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SYMMETRIES OF DEGENERATE CENTER SINGULARITIES OF PLANE VECTOR FIELDS*

СИМЕТРІЇ ВИРОДЖЕНИХ ОСОБЛИВОСТЕЙ ТИПУ ЦЕНТР ВЕКТОРНИХ ПОЛІВ НА ПЛОЩИНІ

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Let $D^2 \subset \mathbb{R}^2$ be a closed unit 2-disk centered at the origin $O \in \mathbb{R}^2$, and F be a smooth vector field such that O is a unique singular point of F and all other orbits of F are simple closed curves wrapping once around O. Thus topologically O is a „center" singularity. Let $\mathcal{D}^+(F)$ be the group of all diffeomorphisms of D^2 which preserve orientation and orbits of F.

Recently the author described the homotopy type of $\mathcal{D}^+(F)$ under the assumption that the 1-jet $j^1 F(O)$ of F at O is non-degenerate. In this paper degenerate case $j^1 F(O)$ is considered. Under additional " non-degeneracy assumptions" on F the path components of $\mathcal{D}^+(F)$ with respect to distinct weak topologies are described. These conditions imply that for each $h \in \mathcal{D}^+(F)$ its path component in $\mathcal{D}^+(F)$ is uniquely determined by the 1-jet of h at O.

Нехай $D^2 \subset \mathbb{R}^2$ — замкнений одиничний двовимірний диск з центром у початку координат $O \in \mathbb{R}^2$ та F — гладке векторне поле таке, що O є єдиною особливою точкою F, а всі інші орбіти — простими замкненими кривими, що огортають O один раз. Таким чином, топологічно O є особливістю типу центр. Нехай $\mathcal{D}^+(F)$ — група всіх дифеоморфізмів D^2, що зберігають орієнтацію та орбіти поля F.

Нещодавно автором було описано гомотопічний тип $\mathcal{D}^+(F)$ за умови, що 1-струмінь $j^1 F(O)$ поля F в O є невиродженим. У цій статті розглядається вироджений випадок $j^1 F(O)$. За додаткової умови невиродженості на F описано компоненти лінійної зв'язності простору $\mathcal{D}^+(F)$ відносно різних слабких топологій. З цих умов випливає, що для кожного $h \in \mathcal{D}^+(F)$ його компонента лінійної зв'язності в $\mathcal{D}^+(F)$ єдиним чином визначається 1-струменем h в O.

1. Introduction. Let $D^2 = \{x^2 + y^2 \leq 1\} \subset \mathbb{R}^2$ be a closed unit 2-disk centered at the origin $O \in \mathbb{R}^2$, $V \subset \mathbb{R}^2$ be a closed subset diffeomorphic to D^2, $z \in \operatorname{Int} V$, and

$$F = F_1 \frac{\partial}{\partial x} + F_2 \frac{\partial}{\partial y}$$

be a C^∞ vector field on V. We will say that F is a *TC vector field on V with topological center at z* if it satisfies the following conditions:

(T_1) z is a unique singular point of F,

(T_2) F is tangent to ∂V, so ∂V is an orbit of F, and

(T_3) all other orbits of F are closed.

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Let F be a TC vector field on V. Then it easily follows from Poincaré – Bendixson theorem [1] that there exists a *homeomorphism*

$$h = (h_1, h_2) : V \to D^2 \tag{1.1}$$

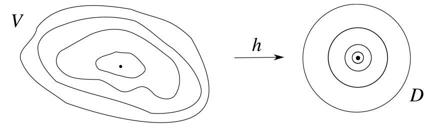

Fig. 1

such that $h(z) = O$ and for every other orbit o of F its image $h(o)$ is a circle of some radius $c \in (0, 1]$ centered at the origin, see Fig. 1. This motivates the term TC which we use, see [2]. Moreover, since the first recurrent map of closed orbits is smooth, see [1], it can be assumed that the restriction $h : V \backslash z \to D^2 \backslash O$ is a C^∞ diffeomorphism.

Let F be a TC vector field on D^2. In this case we will always assume that $F(O) = 0$. Consider the following matrix

$$\nabla F = \begin{pmatrix} \dfrac{\partial F_1}{\partial x}(O) & \dfrac{\partial F_1}{\partial y}(O) \\ \\ \dfrac{\partial F_2}{\partial x}(O) & \dfrac{\partial F_2}{\partial y}(O) \end{pmatrix}.$$

We will call ∇F the *linear part* or the *linearization* of F at O.

Suppose ∇F is non-degenerate. Then it can be shown that there are local coordinates at O in which

$$j^1 F(O) = -ay \frac{\partial}{\partial x} + ay \frac{\partial}{\partial y}$$

for some $a \neq 0$, so the 1-jet of F at O is a “rotation”. This class of singularities is well-studied from many points of view, see e.g. [3 – 8]. In particular, in [5] normal forms of such vector fields are obtained.

Denote by $\mathcal{D}(F)$ the group of C^∞ diffeomorphisms h of D^2 such that $h(o) = o$ for each orbit o of F. Let also $\mathcal{D}^+(F)$ be the subgroup of $\mathcal{D}(F)$ consisting of all orientation preserving diffeomorphisms, and $\mathcal{D}^\partial(F)$ be the subgroup of $\mathcal{D}^+(F)$ consisting of all diffeomorphisms fixed of ∂D^2. We endow $\mathcal{D}(F)$, $\mathcal{D}^+(F)$, and $\mathcal{D}^\partial(F)$ with the weak $\mathbf{W}^\infty$-topologies, see Section 6.

The main result of [2] describes the homotopy types of $\mathcal{D}^\partial(F)$ and $\mathcal{D}^+(F)$ for a TC vector field with non-degenerate ∇F, see (i) of Proposition 2.1.

Moreover, in [9] the author calculated the homotopy type of $\mathcal{D}^+(F)$ for F being a “reduced” Hamiltonian vector field of a real homogeneous polynomial $f : \mathbb{R}^2 \to \mathbb{R}$ in two variables having a local extremum at O, see Example 2.1. Notice that in almost all the cases of f, we have that $\nabla F = 0$.

In the present paper we study $\mathcal{D}^\partial(F)$ and $\mathcal{D}^+(F)$ for TC vector fields with degenerate ∇F but satisfying certain additional “non-degeneracy” assumptions, see Theorems 2.1, 2.2, and 2.3. The obtained results are not so complete as in the non-degenerate case due to the variety of normal forms. We are able to prove that the inclusion $\mathcal{D}^\partial(F) \subset \mathcal{D}^+(F)$ induces isomorphisms of the homotopy groups π_k for $k \geq 1$ and also describe path components of $\mathcal{D}^\partial(F)$ and

$\mathcal{D}^+(F)$ with respect to distinct weak topologies. It turns out that for each $h \in \mathcal{D}^+(F)$ its path component in $\mathcal{D}^+(F)$ is uniquely determined by the 1-jet of h at O.

These results agree with the ones of [9] and they are essentially new for the case when ∇F is degenerate but is not zero.

The principal difference of the presented technique from [2] that we do not require continuity of the inverse of the so-called *shift-map* of F, see § 2.4.

In [10] for each compact surface M the author calculated the homotopy types of the stabilizers and orbits of Morse functions on M with respect to the right action of the diffeomorphism group $\mathcal{D}(M)$ of M. The results of the present paper as well as of [2] will be used in another paper to extend calculations of [10] to a large class of smooth functions with degenerate singularities on surfaces.

2. Formulation of results. Let F be a TC vector field on D^2. Denote by $\mathcal{E}(F)$ the subset of $C^\infty(D^2, D^2)$ which consists of all maps $h : D^2 \to D^2$ satisfying the following conditions:

(i) $h(o) = o$ for every orbit o of F; in particular, $h(O) = O$;

(ii) h is a local diffeomorphism at O, though it can be non-bijective and even degenerate outside O.

Evidently, $\mathcal{E}(F)$ is a subsemigroup of $C^\infty(D^2, D^2)$ with respect to the usual composition of maps. Consider the map

$$\mathbf{j} : \mathcal{E}(F) \to GL\,(2,\mathbb{R}), \quad \mathbf{j}\,(h) = J(h,O),$$

associating to every $h \in \mathcal{E}(F)$ its Jacobi matrix $J(h,O)$ at O. Let

$$L(F) := \mathbf{j}(\mathcal{E}(F))$$

be the image of $\mathbf{j}$. Then a priori $L(F)$ is a subsemigroup of $\mathrm{GL}(2,\mathbb{R})$.

Let $\mathcal{E}^+(F) = \mathbf{j}^{-1}(\mathrm{GL}^+(2,\mathbb{R}))$ be the subset of $\mathcal{E}(F)$ consisting of all maps h with positive Jacobian at O.

Let also $\mathcal{E}^\partial(F) \subset \mathcal{E}^+(F)$ be the subsemigroup consisting of all maps h fixed on ∂D^2, i.e., $h(x) = x$ for all $x \in \partial D^2$. Evidently,

$$\mathcal{D}(F) \subset \mathcal{E}(F), \quad \mathcal{D}^+(F) \subset \mathcal{E}^+(F), \quad \mathcal{D}^\partial(F) \subset \mathcal{E}^\partial(F). \tag{2.1}$$

For $r = 0, 1, \ldots, \infty$ denote by $\mathcal{E}(F)^r$ ($\mathcal{E}^+(F)^r$, etc.) the space $\mathcal{E}(F)$, ($\mathcal{E}^+(F)$, etc.) endowed with the weak $\mathbf{W}^r$-topology, see § 6. Let also $\mathcal{E}_{\mathrm{id}}(F)^r$ ($\mathcal{E}^+_{\mathrm{id}}(F)^r$ etc.) be the path component of the identity map id_{D^2} in $\mathcal{E}(F)$, ($\mathcal{E}^+(F)$, etc.) with respect to the $\mathbf{W}^r$-topology. Evidently, each $h \in \mathcal{E}(F) \backslash \mathcal{E}^+(F)$ (if it exists) changes the orientation of D^2, whence $\mathcal{E}^+(F)^r$ consists of full path components of $\mathcal{E}(F)^r$. In particular,

$$\mathcal{E}^+_{\mathrm{id}}(F)^r = \mathcal{E}_{\mathrm{id}}(F)^r, \quad r = 0, 1, \ldots, \infty.$$

It turns out that it is more convenient to work with $\mathcal{E}(F)$ instead of $\mathcal{D}(F)$. Moreover, the following theorem shows that such a replacement does not loose the information about homotopy types.

