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PERIODIC SOLUTIONS OF NONLINEAR EVOLUTION EQUATIONS WITH W_{λ_0}-PSEUDOMONOTONE MAPS

ПЕРІОДИЧНІ РОЗВ'ЯЗКИ НЕЛІНІЙНИХ ЕВОЛЮЦІЙНИХ РІВНЯНЬ З W_{λ_0}-ПСЕВДОМОНОТОННИМИ ВІДОБРАЖЕННЯМИ

P. O. Kasyanov

Kyiv Nat. Taras Shevchenko Univ.
Volodymyrs'ka Str. 64, 01033, Kyiv, Ukraine
e-mail: kasyanov@univ.kiev.ua

V. S. Mel'nik

Inst. Appl. and System Analysis Nat. Acad. Sci. Ukraine
Peremogy Avenue 37, Build. 35, 03056, Kyiv, Ukraine
e-mail: vsmelnyk@ukr.net

S. Toscano

Univ. Salerno
Via Ponte Don Melillo 1, 84084 Fisciano (Salerno), Italy
e-mail: speranza.toscano@virgilio.it

We consider differential-operator equations with W_{λ_0}-pseudomonotone operators. The problem of studying periodic solutions via the Faedo – Galerkin method has been considered. The important a priory estimates have been obtained. A topological description of resolvent operators is given.

Розглянуто диференціально-операторні рівняння з W_{λ_0}-псевдомонотонними операторами. Розв'язано проблему вивчення періодичних розв'язків методом Фаедо – Гальоркіна. Отримано важливі апріорні оцінки. Наведено певний топологічний опис резольвентних операторів.

We obtain a condition for existence and uniqueness of a periodic solution of a system of nonlinear integro-differential equations with an impulsive effect. The solution is represented as a limit of periodic iterations. We give estimates for the convergence rate and the exact solution.

Одержано умову існування єдиного періодичного розв'язку системи нелінійних інтегро-диференціальних рівнянь з імпульсною дією. Розв'язок подано у вигляді границі періодичних ітерацій. Наведено оцінки швидкості збіжності і точного розв'язку системи.

1. Introduction. One of the most effective approaches to investigate nonlinear problems, defined by partial differential equations with boundary values consists in their into equations in Banach spaces governed by nonlinear operators. In order to study these equations, modern methods of nonlinear analysis have been used [1 – 3]. In [4], by using a special basis, the Cauchy problem for a class of equations with operators of Volterra type has been studied. An important periodic problem for equations with monotone differential operators of Volterra type has been studied in [1]. Periodic solutions for pseudomonotone operators have been considered in [2]. In the present paper we introduce a new construction of bases to prove existence of periodic solutions of differential-operator equations by using the Faedo–Galerkin method for W_{λ_0}-pseudomonotone operators. From the point of view of applications, we have essentially extended the class of operators considered by other authors (see [4–6]).

2. Problem definition. Let $(V_1, \|\cdot\|_{V_1})$ and $(V_2, \|\cdot\|_{V_2})$ be some reflexive separable Banach spaces, continuously embedded in a Hilbert space $(H, (\cdot,\cdot))$ such that

$$V := V_1 \cap V_2 \text{ is dense in the spaces } V_1,\ V_2 \text{ and } H. \tag{2.1}$$

After the identification $H \cong H^*$ we get

$$V_1 \subset H \subset V_1^*, \qquad V_2 \subset H \subset V_2^*, \tag{2.2}$$

with continuous and dense embeddings [1], where $(V_i^*, \|\cdot\|_{V_i})$ is the space topologically conjugate to V_i with respect to the canonical bilinear form $\langle\cdot,\cdot\rangle_{V_i} : V_i^* \times V_i \to \mathbb{R}$, $i = 1, 2$, which coincides on H with the inner product $(\cdot,\cdot)$ on H. Let us consider the functional spaces $X_i = L_{r_i}(S; H) \cap L_{p_i}(S; V_i)$, where $S = [0, T]$, $0 < T < +\infty$, $1 < p_i \le r_i < +\infty$, $i = 1, 2$. The spaces X_i are Banach spaces with the norms $\|y\|_{X_i} = \|y\|_{L_{p_i}(S;V_i)} + \|y\|_{L_{r_i}(S;H)}$. Moreover, X_i is a reflexive space.

