s)\, ds \right\| \le$$

$$\le \|h(t_1) - h(t_2)\| + \int\limits_0^1 |g(t_1, s) - g(t_2, s)|\, \|w(s)\| ds \le$$

$$\le \|h(t_1) - h(t_2)\| + \int\limits_0^1 |g(t_1, s) - g(t_2, s)| k(s) ds.$$

This shows that $N(M)$ is equicontinuous. Similarly, we can prove that $N'(M) = \{u' | u \in \in N(M)\}$ is equicontinuous (see for example (2.2)). Since $M = \mathrm{co}\,(\{x_0\} \bigcup N(M))$, we have that M, M' are equicontinuous. Since M is bounded, from Theorem 1.2.2 in [10] we know that $\alpha(M(t))$ and $\alpha(M'(t))$ are continuous. Since $M = \mathrm{co}\,(\{x_0\} \bigcup N(M))$ and $C \subset M$ is countable, there exists $V = \{v_n : n \ge 1\} \subset N(M)$ such that $C \subset \mathrm{co}\,(\{x_0\} \bigcup V)$, where $v_n(\cdot) = h(\cdot) + \int_0^1 g(\cdot, s)w_n(s)\, ds$ and $w_n(\cdot) \in F(\cdot, x_n(\cdot), x_n'(\cdot))$, $x_n \in M$. Therefore

$$\alpha(\overline{M}(t)) = \alpha(\overline{C}(t)) = \alpha(C(t)) \le \alpha\left(\mathrm{co}\,\left(\{x_0\} \cup V\right)(t)\right) =$$

$$= \alpha(V(t)) = \alpha\left(\left\{ \int\limits_0^1 g(t, s)w_n(s)\, ds : \ n \ge 1 \right\}\right) \le$$

$$\le 2 \int\limits_0^1 \alpha(\{g(t, s)w_n(s) : \ n \ge 1\}) ds.$$

From

$$\alpha(\{g(t, s)w_n(s) : \ n \ge 1\}) \le |g(t, s)| \alpha(F(s, M(s), M'(s)) \le$$
$$\le |g(t, s)| w(s, \max\{\alpha(M(s)), \alpha(M'(s))\}),$$

we know

$$\alpha(M(t)) \le 2 \int\limits_0^1 |g(t, s)| w(s, \max\{\alpha(M(s)), \alpha(M'(s))\}) ds.$$

Similarly, we have

$$\alpha(M'(t)) \le 2 \int\limits_0^1 |g_t(t, s)| w(s, \max\{\alpha(M(s)), \alpha(M'(s))\}) ds.$$

Hence

$$\max\{\alpha(M(t)),\alpha(M'(t))\} \le 2\int_0^1 (|g(t,s)|+|g_t(t,s)|)w(s,\max\{\alpha(M(s)),\alpha(M'(s))\})ds.$$

By (C_5), we know that for any $t \in [0,1]$,

$$\alpha(M(t)) = 0, \quad \alpha(M'(t)) = 0.$$

From Theorem 1.2.4 and Theorem 1.2.6 in [10], we have

$$\alpha_0(M) = 0, \quad \alpha_0(M') = 0, \quad \text{i.e.,} \quad \alpha_1(M) = 0,$$

where α_0 denotes the noncompact measure in $C([0,1],E)$ and α_1 denotes the noncompact measure in $C^1([0,1],E)$. Hence M is relatively compact, i.e., $\overline{M}$ is compact.

Lemma 2.3. *Assume that* $F : T\times E\times E \to 2^E$ *satisfies* (C_3) *and* (C_4). *Then there exists* $U_R \subset C^1([0,1],E)$ *such that* $N : U_R \to 2^{U_R}$.