We will assume throughout that the identity map id_{D^2} is a base point and therefore omit it from the notation. For instance, we denote the n-th homotopy group $\pi_n(\mathcal{E}(F)^r, \mathrm{id}_{D^2})$ simply by $\pi_n\mathcal{E}(F)^r$ and so on.

Theorem 2.1. *Let F be a* TC *vector field on D^2. Let $\mathcal{D}$ denotes one of the groups $\mathcal{D}(F)$, $\mathcal{D}^+(F)$, or $\mathcal{D}^\partial(F)$, and $\mathcal{E}$ be the corresponding semigroup $\mathcal{E}(F)$, $\mathcal{E}^+(F)$, or $\mathcal{E}^\partial(F)$. By $\mathcal{D}^r$ (resp. $\mathcal{E}^r$) we denote the topological space $\mathcal{D}$ (resp. $\mathcal{E}$) endowed with the $\mathbf{W}^r$-topology. Then*

*(1) the inclusion $\mathcal{D}^r \subset \mathcal{E}^r$ is a **weak homotopy equivalence**[1] for $r \geq 1$;*

*(2) in the $\mathbf{W}^0$-topology, the induced map $\pi_0\mathcal{D}^0 \to \pi_0\mathcal{E}^0$ is a **surjection**;*

*(3) for each $r \geq 0$ the semigroup $\pi_0\mathcal{E}^r$ is a **group** and any two path components of $\mathcal{E}^r$ are homeomorphic to each other.*

Remark 2.1. In general, a topological semigroup may have path components which are non homeomorphic to each other. For instance, this is often so for the semigroup of continuous maps $C(X, X)$ of a topological space X with non-trivial homotopy groups, see e.g. [11].

The next result describes the relative homotopy groups of the pair $(\mathcal{E}^+(F), \mathcal{E}^\partial(F))$.

Theorem 2.2. *Let F be a* TC *vector field on D^2. Then for each $r \geq 0$,*

$$\pi_n(\mathcal{E}^+(F)^r, \mathcal{E}^\partial(F)^r) = \begin{cases} \mathbb{Z}, & n = 1, \\ 0, & \textit{otherwise}. \end{cases}$$

*Hence the inclusion $\mathcal{E}^\partial(F) \subset \mathcal{E}^+(F)$ yields **isomorphisms**,*

$$\pi_n\mathcal{E}^\partial(F)^r \to \pi_n\mathcal{E}^+(F)^r, \quad n \geq 2, \tag{2.2}$$

and we also have the following exact sequence:

$$0 \to \pi_1\mathcal{E}^\partial(F)^r \to \pi_1\mathcal{E}^+(F)^r \to \mathbb{Z} \to \pi_0\mathcal{E}^\partial(F)^r \to \pi_0\mathcal{E}^+(F)^r \to 0. \tag{2.3}$$

Our next aim (see Theorem 2.3 below) is to obtain some information about the homotopy groups of $\mathcal{E}(F)$, $\mathcal{E}^+(F)$ and $\mathcal{E}^\partial(F)$. First we recall necessary definitions and some preliminary results.

Shift map. Let $\mathbf{F} : D^2 \times \mathbb{R} \to D^2$ be the flow generated by F and

$$\varphi : C^\infty(D^2, \mathbb{R}) \to C^\infty(D^2, D^2)$$

be the map defined by

$$\varphi(\alpha)(z) = \mathbf{F}(z, \alpha(z))$$

for $\alpha \in \mathbb{C}^\infty(D^2, \mathbb{R})$ and $z \in D^2$. We will call φ the *shift map* along orbits of F and denote its image in $C^\infty(D^2, D^2)$ by $Sh(F)$,

$$Sh(F) := \varphi(C^\infty(D^2, \mathbb{R})) \subset C^\infty(D^2, D^2).$$

[1] Recall that a map $i : \mathcal{D} \to \mathcal{E}$ is a *weak homotopy equivalence* if for each $n \geq 0$ the induced map $i_n : \pi_n(\mathcal{D}, x) \to \pi_n(\mathcal{E}, x)$ of homotopy sets (groups for $n \geq 1$) is a bijection for each $x \in \mathcal{D}$.

Lemma 2.1. *The following inclusions hold true:*

$$Sh(F) \subset \mathcal{E}_{\mathrm{id}}(F)^{\infty} \subset \ldots \subset \mathcal{E}_{\mathrm{id}}(F)^{1} \subset \mathcal{E}_{\mathrm{id}}(F)^{0}. \tag{2.4}$$

If $Sh(F) = \mathrm{E}_{\mathrm{id}}(F)^r$ *for some* $r = 0, 1, \ldots, \infty$, *then*

$$\mathcal{D}_{\mathrm{id}}(F)^{\infty} = \ldots = \mathcal{D}_{\mathrm{id}}(F)^{r}, \tag{2.5}$$

whence the identity maps $\mathrm{id} : \mathcal{D}(F)^{\infty} \to \mathcal{D}(F)^{s}$ *and* $\mathrm{id} : \mathcal{E}(F)^{\infty} \to \mathcal{E}(F)^{s}$ *for* $s \geq r$ *yield the following bijections:*

$$\pi_0\mathcal{D}(F)^{\infty} \approx \ldots \approx \pi_0\mathcal{D}(F)^{r}, \quad \pi_0\mathcal{E}(F)^{\infty} \approx \ldots \approx \pi_0\mathcal{E}(F)^{r}.$$

Proof. The first inclusion in (2.4) follows from [12] (Corollary 21) and the others are evident. The fact that (2.5) is implied by the assumption $Sh(F) = \mathcal{E}_{\mathrm{id}}(F)^r$ is proved in [9].

The lemma is proved.

The following Proposition 2.1 and Example 2.1 describe some results about $\ker \mathrm{j}$, $Sh(F)$, and $\mathcal{E}_{\mathrm{id}}(F)^r$ for TC vector fields. The most complete information is given for the cases when ∇F is non-degenerate and when F is a "reduced" Hamiltonian vector field of some homogeneous polynomial on $\mathbb{R}^2$.

Proposition 2.1. *Let F be a* TC *vector field on D^2.*

(1) If $\nabla F = 0$, *then* $Sh(F) \subset \ker \mathbf{j}$.

(2) Suppose that ∇F is degenerate but is not zero. Then there are local coordinates at O in which $\nabla F = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$ *for some* $a \in \mathbb{R}\backslash\{0\}$. *Define the following subsets of* $\mathrm{GL}^{+}(2, \mathbb{R})$:

$$\mathcal{A}_{++} = \left\{ \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}, \, d \in \mathbb{R} \right\}, \quad \mathcal{A}_{--} = \left\{ \begin{pmatrix} -1 & d \\ 0 & -1 \end{pmatrix}, \, d \in \mathbb{R} \right\},$$

$$\mathcal{A}_{+-} = \left\{ \begin{pmatrix} 1 & d \\ 0 & -1 \end{pmatrix}, \, d \in \mathbb{R} \right\}, \quad \mathcal{A}_{-+} = \left\{ \begin{pmatrix} -1 & d \\ 0 & +1 \end{pmatrix}, \, d \in \mathbb{R} \right\}, \tag{2.6}$$

$$\mathcal{A} = \mathcal{A}_{++} \cup \mathcal{A}_{--} \cup \mathcal{A}_{+-} \cup \mathcal{A}_{-+}.$$

Then

$$\mathbf{j}(Sh(F)) = \mathcal{A}_{++}, \quad \mathbf{j}(\mathcal{E}^{+}(F)) \subset \mathcal{A}_{++} \cup \mathcal{A}_{--}, \quad \mathbf{j}(\mathcal{E}(F)) \subset \mathcal{A}. \tag{2.7}$$

(3) If ∇F is non-degenerate, then there are local coordinates at O in which F is given by

$$F(x, y) = \alpha(x, y) \left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \right) + \overline{X}\frac{\partial}{\partial x} + \overline{Y}\frac{\partial}{\partial y}, \tag{2.8}$$

*where α is a C^{∞}-function such that $\alpha(O) \neq 0$, and $\overline{X}, \overline{Y}$ are **flat** at O. Moreover,*

$$\mathbf{j}^{-1}(\mathrm{SO}(2)) = Sh(F) = \mathcal{E}_{\mathrm{id}}(F)^{\infty} = \ldots = \mathcal{E}_{\mathrm{id}}(F)^{0} = \mathcal{E}^{+}(F),$$

$$\mathcal{D}_{\mathrm{id}}(F)^{\infty} = \ldots = \mathcal{D}_{\mathrm{id}}(F)^{0},$$

the inclusions $\mathcal{D}^{\partial}(F) \subset \mathcal{E}^{\partial}(F)$ and $\mathcal{D}^{+}(F) \subset \mathcal{E}^{+}(F)$ are homotopy equivalences with respect to the $\mathbf{W}^{\infty}$-topologies, $\mathcal{D}^{\partial}(F)$ is contractible, and $\mathcal{D}^{+}(F)$ is homotopy equivalent to a circle.

(4) Let $\theta : D^2\backslash O \to (0, +\infty)$ be the function associating to each $z \in D^2\backslash O$ its period $\theta(z)$ with respect to F. Then θ is C^{∞} on $D^2\backslash O$ and we will call it the ***period function*** *for F.*

In the cases (1) and (2), i.e., when ∇F is degenerate, $\lim\limits_{z\to O} \theta(z) = +\infty$ and thus θ can not be even continuously extended to all of D^2. On the other hand in the case (3) θ extends to a C^{∞}-function on all of $\mathcal{D}^2$ such that $\theta(O) \neq 0$.

