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SINGULAR INTEGRAL INEQUALITIES WITH SEVERAL NONLINEARITIES AND INTEGRAL EQUATIONS WITH SINGULAR KERNELS*

СИНГУЛЯРНІ ІНТЕГРАЛЬНІ НЕРІВНОСТІ З КІЛЬКОМА НЕЛІНІЙНОСТЯМИ ТА СИНГУЛЯРНИМИ ЯДРАМИ

M. Medveď

Comenius Univ.
Mlinská Dolina, 842 48 Bratislava, Slovakia
e-mail: medved@fmph.uniba.sk

We deal with an integral inequality with a power nonlinearity in its left-hand side, with n nonlinearities in its right-hand side, and weakly singular kernels. The obtained result is an extention of the Bihari, Henry, Pachpatte and Pinto inequalities and results obtained by the author. Using these results we prove sufficient conditions for existence of global solutions of some nonlinear Volterra integral equations with singular kernels and n nonlinearities.

Розглянуто інтегральні нерівності зі степеневою нелінійністю в лівій частині, n нелінійностями у правій частині та слабкосингулярними ядрами. Отриманий результат є узагальненням нерівностей, одержаних Біхарі, Генрі, Пачпатт та Пінто, а також результатів автора. На основі цих результатів встановлено достатні умови існування глобальних розв'язків деяких нелінійних інтегральних рівнянь Вольтерри з сингулярними ядрами та n нелінійностями.

1. Introduction. In the paper [1] the integral inequality

$$u(t) \leq c + \sum_{i=1}^{n} \int\limits_{a}^{t} \lambda_i(s)\omega_i(u(s))\mathrm{d}s, \tag{1}$$

has been studied. The aim of this paper is to study a singular version of the inequality (1) and to generalize the results published in the papers [2–5]. A desingularization method developed in these papers was applied in the papers [3, 5–9] to the study of boundary-value problems, stability problems, and the problem of the existence of global solutions of parabolic differential equations. It is applied in the paper [10] to the study of integral inequalities of two and more variables with singular kernels. Using this method a result on the discrete version of an integral inequality with a singular kernel is also proved in [9]. The method is also used in the papers [11–14].

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2. Main result. First let us consider the inequality

$$u(t)^k \le a + \sum_{i=1}^{N} \int_0^t (t-s)^{\beta_i - 1} \lambda_i(s) \omega_i(u(s)) ds +$$

$$+ \sum_{j=N+1}^{n} \int_0^t \lambda_j(s) \omega_j(u(s)) ds, \qquad t \in \langle 0, T \rangle, \tag{2}$$

where $0 < \beta_i < 1$, $i = 1, 2, \ldots, n$, $k \ge 1$, $0 < T < \infty$ and a is a positive constant. The nonsingular version of the integral inequality (2) with one nonlinearity and $k = 2$ has been studied by B. G. Pachpatte in the paper [15]. We recall the following notation introduced in [1]. $A \subset \mathbb{R}$ be a set and $\omega_1, \omega_2 : A \to \mathbb{R} \setminus \{0\}$. If the function $\omega = \dfrac{\omega_2}{\omega_1}$ is nondecreasing on A, then we write $\omega_1 \propto \omega_2$. Obviously the relation $\propto$ is transitive and reflexive.

Lemma 1. *Let $\alpha \in (0,1)$, $\beta > 0$. Then there exists a constant $C = C(\alpha, \beta) > 0$ such that*

$$\int_0^t (t-s)^{-\alpha} e^{\beta s} ds \le c e^{\beta t}, \qquad t \ge 0 \tag{3}$$

(see also [4, p. 352, 354, 358]).

Theorem 1. *We make the following assumptions:*

(H_1) *the functions $\omega_i : \mathbb{R}_+ := \langle 0, \infty) \to \mathbb{R}, i = 1, 2, \ldots, n$, are continuous and positive on $(0, \infty)$ such that $\omega_{i_1} \propto \omega_{i_2} \propto \ldots \propto \omega_{i_n}$ for some $i_1, i_2, \ldots, i_n \in \{1, 2, \ldots, n\}$ simultaneously different,*

(H_2) *the functions $u, \lambda_i : \langle 0, T \rangle \to (0, \infty)$, $i = 1, 2, \ldots, N$, are continuous, nonnegative, and a is a positive constant,*

(H_3) $0 < \beta_i < 1$, $i = 1, 2, \ldots, n$, $\beta_i \ne \beta_j$ *for* $i \ne j$.