Proof. For any $x \in C^1[T,E]$, let

$$h(\cdot)+\int_0^1 g(\cdot,s)w(s)ds \in N(x), \quad w(\cdot) \in F(\cdot,x(\cdot),x'(\cdot)).$$

We know

$$\|N(x)\|_0 \le \|h\|_1 + a_0 \sup\left\{\int_0^1 \|w(s)\|ds : \ w(\cdot) \in F(\cdot,x(\cdot),x'(\cdot))\right\},$$

$$\|N(x)'\|_0 \le \|h\|_1 + a_0 \sup\left\{\int_0^1 \|w(s)\|ds : \ w(\cdot) \in F(\cdot,x(\cdot),x'(\cdot))\right\},$$

where $\|\cdot\|_0$ denotes the norm in $C([0,1],E)$ and $\|\cdot\|_1$ denotes the norm in $C^1([0,1],E)$. Denote $\|h\|_1 = r$, we have

$$\|N(x)\|_1 \le r + a_0 \sup\left\{\int_0^1 \|w(s)\|ds : \ w(\cdot) \in F(\cdot,x(\cdot),x'(\cdot))\right\}.$$

Assume $\|x\|_1 \le \rho$, from (C_4) we know that for ρ enough large, we have

$$\frac{\|N(x)\|_1}{\rho} \le \frac{r}{\rho} + \frac{a_0 \sup\left\{\int_0^1 \|w(x)\|ds : w(\cdot) \in F(\cdot,x(\cdot),x'(\cdot))\right\}}{\rho} \le$$

$$\leq \frac{r}{\rho} + \frac{a_0 \int_0^1 l_\rho(t)dt}{\rho} < 1.$$

This implies that there exists $R > 0$ such that $N(x) \subset \overline{B}_R$ if $\|x\|_1 \leq R$.

From Lemmas 2.1, 2.2, 2.3, and 1.6, we have the following theorem.

Theorem 2.1. *Assume that $F : T \times E \times E \to CK(E)$ satisfies (C_1)–(C_5). Then inclusions (2.1.) has at least a C^1-solution.*

3. Impulsive neutral functional differential inclusion. We first recall that a family $\{C(t) : t \in \in R\}$ of bounded linear operators in the Banach space E is called a strongly continuous cosine family iff

(i) $C(0) = I$ (I is the identity operator in E);

(ii) $C(t+s) + C(t-s) = 2C(t)C(s)$, $s, t \in R$;

(iii) $C(t)y$ is continuous in t on R for each fixed $y \in E$.

If $C(t)$, $t \in R$, is a strongly continuous cosine family in E, then the strongly continuous sine family $S(t), t \in R$, is the one parameter family of operators in E defined by

$$S(t)y = \int_0^t C(s)yds, \quad y \in E, \quad t \in R.$$

The infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in R\}$ is the operator $A : E \to E$ defined by

$$Ay = \left. \frac{d^2}{dt^2} C(t)y \right|_{t=0}.$$

In this section, we study the following initial value problem by using the theory of strongly continuous cosine and sine families:

$$\begin{gathered} \frac{d}{dt}[y'(t) - f(t, y_t)] \in Ay(t) + F(t, y_t), \\ t \in J = [0, b], \quad t \neq t_k, \\ \triangle y \mid_{t=t_k} = I_k(y(t_k^-)), \\ \triangle y' \mid_{t=t_k} = \overline{I}_k(y(t_k^-)), \quad k = 1, 2, \ldots, m, \\ y_0 = \phi, \quad y'(0) = \eta, \end{gathered} \tag{3.1}$$

where A is an infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in R\}$, $F : J \times E \to 2^E, 0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = b, f : J \times E \to E, \phi \in C([-r, 0], E)$, $I_k, \overline{I}_k \in C[E, E], k = 0, 1, \ldots, m$.