Proof. Statement (1) is a particular case of [13] (Lemma 5.3). (2) and (4) are established in [2].

(3) Representation (2.8) is due to F. Takens [5], and all other statements are proved in [2]. Actually, F. Takens has shown that except for (2.8) there is also an infinite series of normal forms for vector fields with a “rotation as 1-jet”, however the orbits of these vector fields are non-closed, and so they are not TC.

The proposition is proved.

Example 2.1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a real homogeneous polynomial in two variables such that $O \in \mathbb{R}^2$ is a unique critical point of f being its global minimum. Then we can write

$$f(x, y) = \prod_{j=1}^{k} Q_j^{\beta_j}(x, y), \tag{2.9}$$

where every Q_j is a positive definite quadratic form, $\beta_j \geq 1$, and

$$\frac{Q_j}{Q_{j'}} \neq \mathrm{const} \quad \text{for} \quad j \neq j'.$$

Then it is easy to see that $D = \prod\limits_{j=1}^{k} Q_j^{\beta_j - 1}$ is the greatest common divisor of partial derivatives f'_x and f'_y. Let $G = -f'_y\dfrac{\partial}{\partial x} + f'_x\dfrac{\partial}{\partial y}$ be the Hamiltonian vector field of f and

$$F = -(f'_y/D)\frac{\partial}{\partial x} + (f'_x/D)\frac{\partial}{\partial y}.$$

Then the coordinate function of F are relatively prime in the ring $\mathbb{R}[x, y]$. We will call F the ***reduced*** Hamiltonian vector field for f.

Fix $\varepsilon > 0$ and put $V = f^{-1}[0, \varepsilon]$. Then F is a TC vector field on V with singularity at O.

If $k = 1$, then ∇F is non-degenerate and a description of $\pi_0\mathcal{E}^{+}(F)^{\infty}$ and $\mathcal{E}_{\mathrm{id}}(F)^{\infty}$ is given by (3) of Proposition 2.1.

If $k \geq 2$, then $\nabla F = 0$. In this case, see [9, 14],

$$\ker \mathbf{j} = Sh(F) = \mathcal{E}_{\mathrm{id}}(F)^{\infty} = \ldots = \mathcal{E}_{\mathrm{id}}(F)^{1} \neq \mathcal{E}_{\mathrm{id}}(F)^{0} = \mathcal{E}^{+}(F),$$

$\mathcal{E}_{\mathrm{id}}(F)^{\infty}$ is contractible with respect to the $\mathbf{W}^{\infty}$-topology, and $\pi_0\mathcal{E}^{+}(F)^{\infty} \approx \mathbb{Z}_{2n}$ for some $n \geq 1$.

Now we can formulate our last result, Theorem 2.3. It gives some information about *weak* homotopy types of $\mathcal{E}(F)$ and $\mathcal{E}^{\partial}(F)$ under certain restrictions on F. The main assumption is the following one:

$$\ker \mathbf{j} \subset Sh(F). \tag{2.10}$$

It means that for every $h \in \mathcal{E}(F)$, whose 1-*jet at O is the identity*, there exists a C^{∞} shift function on all of D^2.

Theorem 2.3. *Let F be a* TC *vector field on D^2 such that ∇F is degenerate and* $\ker \mathbf{j} \subset Sh(F)$. *Let also $r \geq 1$. Then the following statements hold true.*

(1) If $\nabla F = 0$, then $Sh(F) = \ker \mathbf{j}$. Let $\mathrm{id} \in \mathrm{GL}\,(2,\mathbb{R})$ *be the unit matrix. If in addition the path component of* id *in the image $L(F) = \mathbf{j}(\mathcal{E}(F))$ of* $\mathbf{j}$ *coincides with* $\{\mathrm{id}\}$ *(e.g. when $L(F)$ is discrete), then $Sh(F) = \mathcal{E}_{\mathrm{id}}(F)^1$, and therefore* $\mathbf{j}$ *induces the isomorphisms*

$$\pi_0\mathcal{E}^+(F)^r \approx L(F) \cap \mathrm{GL}^+(2,\mathbb{R}), \quad \pi_0\mathcal{E}(F)^r \approx L(F). \tag{2.11}$$

(2) If $\nabla F = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$ *for some $a \neq 0$, then*

$$Sh(F) = \mathcal{E}_{\mathrm{id}}(F)^{\infty} = \ldots = \mathcal{E}_{\mathrm{id}}(F)^1 = \mathbf{j}^{-1}(\mathcal{A}_{++}), \tag{2.12}$$

whence $\mathbf{j}$ *yields a monomorphism, see (2.6),*

$$\pi_0\mathcal{E}(F)^{\infty} \longrightarrow \pi_0\mathcal{A} \approx \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

(3) The inclusion $\mathcal{E}^{\partial}_{\mathrm{id}}(F)^r \subset \mathcal{E}^{+}_{\mathrm{id}}(F)^r$ between the identity path components is a weak homotopy equivalence, whence from Theorem 2.2 we have the isomorphisms

$$\pi_n\mathcal{E}^{\partial}(F)^r \approx \pi_n\mathcal{E}^+(F)^r, \quad n \geq 1,$$

and the following exact sequence:

$$0 \to \mathbb{Z} \to \pi_0\mathcal{E}^{\partial}(F)^r \to \pi_0\mathcal{E}^+(F)^r \to 0. \tag{2.13}$$

(4) Suppose that the image $L(F)$ of $\mathbf{j}$ *is finite. Then $\pi_0\mathcal{E}^{\partial}(F)^r \approx \mathbb{Z}$, $\pi_0\mathcal{E}^+(F)^r \approx \mathbb{Z}_n$ for some $n \geq 0$, and (2.13) has the following form:*

$$0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \xrightarrow{\mathrm{mod}\; n} \mathbb{Z}_n \to 0.$$

If $\mathcal{E}(F) \neq \mathcal{E}^+(F)$, then $\pi_0\mathcal{E}(F)^r \approx \mathbb{D}_n$, the dihedral group.

The proof of Theorems 2.1, 2.2, and 2.3 will be given in Sections 9 – 11. All of them are based on results of [14] described in Section 7 about existence and uniqueness of shift functions for deformations in $\mathcal{E}^+(F)$, see also Proposition 8.1.

3. The inclusion $D(F) \subset \mathcal{E}(F)$. Let F be a TC vector field of D^2. The aim of this section is to prove Lemma 3.1 which allows to change elements of $\mathcal{E}(F)$ outside some neighbourhood of O to produce diffeomorphisms.

Definition 3.1. *A continuous function $f : D^2 \to [0,1]$ will be called a first strong intergal for F if*

(i) f is C^∞ on $D^2 \backslash O$ and has no critical points in $D^2\backslash O$,

(ii) $f^{-1}(0) = O$, $f^{-1}(1) = \partial D^2$, and for $c \in [0,1]$ the set $f^{-1}(c)$ is an orbit of F.

Notice that we do not require that f be C^∞ at O. It also follows from the definition that f takes distinct values on distinct orbits.

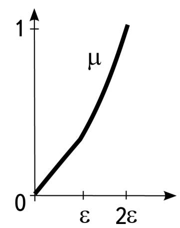

Fig. 2

A first strong integral for F always exists. For instance let $h = (h_1, h_2) : D^2 \to D^2$ be a homeomorphism which maps orbits of F onto concentric circles around O, see (1.1). If h is C^∞ on $D^2\backslash O$, then function $f = h_1^2 + h_2^2$ is the *first strong integral* for F.

For every $c \in (0,1]$ put $U_c = f^{-1}[0,c]$. Then U_c is invariant with respect to F.

Lemma 3.1. *Let $h \in \mathcal{E}(F)$. Then there exists $g \in \mathcal{D}(F)$ such that $h = g$ on some neighbourhood of O.*

Proof. By definition, $h \in \mathcal{E}(F)$ is a diffeomorphism at O, whence there exists $\varepsilon \in (0, 1/2)$ such that $h : U_{2\varepsilon} \to U_{2\varepsilon}$ is a diffeomorphism. Fix any C^∞-diffeomorphism $\mu : [0, 2\varepsilon] \to [0,1]$ such that $\mu = \mathrm{id}$ on $[0,\varepsilon]$, see Fig. 2.

We will now construct a diffeomorphism $\psi : U_{2\varepsilon} \to D^2$ fixed on U_ε and such that $f \circ \psi = \mu \circ f$, i.e., it makes the following diagram commutative:

$$\begin{array}{ccc} U_{2\varepsilon} & \xrightarrow{\psi} & D^2 \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ [0, 2\varepsilon] & \xrightarrow{\mu} & [0,1] \end{array}$$

It follows that if $c \in [0, 2\varepsilon]$ and $o = f^{-1}(c)$ is an orbit of F, then $\psi(o) = f^{-1}(\mu(c))$ is also an orbit of F. Then we can define a diffeomorphism $g : D^2 \to D^2$ by

$$g = \psi \circ h|_{U_{2\varepsilon}} \circ \psi^{-1} : D^2 \xrightarrow{\psi^{-1}} U_{2\varepsilon} \xrightarrow{h} U_{2\varepsilon} \xrightarrow{\psi} D^2.$$

Then $g \in \mathcal{D}(F)$ and since ψ is fixed on U_ε, it follows that $g = h$ on U_ε.

The construction of ψ is standard, see e.g. [15], (Ch. 1, § 3). Consider the gradient vector field ∇f of f defined on $D^2\backslash O$, and let (Φ_t) be the local flow of ∇f. Let $z \in U_{2\varepsilon}$ and γ be the orbit of z with respect to Φ. Then γ intersects the level-set $f^{-1}(\mu(f(z)))$ at a unique point $\psi(z)$, see Fig. 3. Similarly to [16] (Lemma 5.1.3) it can be shown that the correspondence $z \mapsto \psi(z)$ is a diffeomorphism of $U_{2\varepsilon} \to D^2$ if and only if so is μ.