Let us also consider the Banach space $X = X_1 \cap X_2$ with the norm $\|y\|_X = \|y\|_{X_1} + \|y\|_{X_2}$. Since the spaces $L_{q_i}(S; V_i^*) + L_{r_i'}(S; H)$ and X_i^* are isometrically isomorphic, we identify them. Analogously, $X^* = X_1^* + X_2^* = L_{q_1}(S; V_1^*) + L_{q_2}(S; V_2^*) + L_{r_1'}(S; H) + L_{r_2'}(S; H)$, where ${r_i}^{-1} + {r_i'}^{-1} = {p_i}^{-1} + {q_i}^{-1} = 1$. Let us define a duality form on $X^* \times X$,

$$\langle f, y\rangle = \int_S (f_{11}(\tau), y(\tau))_H \, d\tau + \int_S (f_{12}(\tau), y(\tau))_H \, d\tau + \int_S \langle f_{21}(\tau), y(\tau)\rangle_{V_1} \, d\tau +$$

$$+ \int_S \langle f_{22}(\tau), y(\tau)\rangle_{V_2} \, d\tau = \int_S (f(\tau), y(\tau)) \, d\tau,$$

where $f = f_{11} + f_{12} + f_{21} + f_{22}$, $f_{1i} \in L_{r_i'}(S; H)$, $f_{2i} \in L_{q_i}(S; V_i^*)$. For each $f \in X^*$,

$$\|f\|_{X^*} = \inf_{\substack{f = f_{11} + f_{12} + f_{21} + f_{22}:\\ f_{1i} \in L_{r_i'}(S;H),\ f_{2i} \in L_{q_i}(S;V_i^*)\ (i=1,2)}} \max\Big\{ \|f_{11}\|_{L_{r_1'}(S;H)},$$

$$\|f_{12}\|_{L_{r_2'}(S;H)},\ \|f_{21}\|_{L_{q_1}(S;V_1^*)},\ \|f_{22}\|_{L_{q_2}(S;V_2^*)} \Big\}.$$

Let $A : X_1 \to X_1^*$ and $B : X_2 \to X_2^*$ be single-valued nonlinear operators, $L : D(L) \subset \subset X \to X^*$ be a linear closed densly defined operator. We consider the following problem:

$$Ly + A(y) + B(y) = f, \tag{2.3}$$

$$y \in D(L), \tag{2.4}$$

where $f \in X^*$ is arbitrary and fixed.

3. Classes of maps. Let $(Y, \|\cdot\|_Y)$ be a Banach space, W a normed space with a norm $\|\cdot\|_W$. We consider $W \subset Y$ with a continuous embedding. Further, $y_n \rightharpoonup y$ in Y means that y_n weakly converges to y in the space Y. If Y is not reflexive, then $y_n \rightharpoonup y$ in Y^* means that y_n *-weakly converges to y in the space Y^*.

Definition 3.1. *Let $D(A)$ be a subset of Y. A single-valued map $A : D(A) \subset Y \to Y^*$ is called:*

coercive if $\|y\|_Y^{-1}\langle A(y), y\rangle_Y \to +\infty$ as $\|y\|_Y \to \infty$, $y \in D(A)$;

weakly coercive if for each $f \in Y^*$ there exists $R > 0$ such that

$$\langle A(y) - f, y\rangle_Y \geq 0 \text{ as } \|y\|_Y = R, \quad y \in D(A);$$

bounded if for any $L > 0$ there exists $l > 0$ such that

$$\|A(y)\|_{Y^*} \leq l \quad \forall y \in D(A): \quad \|y\|_Y \leq L;$$

locally bounded if for any fixed $y \in D(A)$ there exist constants $m > 0$ and $M > 0$ such that $\|A(\xi)\|_{Y^*} \leq M$ if $\|y - \xi\|_Y \leq m$, $\xi \in D(A)$;

finite-dimensionally locally bounded if for each finite-dimensional subspace $F \subset D(A)$, $A\big|_F$ is locally bounded on $(F, \|\cdot\|_Y)$.