In order to define the concept of a mild solution of the problem, we consider the space $\Omega = \{y : [-r, b] \to E | y_k \in C(J_k, E), k = 0, 1, \ldots, m$ and there exist $y(t_k^-)$ and $y(t_k^+)$, with $y(t_k^-) = y(t_k)$, $k = 1, 2, \ldots, m$, $y(t) = \phi(t)\ \forall t \in [-r, 0]\}$, which is a Banach space with the norm

$$\|y\|_\Omega = \sup\{\|y\|_{J_k} : k = 0, 1, \ldots, m\},$$

$$\|y\|_{J_k} = \sup\{\|y(t)|t \in J_k\|,$$

where y_k is the restriction of y to $J_k = (t_k, t_{k+1}]$, $k = 1, \ldots, m$, and $J_0 = [0, t_1]$. Define y_t as $y_t(s) = y(t+s)$, $s \in [-r, 0]$, for any $y \in \Omega$ and for any $t \in J$. Let $\triangle y|_{t=t_k} = y(t_k^+) - y(t_k^-)$ and $\triangle' y|_{t=t_k} = y'(t_k^+) - y'(t_k^-)$.

Definition 3.1. *$y \in C([-r, b]\backslash\{t_1, t_2, \ldots, t_m\}, E)$ is said to be a mild solution of (3.1.), if $\Delta y\ |_{t=t_k} = I_k(y(t_k^-))$ and $\Delta y'\ |_{t=t_k} = \overline{I}_k(y(t_k^-))$, $k = 1, 2, \ldots, m$, and there exists a $v \in L^1[J, E]$ such that $v(t) \in F(t, y_t)$ a.e. on J, and*

$$y(t) = C(t)\phi(0) + S(t)[\eta - f(0, \phi)] + \int\limits_0^t C(t-s)f(s, y_s)ds + \int\limits_0^t S(t-s)v(s)ds +$$

$$+ \sum_{0<t_k<t} [I_k(y(t_k^-)) + (t - t_k)\overline{I}_k(y(t_k^-))], \quad t \in J.$$

Define

$$S_{F,y} = \{v \in L^1(T, E) : \ v(t) \in F(t, y_t) \quad \text{a.e.} \quad t \in J\},$$

$$Ny = \left\{ h \in \Omega : \ h(t) = \begin{cases} \phi(t), & \text{if } t \in [-r, 0], \\ C(t)\phi(0) + S(t)[\eta - f(0, \phi)] + & \\ + \int_0^t C(t-s)f(s, y_s)ds + \int_0^t S(t-s)v(s)ds + & \\ + \sum\limits_{0<t_k<t} [I_k(y(t_k^-)) + (t - t_k)\overline{I}_k(y(t_k^-))], & \text{if } t \in J. \end{cases} \right\}$$

For the proof of our next main result, we will use the following assumptions:

(H_1) A is the infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in R\}$ which is bounded (i.e., there exists $M_0 > 0$ such that $\|C(t)\| \le M_0\ \forall t \in R$).

(H_2) $F : J \times E \to CP(E)$ is a Caratheodory map, that is, $F(\cdot, x)$ has a strongly measurable selection for any $x \in E$, $F(t, \cdot)$ is u.s.c., for a.e. $t \in [0, 1]$, and for any a bounded set $X \subset \Omega$, there exists $k_1 \in L^1[J, R_+]$ such that $\alpha(F(s, X_s)) \le k_1(s)\alpha(X_s)$.

(H_3) $f(t, u)$ is continuous in the second variable, and there exist $p_1, p_2 \in L^1[J, R_+]$ such that $\|f(t, u)\| \le p_1(t)p_3(\|u\|_\Omega) + p_2(t)$, with $p_3 : J \to R_+$ being nondecreasing, and there exists $k_2 \in L^1([0, b], R_+)$ such that $\alpha(f(s, A)) \le k_2(s)\alpha(A)$.

(H_4) Let $I_k, \overline{I}_k \in C(E, E)$ and there exist $d_k, \overline{d}_k$ such that $\|I_k(x)\| \le d_k$, $\|\overline{I}_k(x)\| \le \overline{d}_k$ for each $x \in E$.