The lemma is proved.

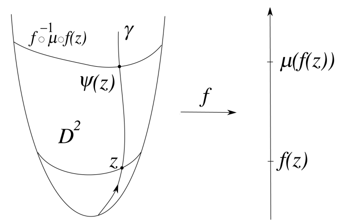

Fig. 3

4. Shift functions. Let M be a smooth (C^∞) manifold, F be a C^∞ vector field on M generating a flow $\mathbf{F} : M \times \mathbb{R} \to M$, and $\varphi : C^\infty(M, \mathbb{R}) \to C^\infty(M, M)$ be the *shift map along orbits of* F defined by $\varphi(\alpha)(z) = \mathbf{F}(z, \alpha(z))$.

If a subset $V \subset M$, a function $\alpha : V \to \mathbb{R}$, and a map $h : M \to M$ are such that $h(z) = \mathbf{F}(z, \alpha(z))$, then we will say that α is a *shift function* for h on V, and that the restriction $h|_V$ is in turn a *shift along orbits of* F *via* α.

For a C^∞-function $\alpha : M \to \mathbb{R}$ we will denote by $F(\alpha)$ the Lie derivative of α along F.

Lemma 4.1 ([12], Theorem 19). *Let* $V \subset M$ *be an open subset,* $\alpha : V \to \mathbb{R}$ *a* C^∞*-function, and* $h : V \to M$ *be a map defined by* $h(z) = \mathbf{F}(z, \alpha(z))$. *Then* h *is a local diffeomorphism at some* $z_0 \in M$ *if and only if* $F(\alpha)(z_0) \neq -1$.

Lemma 4.2 [12]. *Let* $\alpha_g, \alpha_h, \alpha_k : M \to \mathbb{R}$ *be* C^∞*-functions and*

$$g = \varphi(\alpha_g), \quad h = \varphi(\alpha_h), \quad k = \varphi(\alpha_k)$$

be the corresponding shifts. Suppose also that k *is a diffeomorphism. Then the functions*

$$\alpha_{g\circ h} = \alpha_g \circ h + \alpha_h, \quad \alpha_{k^{-1}} = -\alpha_k \circ k^{-1},$$

$$\alpha_{g\circ k^{-1}} = (\alpha_g - \alpha_k) \circ k^{-1}$$

are C^∞ *shift functions for* $g \circ h$, k^{-1}, *and* $g \circ k^{-1}$, *respectively.*

Proof. The formulae for $\alpha_{g\circ h}$ and $\alpha_{k^{-1}}$ coincide with [12] (Equations (8), (9)). They also imply the formula for $\alpha_{g\circ k^{-1}}$.

5. Shift functions for $\mathcal{E}^+(F)$. Let $B = \{(\phi, r) \in \mathbb{R}^2 : 0 \le r \le 1\}$ be a closed strip,

$$\breve{B} = \{(\phi, r) \in \mathbb{R}^2 : 0 < r \le 1\} = B \setminus \{r = 0\}$$

be a half-closed strip in $\mathbb{R}^2$, and $P : B \to D^2$ be the map given by

$$P(r, \phi) = (r\cos\phi, r\sin\phi).$$

Then $P(\breve{B}) = D^2 \backslash O$ and the restriction $P : \breve{B} \to D^2 \backslash O$ is a $\mathbb{Z}$-covering map such that the corresponding group of covering transformations is generated by the following map:

$$\eta : B \to B, \quad \eta(\phi, r) = (\phi + 2\pi, r).$$

It follows that every C^∞ map $h : D^2\backslash O \to D^2\backslash O$ lifts to a P-equivariant (i.e., commuting with η) map $\widetilde{h} : \breve{B} \to \breve{B}$ such that

$$P \circ \widetilde{h} = h \circ P.$$

Such $\widetilde{h}$ is not unique and can be replaced with $\widetilde{h} \circ \eta^n = \eta^n \circ \widetilde{h}$ for any $n \in \mathbb{Z}$.

Remark 5.1. It is well-known that if $h : D^2 \to D^2$ is a C^∞-map being a local diffeomorphism at O, and such that $h^{-1}(O) = O$, then $\widetilde{h}$ extends to a C^∞-map $\widetilde{h} : B \to B$ being a diffeomorphism near the ϕ-axis $\{r = 0\}$. We will not use this fact in the present paper.

Let F be a TC vector field on D^2. Since F is non-singular on $D^2\backslash O$, F lifts to a unique vector field G on $\breve{B}$ such that $F \circ P = TP \circ G$, where $TP : TB \to TD^2$ is the tangent map.

It is easy to see that every orbit $\widetilde{o}$ of G is non-closed, its image $o = P(\widetilde{o})$ is an orbit of F, and the map $P : \widetilde{o} \to o$ is a $\mathbb{Z}$-covering map.

Let $\mathbf{G} : \breve{B} \times \mathbb{R} \to \breve{B}$ be the flow generated by G, then we have the following commutative diagram:

$$\begin{array}{ccc} \breve{B} \times \mathbb{R} & \xrightarrow{\mathbf{G}} & \breve{B} \\ {\scriptstyle P\times \mathrm{id}_{\mathbb{R}}}\big\downarrow & & \big\downarrow{\scriptstyle P} \\ (D^2\backslash O) \times \mathbb{R} & \xrightarrow{\mathbf{F}} & D^2\backslash O \end{array} \tag{5.1}$$

In other words, $\mathbf{F}_t \circ P(\widetilde{z}) = P \circ \mathbf{G}_t(\widetilde{z})$ for all $\widetilde{z} \in \breve{B}$ and $t \in \mathbb{R}$.

In particular, if $\alpha : D^2 \to \mathbb{R}$ is a C^∞-function and $h = \varphi(\alpha)$, i.e., $h(z) = \mathbf{F}(z, \alpha(z))$, then the map $\widetilde{h} : B \to B$ given by $\mathbf{G}(\widetilde{z}, \alpha \circ P(\widetilde{z}))$ is a lifting of h. Indeed,

$$h \circ P(\widetilde{z}) = \mathbf{F}(P(\widetilde{z}), \alpha \circ P(\widetilde{z})) = P \circ \mathbf{G}(\widetilde{z}, \alpha \circ P(\widetilde{z})) = P \circ \widetilde{h}(\widetilde{z}). \tag{5.2}$$

Lemma 5.1. *Let $h \in \mathcal{E}^+(F)$. Then there exists a C^∞ shift function $\beta : D^2\backslash O \to \mathbb{R}$ for h on $D^2\backslash O$, i.e., $h(z) = \mathbf{F}(z, \beta(z))$ for $z \in D^2\backslash O$. Moreover, the set $\{\beta + n\theta : n \in \mathbb{Z}\}$ is the set of all C^∞ shift functions for h on $D^2\backslash O$, where $\theta : D^2\backslash O \to (0, \infty)$ is the period function for F, see (4) of Proposition 2.1.*

If ∇F is degenerate, then any $h \in \mathcal{E}^+(F)$ has at most one C^∞ shift function defined on all of D^2.

Proof. By the definition, $h^{-1}(O) = O$ and h is a local diffeomorphism at O. Then, as noted above, there exists a C^∞ lifting $\widetilde{h} : \breve{B} \to \breve{B}$ of h such that $P \circ \widetilde{h} = h \circ P$. Moreover, h preserves orbits of F, whence $h(\widetilde{o}) = \widetilde{o}$ for each orbit $\widetilde{o}$ of G.

Since the orbits of G are non-closed, there exists a unique C^∞ shift function $\widetilde{\beta} : \breve{B} \to \mathbb{R}$ for $\widetilde{h}$, i.e., $\widetilde{h}(\widetilde{z}) = \mathbf{G}(\widetilde{z}, \widetilde{\alpha}(\widetilde{z}))$ for all $\widetilde{z} \in \breve{B}$. Also notice that $\mathbf{G}_t$ and $\widetilde{h}$ are $\mathbb{Z}$-equivariant. This easily implies that $\widetilde{\beta}$ is $\mathbb{Z}$-invariant, whence it defines a unique C^∞-function $\beta : D^2\backslash O \to \mathbb{R}$ such that $\widetilde{\beta} = \beta \circ P$. Then it follows from (5.2) that β is a shift function for h with respect to F.

Suppose that $\alpha : D^2\backslash O \to \mathbb{R}$ is another C^∞ shift function for h on $D^2\backslash O$. Then $h(z) = \mathbf{F}(z, \alpha(z)) = \mathbf{F}(z, \beta(z))$ for $z \neq 0$, whence the difference $\alpha(z) - \beta(z)$ is a certain integer multiple of the period $\theta(z)$ of z. Since θ and $\alpha - \beta$ are C^∞ on $D^2\backslash O$, it follows that $\alpha - \beta = n\theta$ for some $n \in \mathbb{Z}$.

Conversely, for each $n \in \mathbb{Z}$ and $z \neq O$ we have that

$$\mathbf{F}(z, \beta(z) + n\theta(z)) = \mathbf{F}(\mathbf{F}(z, n\theta(z), \beta(z)) = \mathbf{F}(z, \beta(z)) = h(z).$$

Thus $\beta + n\theta$ is a shift function for h on $D^2\backslash O$.

Finally, suppose that ∇F is degenerate, and $\alpha, \beta : D^2 \to \mathbb{R}$ are two C^∞ shift function for h defined on all of D^2. Then they also shift functions for h on $D^2\backslash O$, whence $\alpha - \beta = n\theta$ for some $n \in \mathbb{Z}$. But, by Proposition 2.1, $\lim\limits_{z\to O} \theta(z) = +\infty$, while $\alpha - \beta$ is C^∞ on all of D^2. Hence $n = 0$, i.e., $\alpha = \beta$.

The lemma is proved.