If $\varepsilon > 0$ and the function $u(t)$ satisfies inequality (1) for $t \in \langle 0, T \rangle$ then

$$u(t) \le \left(W_{i_n}^{-1} \left[W_{i_n}(c_{i_{n-1}}) + \int_0^t \hat{\lambda}_{i_n}(s) ds \right] \right)^{\frac{1}{kr}}, \qquad t \in \langle 0, b \rangle,$$

where

$$W_{i_m}(u) = \int_{u_m}^{u} \frac{d\sigma}{[\omega_{i_m}(\sigma^{\frac{1}{kr}})]^r} u \ge u_m = W_{i_{m-1}}^{-1}(u_{i_{m-1}}), \qquad u_0 > 0,$$

$$r = r(\varepsilon) := q_1 q_2 \dots q_N, \qquad q_i = \frac{1}{\beta_i} + \varepsilon, \quad \varepsilon > 0,$$

$$\beta_i = \frac{1}{1+z_i}, \quad z_i > 0, i = 1, 2, \dots, N,$$

$$W_{i_m}^{-1} \, is \; the \; inverse \; function \; of \; W_{i_m}, \qquad c_0 = (n+1)^{r-1} a^r,$$

$$r = r(\varepsilon) := q_1 q_2 \dots q_N, \quad q_i = \frac{1}{\beta_i} + \varepsilon, \quad \varepsilon > 0, \quad \beta_i = \frac{1}{1+z_i}, \quad z_i > 0,$$

$$c_{i_m} = W_{i_m}^{-1} \left[W_{i_m}(c_{i_{m-1}}) + \int_0^T \hat{\lambda}_{i_m}(s) ds \right], \quad m = 1, 2, \dots, n,$$

$$\hat{\lambda}_i(t) = T^{\hat{q}_i - 1}(n+1)^{r-1} d_i^r e^{-rt} \lambda_i(t)^r, \quad \hat{q}_i = q_1 q_2 \dots q_{i-1} q_{i+1} \dots q_N,$$

$$d_i = c_i^{\frac{1}{p_i}} e^T, \quad i = 1, 2, \dots, N, \quad \hat{\lambda}_j(t) = T^{r-1}(n+1)^{r-1} \lambda_j(s)^r,$$

$$j = N+1, N+2, \dots, n, \quad p_i = \frac{1 + z_i + \varepsilon}{z_i + \varepsilon}, \quad c := c_i = c_i(\alpha_i, p_i) > 0$$

satisfies the inequality (3) with $\alpha = \alpha_i p_i$, $\alpha_i = 1 - \beta_i$, $\beta = \beta_i$ *and the number* $b \in \langle 0, T \rangle$ *is the largest number such that*

$$\|\hat{\lambda}_{i_m}\|_b := \int_0^b \hat{\lambda}_{i_m}(s) ds \leq \int_{c_{m-1}}^{\infty} \frac{d\sigma}{[\omega_{i_m}(\sigma^{\frac{1}{kr}})]^r}, \qquad m = 1, 2, \dots, n.$$

This theorem obviously extends the Bihari lemma [16] to n nonlinearities. This is a singular version of the Pinto inequality [12] (Theorem 1) and it is also a generalization of an inequality proved by J. D. Dauer and N. I. Mahmudov [7] (Theorem 4). They consider the inequality (2) with $N = 1, n = 2, \beta_1 > 0$ and in the proof of their result the method of desingularization developed in [4] is applied.

The following proof of Theorem 1 is based on the method of desingularization of singular integral inequalities (see [4]), the method applied in the study of integral inequalities with power nonlinearities in the left-hand sides in the papers [5, 15] and the Pinto's method [1] (a generalization of Bihari's method to integral inequalities with n nonlinearities).