(H_5) $\limsup_{\rho\to\infty} \displaystyle\int_0^b \frac{M_0 p_3(\rho)p_1(t) + M_0 b l_\rho(t)}{\rho} dt < 1.$

Lemma 3.1 [4]. *Let I be a real compact interval and E be a real Banach space, for all $u \in C[I, E]$, $F(\cdot, u)$ be measurable, $F(t, \cdot)$ upper semicontinuous for a.e. $t \in I$. If $\Gamma : L^1[I, E] \to C[I, E]$ is a linear mapping, then*

$$\Gamma \circ S_F : C[I, E] \to BPC(C[I, E]), \quad \textit{a.e.} \quad y \mapsto (\Gamma \circ S_F)(y) = \Gamma(S_{F,y})$$

has closed graph.

Theorem 3.1. *If the hypotheses* (H_1), (H_2), (H_3), (H_4), (H_5) *are satisfied, then (3.1.) at least has a mild solution.*

Proof. Clearly the fixed point of the operator N is a mild solution of (3.1.). Since F has closed values, (H_4) and Lemma 1.1 in [12] imply $S_{F,y} \neq \varnothing$.

Step 1. We prove that Ny is convex for any $y \in C([-r,b],E)$. Indeed, for any $h_1, h_2 \in Ny$, there exist $v_1, v_2 \in S_{F,y}$ such that

$$h_1(t) = \begin{cases} \phi(t), & \text{if } t \in T_0, \\ C(t)\phi(0) + S(t)[\eta - f(0,\phi)] + \\ + \int_0^t C(t-s)f(s,y_s)ds + \int_0^t S(t-s)v_1(s)ds + \\ + \sum\limits_{0<t_k<t} [I_k(y(t_k^-)) + (t-t_k)\overline{I}_k(y(t_k^-))], & \text{if } t \in J, \end{cases}$$

$$h_2(t) = \begin{cases} \phi(t), & \text{if } t \in T_0, \\ C(t)\phi(0) + S(t)[\eta - f(0,\phi)] + \\ + \int_0^t C(t-s)f(s,y_s)ds + \int_0^t S(t-s)v_2(s)ds + \\ + \sum\limits_{0<t_k<t} [I_k(y(t_k^-)) + (t-t_k)\overline{I}_k(y(t_k^-))], & \text{if } t \in J. \end{cases}$$

For any $\lambda \in [0,1]$, then

$$(\lambda h_1 + (1-\lambda)h_2)(t) = \\ = \begin{cases} \phi(t), & \text{if } t \in T_0, \\ C(t)\phi(0) + S(t)[\eta - f(0,\phi)] + \\ + \int_0^t C(t-s)f(s,y_s)ds + \int_0^t S(t-s)(\lambda v_1(s) + (1-\lambda)v_2(s))ds + \\ + \sum\limits_{0<t_k<t} [I_k(y(t_k^-)) + (t-t_k)\overline{I}_k(y(t_k^-))], & \text{if } t \in J. \end{cases}$$

Since F has convex values, $S_{F,y}$ also has convex values, so $\lambda h_1 + (1-\lambda)h_2 \in Ny$.

Step 2. We prove that N is a bounded operator. For each bounded set $U \subset \Omega$, let $R = \sup\{\|u\|_\Omega : u \in U\}$. Next, we show that $N(U)$ is bounded in Ω. By the definition of N, we only need to show that $N(U)$ is bounded on $[0,b]$. For any $h \in N(U)$, there exist $y \in U$, $v \in S_{F,y}$ such that

$$h(t) = C(t)\phi(0) + S(t)[\eta - f(0,\phi)] + \int\limits_0^t C(t-s)f(s,y_s)ds + \int\limits_0^s S(t,s)v(s)ds + \\ + \sum_{0<t_k<t} [I_k(y(t_k^-)) + (t-t_k)\overline{I}_k(y(t_k^-))], \quad \text{if} \quad t \in J.$$