6. $(\mathbf{K}, r)$-deformations. Let A and B be smooth manifolds. Then the space $C^\infty(A, B)$ admits a series $\{\mathbf{W}^r\}_{r=0}^\infty$ of *weak* topologies, see [17]. The $\mathbf{W}^0$-topology coincides with the *compact open* one. Let $J^r(A, B)$, $r < \infty$, be the manifold of r-jets of maps $A \to B$. Then there is a natural inclusion $i_r : C^\infty(A, B) \subset C^\infty(A, J^r(A, B))$ associating to each $f : A \to B$ its r-jet prolongation $j^r(f) : A \to J^r(A, B)$. Endow $C^\infty(A, J^r(A, B))$ with the $\mathbf{W}^0$-topology. Then the topology on $C^\infty(A, B)$ induced by i_r is called the $\mathbf{W}^r$-topology. Finally, the $\mathbf{W}^\infty$-topology is generated by all $\mathbf{W}^r$ for $0 \le r < \infty$.

Let $\mathcal{X} \subset C^\infty(A, B)$ be a subset, $\mathbf{K}$ be a Hausdorff, locally compact topological space, and $\omega : \mathbf{K} \to \mathcal{X}$ be a map. Then ω induces the following mapping $\Omega : \mathbf{K} \times A \to B$ defined by $\Omega(a, k) = \omega(k)(a)$. Conversely, every map $\Omega : \mathbf{K} \times A \to B$ such that $\Omega(k, \cdot) : A \to B$ belongs to $\mathcal{X}$ induces a map $\omega : \mathbf{K} \to \mathcal{X}$.

Endow $\mathcal{X}$ with the induced $\mathbf{W}^0$-topology. Then it is well known, e.g. [18] (§ 44.IV), that ω is continuous if and only if Ω is so.

Definition 6.1. *Let $r = 0, \ldots, \infty$. Then the map $\Omega : \mathbf{K} \times A \to B$ will be called a (K, r)-deformation in $\mathcal{X}$ if $\Omega_k \in \mathcal{X}$ for all $k \in \mathbf{K}$ and the induced map $\omega : K \to \mathcal{X}$ is continuous whenever $\mathcal{X}$ is endowed with the $\mathbf{W}^r$-topology. In other words, the map $j^r : \mathbf{K} \times A \to J^r(A, B)$ associating to each $(k, a) \in \mathbf{K} \times A$ the r-jet prolongation $j^r\Omega_k(a)$ of Ω_k at a is continuous.*

If $\mathbf{K} = [0, 1]$ then the $(\mathbf{K}, r)$-deformation will be called an r-homotopy.

7. Shift functions for $(\mathbf{K}, r)$-deformations. Let $\mathbf{K}$ be a Hausdorff, locally compact, and path connected topological space, F be a TC vector field on D^2,

$$\omega : \mathbf{K} \to \mathcal{E}^+(F)^r$$

be a continuous map into some $\mathbf{W}^r$-topology of $\mathcal{E}^+(F)$, and

$$\Omega : \mathbf{K} \times D^2 \to D^2, \quad \Omega(k, z) = \omega(k)(z) \tag{7.1}$$

be the corresponding $(\mathbf{K}, r)$-deformation in $\mathcal{E}^+(F)$, so $\Omega_k \in \mathcal{E}^+(F)$ for all $k \in \mathbf{K}$.

Then by Lemma 5.1 for each $k \in \mathbf{K}$ the map Ω_k has a (not unique) C^∞ shift function Λ_k defined on $D^2\backslash O$. Thus we can define a map $\Lambda : \mathbf{K} \times (D^2\backslash O) \to \mathbb{R}$ by $\Lambda(k, z) = \Lambda_k(z)$ which, in general, is not even continuous, though it is C^∞ for each k.

Lemma 7.1 below is a particular case of results of [14], see also [12] (Theorem 25). It shows that Λ_k can always be chosen so that Λ becomes continuous in (k, z).

Definition 7.1. *A $(\mathbf{K}, r)$-deformation $\Lambda : \mathbf{K} \times (D^2\backslash O) \to \mathbb{R}$ satisfying*

$$\Omega(k, z) = \mathbf{F}(z, \Lambda(k, z)) \quad \forall (k, z) \in \mathbf{K} \times (D^2\backslash O) \tag{7.2}$$

will be called a ***shift function*** *for the $(\mathbf{K}, r)$-deformation Ω.*

Lemma 7.1 [14]. *Let $k_0 \in \mathbf{K}$ and Λ_{k_0} be any C^∞ shift function for Ω_{k_0}. Then there exists at most one shift function $\Lambda : \mathbf{K} \times (D^2 \backslash O) \to \mathbb{R}$ for Ω such that $\Lambda(k_0, z) = \Lambda_{k_0}(z)$.*

Moreover, if $\mathbf{K}$ is simply connected, i.e., $\pi_1 \mathbf{K} = 0$, then any shift function Λ_{k_0} for Ω_{k_0} uniquely extends to a shift function

$$\Lambda : \mathbf{K} \times (D^2 \backslash O) \to \mathbb{R}$$

for Ω.

This lemma will be used in the proofs of Theorems 2.1 and 2.2. For the proof of Theorem 2.3 we will also need the following Lemmas 7.2 and 7.3.

Suppose now that $\omega(\mathbf{K}) \subset Sh(F)$, that is, for each $k \in \mathbf{K}$ the map Ω_k has a C^∞ shift function defined on all of D^2. Let $\Lambda : \mathbf{K} \times (D^2 \backslash O) \to \mathbb{R}$ be a shift function for Ω such that Λ_{k_0} for some $k_0 \in \mathbf{K}$ smoothly extends to all of D^2. The following lemma gives sufficient conditions when any other shift function $\Lambda_k = \Lambda(k, \cdot)$ smoothly extends to all of D^2. Again it is a particular case of results of [14].

Lemma 7.2 [14]. *Let $\Omega : \mathbf{K} \times D^2 \to D^2$ be a $(\mathbf{K}, r)$-deformation admitting a shift function $\Lambda : \mathbf{K} \times (D^2 \backslash O) \to \mathbb{R}$. Suppose that*

(i) $\ker \mathbf{j} \subset Sh(F)$,

(ii) $\Omega_0 = \mathrm{id}_{D^2}$ *for some* $k_0 \in \mathbf{K}$, *and*

(iii) $\mathbf{j}\,\Omega_k = \mathrm{id}$, *i.e.,* $\Omega_k \in \ker \mathbf{j} \subset Sh(F)$, *for all* $k \in \mathbf{K}$.

Then for each $k \in \mathbf{K}$ the function $\Lambda_k : D^2 \backslash O \to \mathbb{R}$ extends to a C^∞-function on all of D^2, though the induced function $\Lambda : \mathbf{K} \times D^2 \to \mathbb{R}$ is not necessarily continuous.

Finally, we present a sufficient condition when a map into $Sh(F)$ can be deformed into $\ker \mathbf{j}$.

Lemma 7.3. *Let $\mathbf{K}$ be path connected and simply connected, $r \geq 1$, and $\omega : \mathbf{K} \to Sh(F)$ be a continuous map into the $\mathbf{W}^r$-topology of $Sh(F)$. Suppose ∇F is degenerate and $\ker \mathbf{j} \subset Sh(F)$. Then there exists a homotopy $\mathbf{B} : I \times \mathbf{K} \to Sh(F)$ such that $\mathbf{B}_0 = \omega$, $\mathbf{B}_1(\mathbf{K}) \subset \ker \mathbf{j}$, and $\mathbf{B}_t(k) = \omega(k)$ for all k such that $\omega(k) \in \ker \mathbf{j}$.*

Proof. If $\nabla F = 0$, then $Sh(F) = \ker \mathbf{j}$ and there is nothing to prove.

Suppose that $\nabla F = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$ for some $a \neq 0$. Let $\Omega : \mathbf{K} \times D^2 \to D^2$ be the corresponding $(\mathbf{K}, r)$-deformation in $Sh(F)$. Then, by (2.7),

$$\mathbf{j}\,(\Omega_k) = \begin{pmatrix} 1 & a\,\tau(k) \\ 0 & 1 \end{pmatrix}, \quad k \in I,$$

for some $\tau(k) \in \mathbb{R}$. Since Ω is an r-homotopy with $r \geq 1$, it follows that the function $\tau : \mathbf{K} \to \mathbb{R}$ is continuous. Moreover, $\tau(k) = 0$ if and only if $\Omega_k \in \ker \mathbf{j}$.

Define the homotopy $\mathbf{B} : I \times \mathbf{K} \to Sh(F)$ by

$$\mathbf{B}(t, k)(z) = \mathbf{F}\,(\Omega(k, z), -t\tau(k)).$$

Then it is easy to see that $\mathbf{B}$ satisfies the statement of our lemma.

8. Deformations in $\mathcal{E}^+(F)$. In this section we prove the key Proposition 8.1 that will imply Theorems 2.1, 2.2, and 2.3.