Proof Theorem 1. Let

$$\alpha_i = 1 - \beta_i, \quad \beta_i = \frac{1}{1+z_i}, \quad z_i > 0,$$

$$q_i = \frac{1}{\beta_i} + \varepsilon, \quad p_i = \frac{1 + z_i + \varepsilon}{z_i + \varepsilon}, \quad i = 1, 2, \dots, n.$$

Obviously $\dfrac{1}{p_i} + \dfrac{1}{q_i} = 1$. The Hölder inequality and (3) yield

$$\int_0^t (t-s)^{\beta_i - 1} \lambda_i(s)\omega_i(u(s))ds =$$

$$= \int_0^t (t-s)^{-\alpha_i} e^s \cdot e^{-s} \lambda_i(s)\omega_i(u(s))ds \le$$

$$\le \left(\int_0^t (t-s)^{-\alpha_i p_i} e^{p_i s} ds \right)^{\frac{1}{p_i}} \left(\int_0^t e^{-q_i s} \lambda_i(s)^{q_i} \omega_i(u(s))^{q_i} ds \right)^{\frac{1}{q_i}} \le$$

$$\le d_i e^t \left(\int_0^t \nu_i(s)\omega_i(u(s))^{q_i} ds \right)^{\frac{1}{q_i}},$$

where

$$\nu_i(t) = e^{-q_i t} \lambda_i(t)^{q_i}, \qquad d_i = c_i^{\frac{1}{p_i}} e^T,$$

$$\int_0^t (t-s)^{-\alpha_i p_i} e^{p_i s} ds \le c_i e^{p_i t}, \qquad t \ge 0, \qquad c_i = c_i(\alpha_i, p_i) > 0,$$

see Lemma 1.

From the inequality (2) we obtain

$$u(t)^k \le a + \sum_{i=1}^{N} d_i \left(\int_0^t \nu_i(s)\omega_i(u(s))^{q_i} ds \right)^{\frac{1}{q_i}} + \sum_{j=N+1}^{n} \int_0^t \lambda_j(s)\omega_j(u(s))ds.$$

If we define $r = q_1 q_2 \ldots q_N$ then applying the inequality $(A_1 + A_2 + \ldots + A_{n+1})^r \le (n + +1)^{r-1}(A_1^r + A_2^r + \ldots + A_{n+1}^r),\ A_1, A_2, \ldots, A_{n+1} \ge 0$, we obtain

$$u(t)^{kr} \le (n+1)^{r-1} \left[a^r + \sum_{i=1}^{N} d_i^r \left(\int_0^t \nu_i(s)\omega_i(u(s))^{q_i} ds \right)^{\hat{q}_i} + \right.$$

$$\left. + \sum_{j=N+1}^{n} \left(\int_0^t \lambda_j(s)\omega_j(u(s))ds \right)^r \right], \tag{4}$$

where $\hat{q}_i = q_1 q_2 \dots q_{i-1} q_{i+1} \dots q_N > 1$. The Jensen's inequality in the form

$$\left(\int_0^t g(s)ds\right)^{\delta} \le t^{\delta-1}\int_0^t g(s)^{\delta}ds, \qquad \delta \ge 1,$$

yields

$$\left(\int_0^t \nu_i(s)\omega_i(u(s))^{q_i}ds\right)^{\hat{q}_i} \le T^{\hat{q}_i-1}\int_0^t \nu_i(s)^{\hat{q}_i}\omega_i(u(s)^r ds =$$

$$= T^{\hat{q}_i-1}\int_0^t \mathrm{e}^{-rt}\lambda_i(s)^r ds,$$

$$\left(\int_0^t \lambda_j(s)\omega_j(u(s))ds\right)^{r} \le T^{r-1}\int_0^t \lambda_j(s)^r\omega_j(u(s))^r ds$$

and from inequality (4) it follows that

$$u(t)^{kr} \le c_0 + \sum_{i=1}^{n}\int_0^t \hat{\lambda}_i(s)\omega_i(u(s))^r ds,$$

where

$$c_0 = (n+1)^{r-1}a^r, \;\; \hat{\lambda}_i(t) = T^{\hat{q}_i-1}(n+1)^{r-1}d_i^r e^{-rt}\lambda_i(t)^r, \;\; i = 1,2,\dots,N,$$

$$\hat{\lambda}_j(t) = T^{r-1}(n+1)^{r-1}\lambda_j(t)^r, \quad j = N+1, N+2,\dots,n.$$