Hence

$$\|h(t)\| \le \|C(t)\|\,\|\phi(0)\| + \|S(t)\|\,\|\eta - f(0,\phi)\| + \int_0^t \|C(t-s)f(s,y_s)\|ds +$$

$$+ \int_0^t \|S(t-s)v(s)\|ds + \| \sum_{0<t_k<t} [I_k(y(t_k^-)) + (t-t_k)\overline{I}_k(y(t_k^-))]\| \le$$

$$\le M_0\|\phi\|_0 + M_0 b\|\eta - f(0,\phi)\| + M_0 \int_0^t (p_3(R)p_1(s) + p_2(s))ds +$$

$$+ \left\| \sum_{0<t_k<t} [I_k(y(t_k^-)) + (t-t_k)\overline{I}_k(y(t_k^-))] \right\| =$$

$$= M_0\|\phi\| + M_0 b\|\eta - f(0,\phi)\| + M_0 p_3(R)\|p_1\|_{L^1} + M_0\|p_2\|_{L^1} +$$

$$+ \sum_{k=1}^m [d_k + (b-t_k)\overline{d}_k].$$

Step 3. We prove that N maps a bounded set into an equicontinuous set of Ω. Assume that $U \subset \Omega$ is a bounded set and there exists $M_1 > 0$ such that $\|y\| \le M_1$ for any $y \in U$. By step 2 we know that there exists $M_2 > 0$ such that $\|v\| \le M_2$ for any $y \in U$ and any $v \in N(U)$. Let $y \in U$, $h \in N_y$, so there exists $v \in S_{F,y}$ such that

$$h(t) = C(t)\phi(0) + S(t)[\eta - f(0,\phi)] +$$

$$+ \int_0^t C(t-s)f(s,y_s)ds + \int_0^t S(t-s)v(s)ds +$$

$$+ \sum_{0<t_k<t} [I_k(y(t_k^-)) + (t-t_k)\overline{I}_k(y(t_k^-))].$$

If $\gamma_1 < \gamma_2$, and $\gamma_1, \gamma_2 \in J_k$, we have

$$\|h(\gamma_2) - h(\gamma_1)\| \le \|C(\gamma_2) - C(\gamma_1)\|\,\|\phi(0)\| + \|S(\gamma_2) - S(\gamma_1)\|\,\|\|[\eta - f(0,\phi)]\|\| +$$

$$+ \left\| \int_0^{\gamma_2} [C(\gamma_2 - s) - C(\gamma_1 - s)]f(s,y_s)ds \right\| +$$

$$+ \left\| \int_{\gamma_1}^{\gamma_2} C(\gamma_1 - s) f(s, y_s) ds \right\| + \left\| \int_0^{\gamma_2} (S(\gamma_2 - s) - S(\gamma_1 - s)) v(s) ds \right\| +$$

$$+ \left\| \int_{\gamma_1}^{\gamma_2} S(\gamma_1 - s) v(s) ds \right\| + \left\| \sum_{0<t_i<\gamma_1} (\gamma_2 - \gamma_1) \overline{I}_i(y(t_i^-)) \right\|.$$

This implies that

$$\|h(\gamma_2) - h(\gamma_1)\| \leq \|C(\gamma_2) - C(\gamma_1)\| \, \|\|\phi(0)\| + \|S(\gamma_2) - S(\gamma_1)\| \, \|[\eta - f(0, \phi)]\| +$$

$$+ \left\| \int_0^{\gamma_2} [C(\gamma_2 - s) - C(\gamma_1 - s)][p_1(s) p_3(M_1) + p_2(s)] ds \right\| +$$

$$+ \left\| \int_{\gamma_1}^{\gamma_2} C(\gamma_1 - s)[p_1(s) p_3(M_1) + p_2(s)] ds \right\| +$$

$$+ \left\| \int_0^{\gamma_2} (S(\gamma_2 - s) - S(\gamma_1 - s)) M_2 ds \right\| +$$

$$+ \sum_{0<t_i<\gamma_1} (\gamma_2 - \gamma_1) \overline{d}_i + \left\| \int_{\gamma_1}^{\gamma_2} S(\gamma_1 - s) M_2 ds \right\|. \tag{3.2}$$

As $\gamma_2 - \gamma_1 \to 0$, the right-hand side of (3.2.) tends to zero. Equicontinuity for the cases $\gamma_1 < < \gamma_2 \leq 0$ is $\gamma_1 \leq 0 \leq \gamma_2$ is obvious. Thus $N(U)$ is equicontinuous.