Let $\mathbf{K}$ be a Hausdorff, locally compact topological space and

$$\omega : \mathbf{K} \to \mathcal{E}^+(F)^r$$

be a continuous map into some $\mathbf{W}^r$-topology of $\mathcal{E}^+(F)$. Our aim is to show that under certain mild assumptions ω is homotopic to a map into $\mathcal{D}^\partial(F) = \mathcal{D}^+(F) \cap \mathcal{E}^\partial(F)$ so that the intersections of $\omega(\mathbf{K})$ with $\mathcal{D}^\partial(F)$, $\mathcal{D}^+(F)$ and $\mathcal{E}^\partial(F)$ remain in the corresponding spaces during the homotopy. More precisely the following result holds true:

Proposition 8.1. *Suppose that either*
(i) $\mathbf{K}$ *is a point and* $r \geq 0$, *or*
(ii) $\mathbf{K}$ *is compact, path connected, and simply connected, and* $r \geq 1$.
Let $\mathbf{L} \subset \mathbf{K}$ *be a (possibly empty) subset such that* $\omega(\mathbf{L}) \subset \mathcal{D}^+(F)$, *and* $\mathbf{P} \subset \mathbf{K}$ *be a connected subset such that* $\omega(\mathbf{P}) \subset \mathcal{E}^\partial(F)$. *Thus we can regard* ω *as a map of triples,*

$$\omega : (\mathbf{K}; \mathbf{L}, \mathbf{P}') \to (\mathcal{E}^+(F)^r; \mathcal{D}^+(F)^r, \mathcal{E}^\partial(F)^r).$$

Then there exists a homotopy of triples,

$$\mathbf{A}_t : (\mathbf{K}; \mathbf{L}, \mathbf{P}') \to (\mathcal{E}^+(F)^r; \mathcal{D}^+(F)^r, \mathcal{E}^\partial(F)^r), \quad t \in I,$$

such that

$$\mathbf{A}_0 = \omega \quad \textit{and} \quad \mathbf{A}_1(\mathbf{K}) \subset \mathcal{D}^\partial(F). \tag{8.1}$$

The phrase ***homotopy of triples*** *means that*

$$\mathbf{A}_t(\mathbf{L}) \subset \mathcal{D}^+(F), \quad \mathbf{A}_t(\mathbf{P}) \subset \mathcal{E}^\partial(F), \tag{8.2}$$

and therefore $\mathbf{A}_t(\mathbf{L} \cap \mathbf{P}) \subset \mathcal{D}^+(F) \cap \mathcal{E}^\partial(F) = \mathcal{D}^\partial(F)$ *for all* $t \in I$.

The proof will be given at the end of this section. Let

$$\Omega : \mathbf{K} \times D^2 \to D^2, \quad \Omega(k, z) = \omega(k)(z)$$

be the corresponding $(\mathbf{K}, r)$-deformation in $\mathcal{E}^+(F)$. Then by Lemma 7.1 there exists a shift function $\Lambda : \mathbf{K} \times (D^2 \backslash O) \to \mathbb{R}$ for Ω. The deformation of Ω_k we will be produced via a deformation of Λ_k.

Let $a, b \in (0,1)$ be such that $a < b$, $f : D^2 \to [0,1]$ be the first strong integral for F, see Definition 3.1, and $\nu : [0,1] \to [0,1]$ be a C^∞-function such that $\nu[0,a] = 1$ and $\nu[b,1] = 0$. Define a function $\alpha : \mathbf{K} \times (D^2 \backslash O) \to \mathbb{R}$ by

$$\alpha(k, z) = \nu \circ f(z) \cdot \Lambda(k, z), \quad (k, z) \in \mathbf{K} \times (D^2 \backslash O), \tag{8.3}$$

the map $\Omega' : \mathbf{K} \times D^2 \to D^2$ by

$$\Omega'(k, z) = \begin{cases} \mathbf{F}(z, \alpha(k, z)), & z \neq O, \\ O, & z = O, \end{cases} \tag{8.4}$$

and a homotopy $A : I \times \mathbf{K} \times D^2 \to D^2$ by

$$A(t,k,z) = \begin{cases} \mathbf{F}\,(z,(1-t)\alpha(k,z) + t\Lambda(k,z)), & z \neq O, \\ \\ O, & z = O. \end{cases} \tag{8.5}$$

Lemma 8.1. *For $(t,k) \in I \times \mathbf{K}$ denote $A_t = A(t,\cdot,\cdot) : \mathbf{K} \times D^2 \to D^2$ and $A_{t,k} = A(t,k,\cdot) : D^2 \to D^2$. Then*

(a) $A_0 = \Omega$, $A_1 = \Omega'$, and A_t is a $(\mathbf{K}, r)$-deformation in $\mathcal{E}(F)$ for each $t \in I$.

(b) Ω'_k is fixed on $D^2 \setminus U_b$ for all $k \in \mathbf{K}$. In particular, $A_1 = \Omega'$ is a deformation in $\mathcal{E}^{\partial}(F)$.

(c) If for some $(k,z) \in \mathbf{K} \times D^2$ the map Ω_k is a local diffeomorphism at z, then so is $A_{t,k}$ for each $t \in I$.

(d) Denote $\mathbf{Z} = \Lambda^{-1}(0) \subset \mathbf{K} \times (D^2 \backslash O)$. Thus $\Omega(k,z) = z$ for all $(k,z) \in \mathbf{Z}$. Then

$$A(t,k,z) = z \quad \forall\, t \in I, \quad (k,z) \in \mathbf{Z}.$$

(e) Let $\mathbf{P} \subset \mathbf{K}$ be a connected subset such that Ω_k is fixed on ∂D^2 for each $k \in \mathbf{P}$ and $\Lambda_{k_0}|_{\partial D^2} = 0$ for some $k_0 \in \mathbf{P}$. Then $A_{t,k}$ is also fixed on ∂D^2 for all $(t,k) \in I \times \mathbf{P}$.

Thus A induces a homotopy

$$\mathbf{A}_t : \mathbf{K} \to \mathcal{E}^+(F)^r, \quad \mathbf{A}_t(k)(z) = A(t,k,z) \tag{8.6}$$

such that $\mathbf{A}_0 = \omega$, and $\mathbf{A}_1(\mathbf{K}) \subset \mathcal{E}^{\partial}(F)$.

Proof. Statements (a) and (b) follow from (8.3) – (8.5).

(c) Denote

$$\beta_{t,k}(z) = (1-t)\alpha(k,z) + t\Lambda(k,z) = ((1-t)\nu \circ f(z) + t) \cdot \Lambda_k(z). \tag{8.7}$$

Then by (8.5) $\beta_{t,k}$ is a shift function for $A_{t,k}$ on $D^2 \backslash O$.

The assumption that Ω_k is a local diffeomorphism at z means that

$$F(\Lambda_k)(z) > -1, \tag{8.8}$$

see Lemma 4.1. Therefore by that lemma it suffices to verify that $F(\beta_{t,k})(z) > -1$ for all $t \in I$.

Notice that

$$F\left(((1-t)\nu \circ f + t) \cdot \Lambda_k\right) = (1-t)F(\nu \circ f) + ((1-t)\nu \circ f + t)\, F(\Lambda_k).$$

The first summand is zero since f and, therefore, $\nu \circ f$ are constant along orbits of F. Moreover, $0 \leq \nu(z) \leq 1$, whence we get from (8.8) that the second summand is > -1. Hence $F(\beta_{t,k})(z) > > -1$ for all $k \in \mathbf{K}$.

(d) If $\Lambda(k,z) = 0$ for some $(k,z) \in \mathbf{K} \times (D^2 \backslash O)$, then by (8.7) $\beta_{t,k}(z) = 0$, whence $A(t,k,z) = \mathbf{F}\,(z, \beta_{t,k}(z)) = \mathbf{F}\,(z, 0) = z$.

(e) If Ω_k is fixed on ∂D^2 for some $k \in \mathbf{K}$, then Λ_k takes on ∂D^2 a constant value,

$$\Lambda_k(\partial D^2) = n_k \cdot \theta(\partial D^2),$$

for some $n_k \in \mathbb{Z}$. Since $\mathbf{P}$ is connected, Λ is continuous on $\mathbf{P} \times \partial D^2$, and the set of possible values of Λ_k on ∂D^2 is discrete, it follows that Λ is constant on $\mathbf{P}\times\partial D^2$. In particular, $\Lambda|_{\mathbf{P}\times\partial D^2} = \Lambda_{k_0}|_{\mathbf{P}\times\partial D^2} = 0$. Then by (d) $A_{t,k}$ is fixed on ∂D^2 for all $(k,t) \in I \times \mathbf{P}$.

The lemma is proved.

Proof of Proposition 8.1. We will find $a, b \in (0,1)$ and a shift function $\Lambda : \mathbf{K}\times(D^2\backslash O) \to \mathbb{R}$ for Ω such that the corresponding homotopy $\mathbf{A}_t$ constructed in Lemma 8.1 will satisfy (8.1) and (8.2).

Choice of Λ. Let $\Lambda' : \mathbf{K} \times (D^2\backslash O) \to \mathbb{R}$ be any shift function for Ω. Since Ω_{k_0} is fixed on ∂D^2 for some $k_0 \in \mathbf{P}$, we have that $\Lambda'_{k_0}|_{\partial D^2} = n\theta(\partial D^2)$ for some $n \in \mathbb{Z}$.

Define another function $\Lambda : \mathbf{K} \times (D^2\backslash O) \to \mathbb{R}$ by

$$\Lambda(k,z) = \Lambda'(k,z) - n\theta(\partial D^2).$$

Then Λ is also a shift function for Ω in the sense of (7.2) and satisfies

$$\Lambda_{k_0}|_{\partial D^2} = 0. \tag{8.9}$$

Choice of $a, b \in (0,1)$. Notice that $\Omega_k(U_b) = U_b$ for all $k \in \mathbf{K}$ and $b \in (0,1]$. We claim that *there exists* $b \in (0,1)$ *such that the map* $\Omega_k : U_b \to U_b$ *is a diffeomorphism for all* $k \in I$. Indeed, by the definition of $\mathcal{E}^+(F)$, the map Ω_k is a diffeomorphism at O for each $k \in \mathbf{K}$. This implies existence of b in the case (i), i.e., when $\mathbf{K}$ is a point. In the case (ii) the assumption $r \geq 1$ means that the partial derivatives of Ω_k are continuous functions on $\mathbf{K} \times D^2$. Then existence of b now follows from compactness of $\mathbf{K} \times D^2$.

Take arbitrary $a \in (0,b)$ and let $\mathbf{A}_t$ be a homotopy constructed in Lemma 8.1 for Λ and a, b. We claim that $\mathbf{A}$ satisfies (8.1) and (8.2).

By (a) of Lemma 8.1 $\mathbf{A}_0 = \omega$.