Definition 3.2. *A single-valued map $A : D(A) \subset Y \to Y^*$ is called*

radially continuous if for any fixed $y, \xi \in D(A)$,

$$\lim_{t\to+0}\langle A(y + t\xi), \xi\rangle_Y = \langle A(y), \xi\rangle_Y;$$

an operator with semibounded variation on W (with (Y, W)-semibounded variation) if for all $R > 0$ and all $y_1, y_2 \in D(A)$ with $\|y_1\|_Y \leq R$, $\|y_2\|_Y \leq R$, the inequality

$$\langle A(y_1) - A(y_2), y_1 - y_2\rangle_Y \geq -C(R; \|y_1 - y_2\|'_W)$$

is fulfilled;

λ-pseudomonotone on W (W_λ-pseudomonotone), if for each sequence $\{y_n\}_{n\geq1} \subset W \cap D(A)$ such that $y_n \rightharpoonup y_0$ in W with $y_0 \in D(A)$, the inequality

$$\varlimsup_{n\to\infty}\langle A(y_n), y_n - y_0\rangle_Y \leq 0, \tag{3.1}$$

implies existence of $\{y_{n_k}\}_{k\geq1}$ from $\{y_n\}_{n\geq1}$ such that

$$\varliminf_{k\to\infty}\langle A(y_{n_k}), y_{n_k} - w\rangle_Y \geq \langle A(y_0), y_0 - w\rangle_Y \quad \forall w \in D(A) \cap W; \tag{3.2}$$

λ_0-pseudomonotone on W (W_{λ_0}-pseudomonotone), if for each sequence $\{y_n\}_{n\geq1} \subset W \cap$ $\cap D(A)$ such that

$$y_n \rightharpoonup y_0 \text{ in } W, \quad A(y_n) \rightharpoonup d_0 \text{ in } Y^* \text{ with } y_0 \in D(A) \text{ and } d_0 \in Y^*,$$

it follows from (3.1) that there exists $\{y_{n_k}\}_{k\geq 1} \subset \{y_n\}_{n\geq 1}$ such that (3.2) is true.
The above single-valued map satisfies
property (κ) if for each bounded set D in Y there exists $c \in \mathbb{R}$ such that

$$\langle A(v), v\rangle_Y \geq -c\|v\|_Y \quad \forall v \in D;$$

property (Π) if for each nonempty bounded subset $B \subset D(A)$ and for each $k > 0$ such that

$$\langle A(y), y\rangle_Y \leq k \quad \text{for each } y \in B,$$

it follows that there exists $K > 0$ such that

$$\|A(y)\|_{Y^*} \leq K \quad \text{for all } y \in B.$$

Here $C(r_1; \cdot) : \mathbb{R}_+ \to \mathbb{R}$ is a continuous function for each $r_1 \geq 0$ and such that $\tau^{-1}C(r_1; \tau r_2) \to 0$ as $\tau \to +0\ \forall\, r_1, r_2 \geq 0$ and $\|\cdot\|'_W$ is a (semi-)norm on Y, that is compact with respect to $\|\cdot\|_W$ on W and continuous with respect to $\|\cdot\|_Y$ on Y!

Remark 3.1. The idea of passing to subsequences in the latter definition was adopted from Skrypnik's work [7].

Let $Y = Y_1 \cap Y_2$, where $(Y_1, \|\cdot\|_{Y_1})$ and $(Y_2, \|\cdot\|_{Y_2})$ are some Banach spaces.