Step 4. We prove that $N(D)$ is relatively compact for each compact set $D \subset \Omega$. We only need to show that for any $\{h_n : n \geq 1\} \subset N(D)$, which has a convergent subsequence, i.e., $\{h_n : n \geq 1\} \subset N(D)$ is relatively compact. Assume

$$h_n(t) = C(t)\phi(0) + S(t)[\eta - f(0, \phi)] + \int_0^t C(t - s) f(s, y_{ns}) ds + \int_0^t S(t - s) v_n(s) ds +$$

$$+ \int_0^t S(t - s) v_n(s) ds + \sum_{0<t_k<t} \left[I_k(y_n(t_k^-)) + (t - t_k) \overline{I}_k(y_n(t_k^-)) \right],$$

where $h_n \in N(y_n)$, $y_n \in D$, $v_n \in S_{F,y_n}$. From the conclusion in step 3 and Ascoli–Arzela Theorem, we only need to prove that $\{h_n(t) : n \geq 1\}$ is relatively compact for any $t \in J$. By

Lemma 1.2, (H_2) and (H_4), we know

$$\alpha(\{h_n(t) : n \geq 1\}) \leq \alpha(C(t)\phi(0)) + \alpha(S(t)[\eta - f(0,\phi)]) +$$

$$+ \alpha\left(\left\{\int_0^t C(t-s)f(s,y_{ns})ds : n \geq 1\right\}\right) +$$

$$+ \alpha\left(\left\{\int_0^t S(t-s)v_n(s)ds : n \geq 1\right\}\right) +$$

$$+ \alpha\left(\left\{\sum_{0<t_k<t}[I_k(y_n(t_k^-)) + (t-t_k)\overline{I}_k(y_n(t_k^-))] : n \geq 1\right\}\right) \leq$$

$$\leq 2\int_0^t \alpha(\{C(t-s)f(s,y_{ns}) : n \geq 1\})ds +$$

$$+ 2\int_0^t \alpha(\{S(t-s)v_n(s) : n \geq 1\})ds =$$

$$= 2\int_0^t \alpha(\{S(t-s)v_n(s) : n \geq 1\})ds \leq$$

$$\leq 2\int_0^t M_0 b\alpha(\{y_n(s) : n \geq 1\})k_1(s)ds = 0.$$

Step 5. We prove that N has closed graph. Assume $y_n \in \Omega, y_n \to y, h_n \in Ny_n, h_n \to h$, we will show $h \in Ny$, i.e., we only need to prove that there exists $v \in S_{F,y}$ such that

$$h(t) = C(t)\phi(0) + S(t)[\eta - f(0,\phi)] + \int_0^t C(t-s)f(s,y_s)ds +$$

$$+ \int_0^t S(t-s)v(s)ds + \sum_{0<t_k<t}[I_k(y(t_k^-)) + (t-t_k)\overline{I}_k(y(t_k^-))]. \qquad (3.3)$$

Assume $v_n \in S_{F,y_n}$ is such that

$$h_n(t) = C(t)\phi(0) + S(t)[\eta - f(0,\phi)] + \int_0^t C(t-s)f(s,y_{ns})ds +$$

$$+ \int_0^t S(t-s)v_n(s)ds + \sum_{0<t_k<t}[I_k(y_n(t_k^-)) + (t-t_k)\overline{I}_k(y_n(t_k^-))].$$