Let us prove that $\mathbf{A}_1(\mathbf{K}) \subset \mathcal{D}^\partial(F)$, i.e., for each $k \in \mathbf{K}$ the map $\mathbf{A}_{1,k} = \Omega'_k$ is a diffeomorphism of D^2 fixed on ∂D^2. By (b) of Lemma 8.1 Ω'_k is fixed even on $D^2\backslash U_b$. Moreover, by the assumption on b, we have that $\Omega_k : U_b \to U_b$ is a diffeomorphism, whence, by (c) of Lemma 8.1, $A_{t,k} = \mathbf{A}_t(k)$ is also a self-diffeomorphism of U_b and therefore of all D^2.

To show that $\mathbf{A}_t(\mathbf{L}) \subset \mathcal{D}^+(F)$ notice that by the assumption $\Omega_l : D^2 \to D^2$ is a diffeomorphism for all $l \in \mathbf{L}$. Then again, by (c) of Lemma 8.1, $A_{t,l} = \mathbf{A}_t(l)$ is also a self-diffeomorphism D^2 for all $l \in \mathbf{L}$, i.e., $\mathbf{A}_t(\mathbf{L}) \subset \mathcal{D}^+(F)$.

Finally, the inclusion $\mathbf{A}_t(\mathbf{P}) \subset \mathcal{E}^\partial(F)$ follows from (8.9) and (e) of Lemma 8.1.

9. Proof of Theorem 2.1. First we prove (1) and (2) for the inclusions $\mathcal{D}^+(F) \subset \mathcal{E}^+(F)$ and $\mathcal{D}^\partial(F) \subset \mathcal{E}^\partial(F)$. Then we establish (3) and deduce from it (1) and (2) for the inclusion $D(F) \subset \mathcal{E}(F)$.

(1) We have to show that $\pi_n(\mathcal{E}^r, \mathcal{D}^r) = 0$ for all $n \geq 0$ if $r \geq 1$. Then the result will follow from the exact homotopy sequence for the pair $(\mathcal{E}^r, \mathcal{D}^r)$.

Let $\omega : (I^n, \partial I^n) \to (\mathcal{E}^r, \mathcal{D}^r)$ be a continuous map representing some element of the relative homotopy set $\pi_n(\mathcal{E}^r, \mathcal{D}^r)$. Our aim is to show that ω is homotopic as a map of pairs to a map into $\mathcal{D}$, i.e., $\omega = 0$ in $\pi_n(\mathcal{E}^r, \mathcal{D}^r)$, whence we will get $\pi_n(\mathcal{E}^r, \mathcal{D}^r) = 0$.

Inclusion $\mathcal{D}^+(F) \subset \mathcal{E}^+(F)$**.** If $r \geq 1$, then applying Proposition 8.1 to the case $\mathbf{K} = I^n$, $\mathbf{L} = \partial I^n$ we obtain that ω is homotopic as a map of pairs to a map into $\mathcal{D}^+(F)$.

Inclusion $\mathcal{D}^{\partial}(F) \subset \mathcal{E}^{\partial}(F)$**.** Since

$$(\mathcal{E}^{\partial}(F), \mathcal{D}^{\partial}(F)) \subset (\mathcal{E}^{+}(F), \mathcal{D}^{+}(F)),$$

we see that ω is also an element of $\pi_n(\mathcal{E}^{+}(F)^r, \mathcal{D}^{+}(F)^r)$, which, as just shown, is trivial. Then Proposition 8.1 can be applied to the case $\mathbf{K} = \mathbf{P} = I^n$ and $\mathbf{L} = \partial I^n$, and we obtain that ω is homotopic as a map of pairs $(K, L) \to (\mathcal{E}^{\partial}(F)^r, \mathcal{D}^{\partial}(F)^r)$ to a map into $\mathcal{D}^{\partial}(F)^r$. Hence $\omega = 0$ in $\pi_n(\mathcal{E}^{\partial}(F)^r, \mathcal{D}^{\partial}(F)^r)$.

(2) We have to show that the map $\pi_0\mathcal{D}^0 \to \pi_0\mathcal{E}^0$ is surjective for all $r \geq 0$. Let $h \in \mathcal{E}^{+}(F)$. It can be regarded as a map from the set $\mathbf{K}$ consisting of a unique point into $\mathcal{E}(F)$,

$$\omega : \mathbf{K} \to \mathcal{E}, \quad \omega(\mathbf{K}) = h.$$

Then applying (i) of Proposition 8.1 we obtain that ω is C^{∞}-homotopic to a map into $\mathcal{D}^{\partial}(F)$, whence the inclusion $\mathcal{D}^{\partial}(F) \subset \mathcal{E}^{+}(F)$ yields a surjecitve map $\pi_0\mathcal{D}^{\partial}(F)^r \to \pi_0\mathcal{E}^{+}(F)^r$ for all $r \geq 0$. Therefore in the following diagram induced by inclusions all arrows are surjective:

$$\begin{array}{ccc} \pi_0\mathcal{D}^{\partial}(F)^r & \longrightarrow & \pi_0\mathcal{E}^{\partial}(F)^r \\ \Big\downarrow & & \Big\downarrow \\ \pi_0\mathcal{D}^{+}(F)^r & \longrightarrow & \pi_0\mathcal{E}^{+}(F)^r \end{array} \tag{9.1}$$

The proof of the surjectivity $\pi_0 D(F)^r \to \pi_0\mathcal{E}(F)^r$ is the same as in (1).

(3) It is well known and is easy to prove that *for a topological semigroup $\mathcal{E}$ the set $\pi_0\mathcal{E}$ of path components of $\mathcal{E}$ admits a semigroup structure such that the natural projections $\mathcal{E} \to \pi_0\mathcal{E}$ is a semigroup homomorphism. If $\mathcal{E}$ is a group, then so is $\pi_0\mathcal{E}$.*

If $\mathcal{D} \subset \mathcal{E}$ is a subsemigroup, then the induced map $\pi_0\mathcal{D} \to \pi_0\mathcal{E}$ is a semigroup homomorphism.

In our case $\mathcal{E}^r$ is a topological semigroup and $\mathcal{D}^r$ is a topological group. From (1) we get that for $r \geq 1$ the homomorphism $\pi_0\mathcal{D} \to \pi_0\mathcal{E}$ is a bijection, whence it is a semigroup isomorphism. But $\pi_0\mathcal{D}$ is a group, whence so is $\pi_0\mathcal{E}$.

Let us prove that *all path components of $\mathcal{E}$ are homeomorphic to each other.* By (1) and (2) the map $i_0 : \pi_0\mathcal{D} \to \pi_0\mathcal{E}$ is surjective for each of $\mathbf{W}^r$-topologies, $r \geq 0$. In particular, this implies that each path component of $\mathcal{E}$ contains an invertible element. Now the result is implied by the following statement.

Claim 9.1. *Let $\mathcal{E}$ be a topological semigroup such that each path component of $\mathcal{E}$ contains an invertible element. Then all path components of $\mathcal{E}$ are homeomorphic each other.*

Moreover, let $\mathcal{D}$ be the subgroup consisting of all invertible elements. Then for any two path components $\mathcal{E}_1$ and $\mathcal{E}_2$ there exists a homeomorphism $Q : \mathcal{E}_1 \to \mathcal{E}_2$ such that $Q(\mathcal{E}_1 \cap \mathcal{D}) = \mathcal{E}_2 \cap \mathcal{D}$.

Proof. Let $h_1 \in \mathcal{E}_1$ and $h_2 \in \mathcal{E}_2$ be any invertible elements. Then we can define a homeomorphism $Q : \mathcal{E}_1 \to \mathcal{E}_2$ by $Q(h) = h_2 \cdot h_1^{-1} \cdot h$. Evidently it is continuous, its inverse is given by $Q^{-1}(g) = h_1 \cdot h_2^{-1} \cdot g$, and $Q(\mathcal{E}_1 \cap \mathcal{D}) = \mathcal{E}_2 \cap \mathcal{D}$.

(1) and (2) for the inclusion $\mathcal{D}(F) \subset \mathcal{E}(F)$. Put

$$\mathcal{D}' = \mathcal{D}(F) \backslash \mathcal{D}^{+}(F), \quad \mathcal{E}' = \mathcal{E}(F) \backslash \mathcal{E}^{+}(F).$$

Then $\mathcal{E}'$ consists of full path components of $\mathcal{E}(F)$ with respect each of $\mathbf{W}^r$-topologies. Hence we have to prove our statement for the inclusion $\mathcal{D}' \subset \mathcal{E}'$. We can also assume that $\mathcal{E}' \neq \varnothing$. Then it follows from Lemma 3.1 that $\mathcal{D}' \neq \varnothing$ as well. Let $g \in \mathcal{D}'$. Then we can define a map $Q : \mathcal{E}' \to \mathcal{E}^+(F)$ by $Q(h) = g^{-1} \circ h$ for $h \in \mathcal{E}'$. Evidently, Q is a homeomorphism onto with respect to any of $\mathbf{W}^r$-topologies. Moreover, $Q(\mathcal{D}') = \mathcal{D}^+(F)$. Hence, $\pi_n(\mathcal{E}', \mathcal{D}') = \pi_n(\mathcal{E}^+(F), \mathcal{D}^+(F))$. It remains to note that by (1) and (2) $\pi_n(\mathcal{E}^+(F), \mathcal{D}^+(F)) = 0$ if either $r \geq 1$ and $n \geq 0$, or $r = 0$ and $n = 0$.

10. Proof of Theorem 2.2. The proof is similar to the one given in Section 9. Let

$$\omega : (I^n, \partial I^n) \to (\mathcal{E}^+(F)^r, \mathcal{E}^+(F)^r)$$

be a continuous map being a representative of some element in the relative homotopy set $\pi_n(\mathcal{E}^+(F)^r, \mathcal{E}^\partial(F)^r)$.

We have to show that ω is homotopic as a map of pairs to a map into $\mathcal{E}^\partial(F)$. Again we will apply Proposition 8.1 but now the situation is more complicated.