This inequality can also be written as

$$u(t)^{kr} \le c_0 + \sum_{m=1}^{n}\int_0^t \hat{\lambda}_{i_m(s)}\omega_{i_m}(u(s))^r ds,$$

where the condition (H_1) is satisfied for some $i_1, i_2, \dots, i_m$.

We shall proceed in the sequel by induction with respect to n. For $n = 1$ the result follows from [5] (Theorem 2). Let $n = 2$, i.e., we have the inequality

$$u(t)^{kr} \le \gamma + \int_0^t \hat{\lambda}_{i_1}(s)\omega_{i_1}(u(s))^r ds + \int_0^t \hat{\lambda}_{i_2}(s)\omega_{i_2}(u(s))^r ds. \tag{5}$$

Without loss of generality one can assume that $i_1 = 1, i_2 = 2$, i.e., $\omega_1 \propto \omega_2$.

Let

$$z_1(t) = \int_0^t \hat{\lambda}_1(s)\omega_1(u(s))^r ds, \quad z_2(t) = \gamma + \int_0^t \hat{\lambda}_2(s)\omega_2(u(s))^r ds, \quad z(t) = z_1(t) + z_2(t).$$

Obviously,

$$u(t)^{kr} \le z(t) \tag{6}$$

and therefore we have

$$\frac{\mathrm{d}}{dt}(W_1(z(t)) = \frac{dz/dt}{\omega_1(z(t))^r} \le \hat{\lambda}_1(t) + \hat{\lambda}_2(t)\left[\omega(z(t)^{\frac{1}{kr}})\right]^r,$$

where $\omega = \dfrac{\omega_2}{\omega_1}$. Hence by integrating over $\langle 0, T\rangle$, we obtain

$$W_1(z(t)) \le W_1(c_0) + \int_0^t \hat{\lambda}(s)ds + \int_0^t \hat{\lambda}(s)\left[\omega(z(s)^{\frac{1}{kr}})\right]^r ds.$$

Putting $v(t) = W_1(z(t))$ we have

$$v(t) \le W_1(c_0) + \int_0^T \hat{\lambda}_1(s)ds + \int_0^t \hat{\lambda}_2(s)\omega(W_1^{-1}(v(s))^r ds,$$

i.e.,

$$v(t) \le W_1(c_0) + \int_0^t \hat{\lambda}_2(s)\hat{\omega}(v(s))ds,$$

where $\hat{\omega}(u) = \left(\omega([W_1^{-1}(u)]^{\frac{1}{kr}})\right)^r$. By the Bihari lemma [16]

$$v(t) \le \widehat{W}^{-1}\left(\widehat{W}(W_1(c_0)) + \int_0^T \hat{\lambda}_1(s)ds + \int_0^t \hat{\lambda}_2(s)ds\right),$$

where

$$\widehat{W}(u) = \int_{u_0}^u \frac{\mathrm{d}\sigma}{\hat{\omega}(\sigma)}, \qquad u \ge u_0 > 0,$$

and consequently we obtain from (6) that

$$u(t) \le z(t)^{\frac{1}{kr}} = \left(W_1^{-1}(v(t)\right)^{\frac{1}{kr}} \le \left(W_1^{-1} \circ \hat{W}^{-1}\left(\hat{W}(c_0) + \int_0^t \hat{\lambda}_2(s)ds\right)\right)^{\frac{1}{kr}}. \tag{7}$$