Define $\Gamma: \ L^1[J,E] \to C[J,E]$ by

$$\Gamma(v)(t) = \int_0^t S(t-s)v(s)ds.$$

Then Γ is a linear bounded operator and hence $\Gamma \circ S_F : \Omega \to \Omega$ has closed graph from Lemma 3.1. Hence

$$h_n(t) - C(t)\phi(0) - S(t)[\eta - f(0,\phi)] - \int_0^t C(t-s)f(s,y_{ns})ds -$$

$$- \sum_{0<t_k<t}[I_k(y_n(t_k^-)) + (t-t_k)\overline{I}_k(y_n(t_k^-))] \longrightarrow$$

$$\longrightarrow h(t) - C(t)\phi(0) - S(t)[\eta - f(0,\phi)] - \int_0^t C(t-s)f(s,y_s)ds -$$

$$- \sum_{0<t_k<t}[I_k(y(t_k^-)) + (t-t_k)\overline{I}_k(y(t_k^-))], \quad n \to \infty.$$

Therefore, there exists $v \in S_{F,y}$ such that

$$h(t) - C(t)\phi(0) - S(t)[\eta - f(0,\phi)] - \int_0^t C(t-s)f(s,y_s)ds -$$

$$- \sum_{0<t_k<t}[I_k(y(t_k^-)) + (t-t_k)\overline{I}_k(y(t_k^-))] = \int_0^t S(t-s)v(s)ds.$$

That is (3.3) holds.

Step 6. We prove that there exists $U = \{u \in \Omega : \ \|u\|_\Omega < R\}$ such that $N(\overline{U}) \subset \overline{U}$. Indeed, for any $y \in \Omega$ and $h \in N(y)$, let

$$h(t) = C(t)\phi(0) + S(t)[\eta - f(0,\phi)] + \int_0^t C(t-s)f(s,y_s)ds + \int_0^t S(t-s)v(s)ds+$$

$$+ \sum_{0<t_k<t} [I_k(y(t_k^-)) + (t-t_k)\overline{I}_k(y(t_k^-))],$$

here $v \in S_{F,y}$. We know

$$\|h(t)\| \le \|C(t)\phi(0)\| + \|S(t)[\eta - f(0,\phi)]\| + \left\| \int_0^t C(t-s)f(s,y_s)ds+\right.$$

$$\left. + \int_0^t S(t-s)v(s)ds \right\| + \left\| \sum_{0<t_k<t} [I_k(y(t_k^-)) + (t-t_k)\overline{I}_k(y(t_k^-))] \right\|.$$

Therefore, for any $\|y\|_\Omega \le \rho$,

$$\|N(y)\|_\Omega \le M_0\|\phi\| + M_0 b\|[\eta - f(0,\phi)]\| + M_0 \int_0^t \|f(s,y_s)\|ds+$$

$$+ M_0 b \int_0^t \|v(s)\|ds + \sum_{k=1}^m [d_k + (b-t_k)\overline{d}_k] \le$$

$$\le N_0 + M_0 \int_0^b p_3(\rho)p_1(s)ds + M_0 b \int_0^b l_\rho(s)ds,$$

where $N_0 = M_0\|\phi(0)\| + M_0 b\|\eta - f(0,\phi\| + \sum_{k=1}^m [d_k + (b-t_k)\overline{d}_k] + M_0 \int_0^b p_2(s)ds$. Thus, for any $\|y\|_\Omega \le \rho$, we know

$$\frac{\|N(y)\|_\Omega}{\rho} \le \frac{N_0}{\rho} + \int_0^b \frac{M_0 p_3(\rho)p_1(s) + M_0 b l_\rho(s)}{\rho}\, ds.$$

From the condition (H_5), there exists a large ρ such that

$$\frac{\|N(y)\|_\Omega}{\rho} \le \frac{N_0}{\rho} + \int_0^b \frac{M_0 p_3(\rho)p_1(s) + M_0 b l_\rho(s)}{\rho}\, ds < 1.$$

This implies that there exists $R > 0$ such that $N(\overline{U}) \subset \overline{U}$.