Definition 3.3. *A pair of single-valued maps $A : D(A) \subset Y_1 \to Y_1^*$ and $B : D(B) \subset Y_2 \to Y_2^*$ is called s-mutually bounded, if for each $M > 0$ and each bounded set $D \subset Y$ there exists $K > 0$ such that from*

$$y \in D(A) \cap D(B) \cap D \quad \text{and} \quad \langle A(y), y\rangle_{Y_1} + \langle B(y), y\rangle_{Y_2} \leq M,$$

we have

$$\text{or} \quad \|A(y)\|_{Y_1^*} \leq K \quad \text{or} \quad \|B(y)\|_{Y_2^*} \leq K.$$

Remark 3.2. A bounded map $A : Y \to Y^*$ satisfies property (κ) and property (Π); a λ-pseudomonotone on W map is λ_0-pseudomonotone on W. The converse statement is true for bounded single-valued maps.

If one of the operators of the pair $(A; B)$ is bounded, then the pair $(A; B)$ is s-mutually bounded. Moreover, if a pair $(A; B)$ is s-mutually bounded, then the operator $C = A + B : X \to X^*$ has property (Π).

If a pair of operators is s-mutually bounded and each of them satisfies condition (Π), then the sum of the given operators also satisfies condition (Π).

Now let $W = W_1 \cap W_2$, where $(W_1, \|\cdot\|_{W_1})$ and $(W_2, \|\cdot\|_{W_2})$ are Banach spaces such that $W_i \subset Y_i$ with a continuous embedding.

Lemma 3.1. *Let $A : Y_1 \to Y_1^*$ and $B : Y_2 \to Y_2^*$ be s-mutually bounded λ_0-pseudomonotone operators on W_1 and W_2, respectively. Then the map $C := A+B : Y \to Y^*$ is λ_0-pseudomonotone on W.*

Remark 3.3. If a pair $(A; B)$ is not s-mutually bounded, then the above proposition takes place only for maps that are λ-pseudomonotone on W_1 and W_2, respectively.

Proof. Let $y_n \rightharpoonup y_0$ in W (it means that $y_n \rightharpoonup y_0$ in W_1 and $y_n \rightharpoonup y_0$ in W_2), $C(y_n) \rightharpoonup d_0$ in Y^* and inequality (3.1) hold. Since the pair $(A; B)$ is s-mutually bounded, from the estimate

$$\langle C(y_n), y_n\rangle_Y = \langle A(y_n) + B(y_n), y_n\rangle_Y = \langle A(y_n), y_n\rangle_{Y_1} + \langle B(y_n), y_n\rangle_{Y_2} \le k$$

we have that either $\|A(y_n)\|_{Y_1^*} \le C$ or $\|B(y_n)\|_{Y_2^*} \le C$. Then passing to a subsequence if necessary, we claim that

$$A(y_n) \rightharpoonup d_0' \text{ in } Y_1^* \quad \text{and} \quad B(y_n) \rightharpoonup d_0'' \text{ in } Y_2^*. \tag{3.3}$$

From inequality (3.1) we have

$$\varliminf_{n\to\infty} \langle B(y_n), y_n - y_0\rangle_{Y_2} + \varlimsup_{n\to\infty} \langle A(y_n), y_n - y_0\rangle_{Y_1} \le \varlimsup_{n\to\infty} \langle C(y_n), y_n - y_0\rangle_Y \le 0,$$

or, symmetrically,

$$\varliminf_{n\to\infty} \langle A(y_n), y_n - y_0\rangle_{Y_1} + \varlimsup_{n\to\infty} \langle B(y_n), y_n - y_0\rangle_{Y_2} \le \varlimsup_{n\to\infty} \langle C(y_n), y_n - y_0\rangle_Y \le 0.$$

Let us consider the last inequality. It is obvious that there exists a subsequence $\{y_m\}_m \subset \subset \{y_n\}_{n\ge 1}$ such that

$$0 \ge \varlimsup_{n\to\infty} \langle B(y_n), y_n - y_0\rangle_{Y_2} + \varliminf_{n\to\infty} \langle A(y_n), y_n - y_0\rangle_{Y_1} \ge$$

$$\ge \varlimsup_{m\to\infty} \langle B(y_m), y_m - y_0\rangle_{Y_2} + \lim_{m\to\infty} \langle A(y_m), y_m - y_0\rangle_{Y_1}. \tag{3.4}$$