Obviously

$$\widehat{W}(u) = \int\limits_{u_0}^{u} \frac{d\sigma}{\hat{\omega}(\sigma)} = \int\limits_{u_0}^{u} \frac{d\sigma}{\omega([W_1^{-1}(\sigma)]^{\frac{1}{kr}})^r} =$$

$$= \int\limits_{u_1}^{W_1^{-1}(u)} \frac{d\tau}{[\omega_2(\tau^{\frac{1}{kr}})]^r} = W_2 \circ W_1^{-1}(u),$$

where $u_1 = W_1^{-1}(u_0)$ and therefore (7) yields

$$u(t) \leq W_2^{-1}\left[W_2(c_1) + \int\limits_0^t \hat{\lambda}_2(s)ds\right]$$

where $c_1 = W_1^{-1}(W_1(c_0) + \int \hat{\lambda}_1(s)ds)$. In the case $n > 2$ one can proceed by induction in the same way as in the Pinto's proof of [1] and we omit this part of the proof.

3. Applications. In this section we apply Theorem 1 to a study of properties of solutions of the integral equation

$$\Phi(t) = f(t) + \sum_{i=1}^{N} \int\limits_0^t (t-s)^{\beta_i - 1} P_i(s, \Phi(s))ds +$$

$$+ \sum_{j=N+1}^{n} \int\limits_0^t P_j(s, \Phi(s))ds, \tag{8}$$

where $\Phi, f, P_i, i = 1, 2, \ldots, n$, are vector-valued functions, and the scalar integral equation of the form

$$x(t)^k = h(t) + \sum_{i=1}^{N} \int\limits_0^t (t-s)^{\beta_i - 1} Q_i(s, x(s))ds +$$

$$+ \sum_{j=N+1}^{n} \int\limits_0^t Q_j(s, x(s))ds, \qquad k > 1. \tag{9}$$

By a solution of (8) and (9), respectively, we mean a continuous solution defined on an interval $\langle 0, T)$, where $0 < T \leq \infty$. If $T = \infty$, the solution is called global. A result on the existence of global solutions of (8) with $n = 1$ is proved in [5] and it is applied there in the proof of a theorem on the existence of global solutions of some evolution equations. The special case

$$x(t) = h(t) + \int_0^t (t-s)^{\beta-1} w(x(s)) ds \tag{10}$$

of the equation (8), where x, h, w are scalar functions, has been studied in several papers, e. g. in [9, 17]. In the paper [18] a necessary and sufficient condition for the existence of global solutions of this equation is proved and this result is used in the paper [19] in the proofs of some results on the existence of global solutions to a semilinear parabolic evolution equations. Scalar equations with power nonlinearities in the left-hand sides, i.e., equations of the form (9), however with $n = 1$, are studied e. g. in [20]. In this paper the function $Q_1 = Q$ is supposed to be linear, $h(t) \equiv 0$ and there is a function of t before the integral. The authors have proved some results on lower and upper estimates of solutions of such equations. Equations of such type are arising in various applications, e. g., in the nonlinear theory of wave propagation [21].

Theorem 2. *Let $f \in C(\mathbb{R}_+, \mathbb{R}^M)$, $P_i \in C(R_+ \times \mathbb{R}^M, \mathbb{R}^M)$ with*

$$\|P_i(t,v)\| \le F_i(t)\omega_i(\|v\|), \qquad (t,v) \in \mathbb{R}_+ \times \mathbb{R}^M, \quad i = 1, 2, \ldots, n, \tag{11}$$

where $F_i \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $\omega_i, i = 1, 2, \ldots, n$, are functions satisfying the condition (H_1) of Theorem 1. Assume that there exists an $\varepsilon > 0$ such that

$$\int_0^\infty \frac{\tau^{r-1} d\tau}{\omega_i(\tau)^r} = \infty, \qquad i = 1, 2, \ldots, n, \tag{12}$$

where $r = r(\varepsilon) := q_1 q_2 \ldots q_N$, $q_i := \dfrac{1}{\beta_i} + \varepsilon$, $0 < \beta_i < 1$, $i = 1, 2, \ldots, N$. Then

$$\limsup_{t \to T^-} \|\Phi(t)\| < \infty$$

for any solution $\Phi : \langle 0, T) \to \mathbb{R}^M$ of the equation (8) with $0 < T < \infty$.