For $n \neq 1$ denote $\mathbf{K} = I^n$ and $\mathbf{P} = \partial I^n$. Then $\mathbf{P}$ is path connected and, by Proposition 8.1, ω is homotopic as a map of pairs,

$$(\mathbf{K}, \mathbf{P}) \to (\mathcal{E}^+(F)^r, \mathcal{E}^\partial(F)^r),$$

to a map into $\mathcal{E}^\partial(F)$. In this case we can take b arbitrary, and therefore the arguments hold for the case $r = 0$ as well. This implies $\pi_n(\mathcal{E}^+(F)^r, \mathcal{E}^\partial(F)^r) = 0$ for all $n \neq 1$ and $r \geq 0$.

Suppose $n = 1$. Then $I^1 = [0,1]$ and $\partial I^1 = \{0,1\}$ is not connected, so Proposition 8.1 can be applied only to each of the path components $\{0\}$ and $\{1\}$ of ∂I^1. Actually this is the reason why

$$\pi_1(\mathcal{E}^+(F)^r, \mathcal{E}^\partial(F)^r, \mathrm{id}_{D^2}) \approx \mathbb{Z}. \tag{10.1}$$

To prove (10.1), use $0 \in I^1$ and $\mathrm{id}_{D^2} \in \mathcal{E}^+(F)$ as base points, and thus assume that $\omega(0) = \mathrm{id}_{D^2}$. Consider the (I^1, r)-deformation in $\mathcal{E}^+(F)^r$ corresponding to ω,

$$\Omega : I^1 \times D^2 \to D^2, \quad \Omega(k, z) = \omega(k)(z).$$

Then $\Omega_0 = \mathrm{id}_{D^2}$ and therefore the zero function $\Lambda_0 = 0$ is a shift function for Ω. By Lemma 7.1, Λ_0 extends to a *unique* (I^1, r)-deformation

$$\Lambda : I^1 \times (D^2 \backslash O) \to \mathbb{R}$$

being a shift function for Ω on $D^2 \backslash O$ in the sense of that lemma. In particular the last function Λ_1 is a shift function for $\Omega_1 \in \mathcal{E}^\partial(F)$ which is fixed on ∂D^2. Then by Proposition 8.1 applied to $\mathbf{P} = \{1\}$, we get that Λ_1 takes constant value on ∂D^2 being an integer multiple of the period of orbit ∂D^2 with respect to F. Thus

$$\Lambda(\partial D^2) = \rho_\omega \cdot \theta(\partial D^2)$$

for some $\rho_\omega \in \mathbb{Z}$. Evidently, ρ_ω counts the number of "full rotations" of ∂D^2 during the homotopy Ω. We claim that the correspondence $\rho : \omega \mapsto \rho_\omega$ yields an isomorphism (10.1).

It is easy to see that ρ induces a *surjective homomorphism*

$$R : \pi_1(\mathcal{E}^+(F)^r, \mathcal{E}^\partial(F)^r) \to \mathbb{Z}.$$

To show that R is a monomorphism suppose that $\rho(\omega) = 0$, so $\Lambda_0(\partial D^2) = \Lambda_1(\partial D^2) = 0$. Then it follows from (d) of Lemma 8.1 that

$$A(t,0,z) = A(t,1,z) = z$$

for all $t \in I$ and $z \in \partial D^2$. In other words, $\mathbf{A}_t(0), \mathbf{A}_t(1) \in \mathcal{E}^\partial(F)^r$ for all $t \in I$.

Thus ω is homotopic to a map into $\mathcal{E}^\partial(F)$ via a homotopy relatively to ∂I^1, and therefore it represents a trivial element of $\pi_1(\mathcal{E}^+(F)^r, \mathcal{E}^\partial(F)^r)$. This implies that R is an isomorphism.

11. Proof Theorem 2.3. Let F be a TC vector field on D^2 such that ∇F is degenerate and $\ker \mathbf{j} \subset Sh(F)$.

(1) If $\nabla F = 0$, then the relation $Sh(F) = \ker \mathbf{j}$ follows from the assumption $Sh(F) \supset \ker \mathbf{j}$ and (1) of Proposition 2.1.

Suppose that $\{\mathrm{id}\}$ is the path component of id in $L(F)$. Since j is continuous from $\mathbf{W}^r$-topology of $\mathcal{E}(F)$ for $r \geq 1$, it follows that $\mathcal{E}_{\mathrm{id}}(F)^r \subset \ker \mathbf{j} = Sh(F) \subset \mathcal{E}_{\mathrm{id}}(F)^r$. This also implies (2.11).

(2) Suppose that $\nabla F = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$ for some $a \neq 0$. We have to show that $Sh(F) = \mathcal{E}_{\mathrm{id}}(F)^1 = \mathbf{j}^{-1}(A_{++})$.

It follows from the definition, see (2.6), that $\mathcal{A}$ is a group, $\mathcal{A}_{++}$ is its unity component in $GL^+(2,\mathbb{R})$, and A_{--}, A_{+-}, A_{-+} are another path components of $\mathcal{A}$. Since $\mathbf{j}$ is continuous in the $\mathbf{W}^r$-topology of $\mathcal{E}^+(F)$ for $r \geq 1$, it follows that the inverse images of these path components are open-closed in $\mathcal{E}(F)$. On the other hand, $Sh(F)$ is path connected in all $\mathbf{W}^r$-topologies, as a continuous image of a path connected space $C^\infty(M,\mathbb{R})$, whence

$$Sh(F) \subset \mathcal{E}_{\mathrm{id}}(F)^1 \subset \mathbf{j}^{-1}(A_{++},).$$

Conversely, let $h \in \mathbf{j}^{-1}(A_{++},)$, so $\mathbf{j}\,(h) = \begin{pmatrix} 1 & a\tau \\ 0 & 1 \end{pmatrix}$ for some $\tau \in \mathbb{R}$. We have to show that $h \in Sh(F)$.

Evidently $\mathbf{j}\,(h)$ coincides with

$$\exp(\tau \cdot \nabla F) = \exp \begin{pmatrix} 0 & a\tau \\ 0 & 0 \end{pmatrix}.$$

Consider the flow $(\mathbf{F}_t)$ of F. Then $\mathbf{j}\,(\mathbf{F}_t) = \exp \begin{pmatrix} 0 & at \\ 0 & 0 \end{pmatrix}$ for all $t \in \mathbb{R}$. Hence $\mathbf{j}\,(\mathbf{F}_\tau) = \mathbf{j}\,(h)$,

Define the map $g : D^2 \to D^2$ by $g(z) = \mathbf{F}(h(z), -\tau) = \mathbf{F}_{-\tau} \circ h$. Then $\mathbf{j}\,(g) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, i.e., $g \in \ker \mathbf{j} \subset Sh(F)$.

In other words, $g(z) = \mathbf{F}(z, \alpha(z))$ for some $\alpha \in C^\infty(D^2, \mathbb{R})$. Put $\beta(z) = \alpha(z) + \tau$. Then $h(z) = \mathbf{F}\,(z, \beta(z))$, i.e., $h \in Sh(F)$.

(3) Due to (2.2) we have only to show that the mapping

$$i_1 : \pi_1 \mathcal{E}^{\partial}(F)^r \to \pi_1 \mathcal{E}^{+}(F)^r$$

induced by the inclusion is an isomorphism. Moreover, by exactness of the sequence (2.3) it remains to show that i_1 is surjective.

Let $\omega : I \to \mathcal{E}^{+}(F)^r$ be a continuous map representing a loop in $\mathcal{E}^{+}(F)^r$, i.e.,

$$\omega(0) = \omega(1) = \mathrm{id}_{D^2}. \tag{11.1}$$

We have to show that ω is r-homotopic relatively to ∂I to a map into $\mathcal{E}^{\partial}(F)^r$.

It follows from (11.1) that $\omega(I)$ is contained in $\mathcal{E}^{+}_{\mathrm{id}}(F)^r$ which, by (1) and (2), coincides with $Sh(F)$. Thus $\omega(I) \subset Sh(F)$. Moreover, $\omega(\partial I) \subset \ker \mathbf{j}$. Then, by Lemma 7.3, ω is homotopic to a map into $\ker \mathbf{j}$ relatively to ∂I. Hence we can assume that ω is a loop in $\ker \mathbf{j}$.

Consider the (I, r)-deformation corresponding to ω,

$$\Omega : I \times D^2 \to D^2, \quad \Omega(t, z) = \omega(t)(z).$$

Then $\Omega_0 = \Omega_1 = \mathrm{id}_{D^2}$ and $\Omega_k \in \ker \mathbf{j} \subset Sh(F)$ for all $k \in I$.

In particular, every Ω_k has a C^∞ shift function $\Lambda_k : D^2 \to \mathbb{R}$ defined on all of D^2. Since ∇F is degenerate, we have by Lemma 5.1 that such Λ_k is unique. In particular, $\Lambda_0 = \Lambda_1 = 0$.

Then it follows from Lemma 7.2 that the map $\Lambda : I \times (D^2 \backslash O) \to \mathbb{R}$ defined by $\Lambda(k, z) = \Lambda_k(z)$ is a (I, r)-deformation being a shift function for Ω.

Take any $a, b \in (0, 1)$ such that $a < b$ and consider the homotopy $\mathbf{A}_t$ of ω into $\mathcal{E}^{\partial}(F)$ defined by (8.5). Since $\Omega(0, z) = \Omega(1, z) = z$ for all $z \in D^2$, we obtain from (d) of Proposition 8.1 that $\mathbf{A}_t(0, z) = \mathbf{A}_t(1, z) = z$ for all $t \in I$. In other words $\mathbf{A}_t$ is a homotopy relatively ∂I.

(4) This statement follows from (1) – (3) and the well known fact that any finite subgroup of $GL^{+}(2, \mathbb{R})$ is cyclic.

Theorem 2.3 is proved.

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