Hence, it follows that

$$\text{either} \quad \lim_{m\to\infty} \langle A(y_m), y_m - y_0\rangle_{Y_1} \le 0 \quad \text{or} \quad \varlimsup_{m\to\infty} \langle B(y_m), y_m - y_0\rangle_{Y_2} \le 0.$$

Without loss of generality we assume that $\lim\limits_{m\to\infty} \langle A(y_m), y_m - y_0\rangle_{Y_1} \le 0$. Then, from (3.3) and λ_0-pseudomonotony of A on W_1, it follows that there exists $\{y_{m_k}\}_{k\ge 1}$ in $\{y_m\}_m$ such that

$$\lim_{k\to\infty} \langle A(y_{m_k}), y_{m_k} - v\rangle_{Y_1} \ge \langle A(y_0), y_0 - v\rangle_{Y_1} \quad \forall v \in Y_1 \cap W_1. \tag{3.5}$$

If we take in the last relation $v = y_0$ we obtain that $\langle A(y_{m_k}), y_{m_k} - y_0\rangle_{Y_1} \to 0$ as $k \to +\infty$. Then, due to (3.4), $\varlimsup\limits_{k\to\infty} \langle B(y_{m_k}), y_{m_k} - y_0\rangle_{Y_2} \le 0$.

In virtue of λ_0-pseudomonotony of B on W_2, passing to a subsequence $\{y_{m_k'}\} \subset \{y_{m_k}\}_{k\ge 1}$, we find

$$\varliminf_{k\to\infty} \langle B(y_{m_k'}), y_{m_k'} - w\rangle_{Y_2} \ge \langle B(y_0), y_0 - w\rangle_{Y_2} \quad \forall w \in Y_2 \cap W_2. \tag{3.6}$$

Then from relations (3.5) and (3.6) we finally obtain

$$\varliminf_{k\to\infty} \langle C(y_{m'_k}), y_{m'_k} - x\rangle_Y \geq \varliminf_{k\to\infty} \langle A(y_{m'_k}), y_{m'_k} - x\rangle_{Y_1} + \varliminf_{k\to\infty} \langle B(y_{m'_k}), y_{m'_k} - x\rangle_{Y_2} \geq$$

$$\geq \langle A(y_0), y_0 - x\rangle_{Y_1} + \langle B(y_0), y_0 - x\rangle_{Y_2} = \langle C(y_0), y_0 - x\rangle_Y \quad \forall x \in W \cap Y.$$

The lemma is proved.

Lemma 3.2. *Let $A : Y_1 \to Y_1^*$ and $B : Y_2 \to Y_2^*$ be coercive maps, that satisfy condition (κ). Then the operator $C := A + B : Y \to Y^*$ is coercive.*

Proof. We obtain this statement arguing by contradiction. Let $\{x_n\}_{n\geq 1} \subset Y$ with $x_m \neq \bar{0}$ and $\|x_n\|_Y = \|x_n\|_{Y_1} + \|x_n\|_{Y_2} \to +\infty$ as $n \to \infty$, but

$$\sup_{n\geq 1} \frac{\langle C(x_n), x_n\rangle_Y}{\|x_n\|_Y} < +\infty. \tag{3.7}$$