Proof. Assume that $\Phi : \langle 0, T) \to \mathbb{R}^n$ is a continuous solution of the equation (8), where $0 < T < \infty$ with $\limsup_{t \to T^-} ||\Phi(t)|| = \infty$. The condition (11) yields

$$\|\Phi(t)\| \le \|f(t)\| + \sum_{i=1}^N \int_0^t (t-s)^{\beta_i - 1} F_i(s) \omega_i(||\Phi(s)||) ds +$$

$$+ \sum_{j=N+1}^n \int_0^t F_j(s) \omega_j(||\Phi(s)||) ds$$

and applying Theorem 1 we obtain

$$W_{i_n}(\|\Phi(t)\|) \le W_{i_n}(\kappa_{i_n}) + \int_0^t \hat{F}_{i_n}(s) ds,$$

where

$$\kappa_0 = (n+1)^{r-1}\eta^r, \qquad \eta = \max_{t\in\langle 0,T\rangle} \|f(t)\|,$$

$$\kappa_{i_m} = W_{i_m}^{-1}\left[W_{i_m}(\kappa_{i_{m-1}}) + \int\limits_0^T \hat{F}_{i_m}(s)ds\right],$$

$$W_{i_m}(u) = \int\limits_{u_m}^{u} \frac{d\sigma}{[\omega_{i_m}(\sigma^{\frac{1}{r}})]^r}, \quad u \ge u_m = W_{i_{m-1}}^{-1}(u_{m-1}), \quad u_0 = \|\Phi(0)\|^r,$$

$$\hat{F}_{i_m}(t) = T^{\hat{q}_{i_m}-1}(n+1)^{r-1}d_{i_m}^r F_{i_m}(t),$$

$$\hat{q}_i = q_1 \dots q_{i-1}q_{i+1}\dots q_n, d_{i_m} = c_{i_m}^{\frac{1}{p_{i_m}}} e^T, \quad m = 1, 2, \dots, n, c_{i_m}, p_{i_m}$$

are constants from Theorem 1 and the functions W_{i_m}, $m = 1, 2, \dots, n$, are as in Theorem 1. This yields

$$\int\limits_{u_n}^{\|\Phi(t)\|^r} \frac{d\sigma}{[\omega_{i_n}(\sigma^{\frac{1}{r}})]^r} \le W_{i_n}(\kappa_{i_n}) + \int\limits_0^T \hat{F}_{i_n}(s)ds < \infty.$$

We remark that all the numbers $u_m, m = 1, 2, \dots, n$, are well defined because of the divergence of all integrals from (12). However

$$\limsup_{t\to T^-} \int\limits_{u_n}^{\|\Phi(t)\|^r} \frac{d\sigma}{[\omega_{i_n}(\sigma^{\frac{a}{r}})]^r} = r \limsup_{t\to T^-} \int\limits_{u_0}^{\|\Phi(t)\|} \frac{\tau^{r-1}d\tau}{\omega_{i_n}(\tau)^r} =$$

$$= r\int\limits_0^\infty \frac{\tau^{r-1}d\tau}{\omega_{i_n}(\tau)^r} = \infty$$

and this is a contradiction with (11).

The theorem is proved.

Theorem 3. *Let $h \in C(\mathbb{R}_+, \mathbb{R})$, $Q_i \in C(\mathbb{R}_+, \mathbb{R})$ with*

$$|Q_i(t, w)) \le G_i(t)\omega_i(|w|), \qquad (t, w) \in \mathbb{R}_+ \times \mathbb{R}, i = 1, 2, \dots, n,$$

where $G_i \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $\omega_i, i = 1, 2, \dots, n$, are functions satisfying the condition (H_1) and the condition

$$\int\limits_0^\infty \frac{\tau^{kr-1}d\tau}{\omega_i(\tau)^{kr}} = \infty, \qquad i = 1, 2, \dots, n,$$

where $r = r(\varepsilon)$ is as in Theorem 2 and $k \ge 1$. Then $\limsup_{t\to T^-} |\Psi(t)| \le \infty$ for any solution $\Psi : \langle 0, T) \to \mathbb{R}$ of the equation (9) with $0 < T < \infty$.

This theorem can be proved applying Theorem 1 in the same way as we have proved Theorem 2.

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