Step 7. For the U defined in step 6, if $M \subset \overline{U}, M \subset \mathrm{co}\,(\{0\} \bigcup N(M)), \overline{M} = \overline{C}$ and $C \subset M$ is countable, then clearly there exists $H = \{h_n : n \geq 1\} \subset N(M)$

$$h_n(t) = C(t)\phi(0) + S(t)[\eta - f(0,\phi)] + \int\limits_0^t C(t-s)f(s, y_{ns})ds + \int\limits_0^t S(t-s)v_n(s)ds +$$

$$+ \sum_{0<t_k<t} \left[I_k(y_n(t_k^-)) + (t-t_k)\overline{I}_k(y_n(t_k^-)) \right],$$

where $v_n \in S_{F,y_n}$ and $y_n \in M$ are such that $C \subset \mathrm{co}\,(\{0\} \bigcup H)$. Thus

$$\alpha(\overline{M}(t)) = \alpha(\overline{C}(t)) = \alpha(C(t)) \leq \alpha(\mathrm{co}\,(\{0\} \bigcup H)(t)) = \alpha(H(t)).$$

If $t \in [-r, 0]$, clearly $\alpha(M(t)) = 0$.

When $t \in J_0$, we have

$$\alpha(M(t)) = \alpha(\overline{M}(t)) \leq \alpha(H(t)) \leq$$

$$\leq 2M_0 \int\limits_0^t k_2(s)\alpha(M(s))ds + 2M_0 b \int\limits_0^t k(s)\alpha(M(s))ds \leq$$

$$\leq 2(M_0 + aM_0 b) \int\limits_0^t (k_2(s) + k_1(s))\alpha(M(s))ds. \tag{3.4}$$

From step 3, we know that $N(\overline{U})$ is equicontinuous. From $M \subset \mathrm{co}\,(\{0\} \bigcup N(M))$, M is equicontinuous. Theorem 1.2.2 in [10] implies that $\alpha(M(t))$ is continuous. By Gronwall Lemma in [13], (3.4) implies $\alpha(M(t)) = 0$ for $t \in J_0$. Thus we know $\alpha(y_n(t)) = 0$ for any $t \in J_0$, moreover $\alpha(I_1(y_n(t_1))) = 0, \alpha(\overline{I}_1(y_n(t_1))) = 0$. When $t \in J_1$,

$$\alpha(M(t)) = \alpha(\overline{M}(t)) \leq \alpha(H(t)) \leq$$

$$\leq 2M_0 \int\limits_0^t k_2(s)\alpha(M(s))ds + 2M_0 b \int\limits_0^t k(s)\alpha(M(s))ds +$$

$$+ \alpha(\{[I_1(y_n(t_1^-)) + (t-t_k)\overline{I}_1(y_n(t_1^- 1))] : n \geq 1\}) \leq$$

$$\leq 2(M_0 + aM_0 b) \int\limits_0^t (k_2(s) + k_1(s))\alpha(M(s))ds + \alpha(\{I_1(y_n(t_1)) : n \geq 1\}) +$$

$$+ b\alpha(\{\overline{I}_1(y_n(t_1)) : n \geq 1\}) =$$

$$= 2(M_0 + aM_0 b) \int\limits_0^t (k_2(s) + k_1(s))\alpha(M(s))\, ds.$$

Again using Gronwall Lemma, we know $\alpha(M(t)) = 0$ for any $t \in J_1$, thus $\alpha(I_2(y_n(t_2))) = 0$, $\alpha(\overline{I}_2(y_n(t_2))) = 0$. Similarly, from the continuouty property of $I_k, \overline{I}_k$ and Gronwall Lemma, we have $\alpha(M(t)) = 0$ for any $t \in J_k : k = 1, 2, 3, \ldots, m$. Clearly $\alpha(M(t)) = 0$ holds for any $t \in [-r, b]$. So $\alpha(M) = 0$ by Theorem 1.2.4 in [10]. Hence M is compact in Ω. As a consequence of Lemma 1.6 we deduce that N has a fixed point which is a mild solution of problem (3.1).

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