Let $\gamma_A(r) := \inf\limits_{\|v\|_{Y_1}=r} \dfrac{\langle A(v), v\rangle_{Y_1}}{\|v\|_{Y_1}}$, $\gamma_B(r) := \inf\limits_{\|w\|_{Y_2}=r} \dfrac{\langle B(w), w\rangle_{Y_2}}{\|w\|_{Y_2}}$, $r > 0$. We remark that $\gamma_A(r) \to +\infty$, $\gamma_B(r) \to +\infty$ as $r \to +\infty$. In the case $\|x_m\|_{Y_1} \to +\infty$ as $m \to \infty$ and $\|x_m\|_{Y_2} \leq c \ \forall m \geq 1$, we get $\dfrac{\langle A(x_n), x_n\rangle_{Y_1}}{\|x_n\|_Y} \geq \gamma_A(\|x_n\|_{Y_1}) \dfrac{\|x_n\|_{Y_1}}{\|x_n\|_Y} \to +\infty$ as $m \to +\infty$ and, moreover,

$$\frac{\langle B(x_n), x_n\rangle_{Y_2}}{\|x_n\|_Y} \geq -c_1 \frac{\|x_n\|_{Y_2}}{\|x_n\|_Y} \to 0 \quad \text{as } n \to \infty,$$

where $c_1 \in \mathbb{R}$ is a constant as in condition (κ) with $D = \{y \in Y_2 \mid \|y\|_{Y_2} \leq c\}$. Consequently,

$$\frac{\langle C(x_n), x_n\rangle_Y}{\|x_n\|_Y} = \frac{\langle A(x_n), x_n\rangle_{Y_1}}{\|x_n\|_Y} + \frac{\langle B(x_n), x_n\rangle_{Y_2}}{\|x_n\|_Y} \to +\infty \quad \text{as} \quad n \to \infty.$$

This is in contradiction with (3.7).

If $\|x_n\|_{Y_1} \leq c \ \forall n \geq 1$ and $\|x_n\|_{Y_2} \to \infty$ as $n \to \infty$ the reasoning is the same.

When $\|x_n\|_{Y_1} \to \infty$ and $\|x_n\|_{Y_2} \to \infty$ as $n \to \infty$, we get the contradiction

$$+\infty > \sup_{n\geq 1} \frac{\langle C(x_n), x_n\rangle_Y}{\|x_n\|_Y} \geq \gamma_A(\|x_n\|_{Y_1}) \frac{\|x_n\|_{Y_1}}{\|x_n\|_{Y_1} + \|x_n\|_{Y_2}} +$$

$$+ \gamma_B(\|x_n\|_{Y_2}) \frac{\|x_n\|_{Y_2}}{\|x_n\|_{Y_1} + \|x_n\|_{Y_2}} \geq \min\{\gamma_A(\|x_n\|_{Y_1}), \gamma_B(\|x_n\|_{Y_2})\} \to +\infty.$$

The lemma is proved.

Remark 3.4. Under the conditions of the latter lemma, it follows that the operator $C = A + B : Y \to Y^*$ is weakly coercive.

Definition 3.4. *An operator $L : D(L) \subset Y \to Y^*$ is called*
monotone, if for each $y_1, y_2 \in D(L)$ $\langle Ly_1 - Ly_2, y_1 - y_2\rangle_Y \geq 0$;

maximal monotone, if it is monotone and from $\langle w - Lu, v - u\rangle_Y \geq 0$ *for all* $u \in D(L)$ *it follows that* $v \in D(L)$ *and* $Lv = w$.

Remark 3.5. If the reflexive Banach space Y is strictly convex with its conjugate then [2] (Lemma 3.1.1) the linear operator $L : D(L) \subset Y \to Y^*$ is maximal monotone and densly defined if and only if L is a closed unbounded operator such that

$$\langle Ly, y\rangle_Y \geq 0 \quad \forall y \in D(L) \quad \text{and} \quad \langle L^* y, y\rangle_Y \geq 0 \quad \forall y \in D(L^*),$$

where $L^* : D(L^*) \subset Y \to Y^*$ is the conjugate operator of L in the sense of the theory of unbounded operators [8].

4. Auxiliary statements. From (2.1) and (2.2), $V = V_1 \cap V_2 \subset H$ with a continuous and dense embedding. As V is a separable Banach space, there exists a complete in V, and consequently in H, countable system of vectors, $\{h_i\}_{i\geq 1} \subset V$.

Let for each $n \geq 1$, $H_n = \operatorname{span}\{h_i\}_{i=1}^n$. On H_n we consider the inner product induced from H that we again denote by $(\cdot,\cdot)$. Let $P_n : H \to H_n \subset H$ be the operator of orthogonal projection from H on H_n, i.e.,

$$\forall h \in H : \quad P_n h = \operatorname*{argmin}_{h_n \in H_n} \|h - h_n\|_H.$$

Definition 4.1. *We say that a triple* $(\{h_i\}_{i\geq 1}; V; H)$ *satisfies condition* (γ) *if* $\sup\limits_{n\geq 1} \|P_n\|_{\mathcal{L}(V,V)} < +\infty$, *i.e., there exists* $C \geq 1$ *such that*

$$\forall v \in V\ \forall n \geq 1 : \quad \|P_n v\|_V \leq C\|v\|_V.$$

Remark 4.1. When the system of vectors $\{h_i\}_{i\geq 1} \subset V$ is orthogonal in H, condition (γ) means that the given system is a Schauder basis in the space V (in particular in V_1 and in V_2) [9].

Remark 4.2. Since $P_n \in \mathcal{L}(V, V)$, its conjugate operator $P_n^* \in \mathcal{L}(V^*, V^*)$ and $\|P_n\|_{\mathcal{L}(V,V)} = \|P_n^*\|_{\mathcal{L}(V^*,V^*)}$. It is clear that for each $h \in H$ $P_n h = P_n^* h$. Hence, we identify P_n with P_n^*. Then condition (γ) means that for each $v \in V$ and $n \geq 1$, $\|P_n v\|_{V^*} \leq C \cdot \|v\|_{V^*}$.

Due to the equivalence between H^* and H, it follows that $H_n^* \cong H_n$. For each $n \geq 1$ we consider the Banach space $X_n = L_{p_0}(S; H_n) \subset X$, where $p_0 := \max\{r_1, r_2\}$ with the norm $\|\cdot\|_{X_n}$ induced from the space X. This norm is equivalent to the natural norm in $L_{p_0}(S; H_n)$ [1].

The space $L_{q_0}(S; H_n)$ $(q_0^{-1} + p_0^{-1} = 1)$ with the norm

$$\|f\|_{X_n^*} := \sup_{x\in X_n\setminus\{\bar{0}\}} \frac{|\langle f, x\rangle|}{\|x\|_X} = \sup_{x\in X_n\setminus\{\bar{0}\}} \frac{|\langle f, x\rangle_{X_n}|}{\|x\|_{X_n}}$$

is isometrically isomorphic to the conjugate space X_n^* of X_n (further the given spaces are identified); moreover, the map

$$X_n^* \times X_n \ni f, x \to \int\limits_S (f(\tau), x(\tau))_{H_n} d\tau = \int\limits_S (f(\tau), x(\tau)) d\tau = \langle f, x\rangle_{X_n}$$

is a duality form on $X_n^* \times X_n$. This is true due to $X_n^* = L_{q_0}(S; H_n) \subset L_{q_0}(S; H) \subset L_{r_1'}(S; H) + L_{r_2'}(S; H) + L_{q_1}(S; V_1^*) + L_{q_2}(S; V_2^*) = X^*$ (see [1]). Let us remark that $\langle \cdot, \cdot \rangle \big|_{X_n^* \times X_n} = \langle \cdot, \cdot \rangle_{X_n}$.

Proposition 4.1. *For each* $n \geq 1$, $X_n = P_n X$, *i.e.,* $X_n = \{P_n y(\cdot) \mid y(\cdot) \in X\}$, *and we have*

$$\langle f, P_n y \rangle = \langle f, y \rangle \quad \forall y \in X \quad \textit{and} \quad f \in X_n^*.$$

Moreover, if triples $(\{h_j\}_{j \geq 1}; V_i; H)$, $i = 1, 2$, *satisfy condition* (γ) *with* $C = C_i$, *then*

$$\|P_n y\|_X \leq \max\{C_1, C_2\} \cdot \|y\|_X \quad \forall y \in X \quad \textit{and} \quad n \geq 1.$$

Proof. For each $y \in X$ let $y_n(\cdot) := P_n y(\cdot)$, i.e., $y_n(t) = P_n y(t)$ for almost all (a.a.) $t \in S$. Since P_n is linear and continuous on V_1, V_2 and H, we have that $y_n \in X_n \subset X$. It follows from condition (γ) and the definitions of $\|\cdot\|_{L_{p_i}(S;V_i)}$ and $\|\cdot\|_{L_{r_i}(S;H)}$ that $\|y_n\|_{L_{p_i}(S;V_i)} \leq C_i \|y\|_{L_{p_i}(S;V_i)}$ and $\|y_n\|_{L_{r_i}(S;H)} \leq \|y\|_{L_{r_i}(S;H)}$. Thus $\|y_n\|_X \leq \max\{C_1, C_2\} \|y\|_X$.

Now we prove that for all $f \in X_n^*$ $\langle f, y_n \rangle = \langle f, y \rangle$. As $f \in L_{q_0}(S; H_n)$, we have

$$\langle f, y \rangle = \int_S (f(\tau), y(\tau)) d\tau = \int_S (f(\tau), P_n y(\tau)) d\tau = \int_S (f(\tau), y_n(\tau)) d\tau = \langle f, y_n \rangle,$$

since for all $n \geq 1$, $h \in H$ and $v \in H_n$, $(h - P_n h, v) = (h - P_n h, v)_H = 0$.

The proposition is proved.

For each $n \geq 1$ we denote by I_n the canonical embedding of X_n in X ($\forall x \in X_n$ $I_n x = x$), $I_n^* : X^* \to X_n^*$ its conjugate operator. We remark that

$$\|I_n\|_{\mathcal{L}((X_n, \|\cdot\|_X); (X, \|\cdot\|_X))} = \|I_n^*\|_{\mathcal{L}((X^*, \|\cdot\|_{X^*}); (X_n^*, \|\cdot\|_{X_n^*}))} = 1.$$

Proposition 4.2. *For each* $n \geq 1$ *and* $f \in X^*$, $(I_n^* f)(t) = P_n f(t)$ *for a.a.* $t \in S$. *Moreover, if triples* $(\{h_j\}_{j \geq 1}; V_i; H)$, $i = 1, 2$, *satisfy condition* (γ) *with* $C = C_i$, *then*

$$\textit{for all } f \in X^* \textit{ and } n \geq 1 \qquad \|I_n^* f\|_{X^*} \leq \max\{C_1, C_2\} \|f\|_{X^*},$$

i.e.,

$$\sup_{n \geq 1} \|I_n^*\|_{\mathcal{L}(X^*; X^*)} \leq \max\{C_1, C_2\}.$$

Proof. Let $n \geq 1$ and $f \in X^*$ be fixed. Let us show that for a.a. $t \in S$, $(I_n^* f)(t) = P_n f(t)$. From Remark 4.2 it follows that for each $x \in X_n$

$$\langle I_n^* f, x \rangle = \langle f, x \rangle = \int_S (f(\tau), x(\tau)) d\tau =$$

$$= \int_S (f(\tau) - P_n f(\tau), x(\tau)) d\tau + \int_S (P_n f(\tau), x(\tau)) d\tau = \int_S (P_n f(\tau), x(\tau)) d\tau, \qquad (4.1)$$

since for all $u \in H_n$ and $v \in V^*$, $(v - P_n v, u) = 0$.

Let us prove the last equality. For each $v \in V^*$ there exists a sequence $\{v_k\}_{k\geq 1} \subset H$ such that $v_k \to v$ in V^* as $k \to \infty$. It is clear that for each $k \geq 1$, $(v_k - P_n v_k, u) = (v_k - P_n v_k, u)_H = 0$. From the continuity of $(\cdot, \cdot)$ on $V^* \times V$, it follows that $(v - P_n v, u) = \lim\limits_{k\to\infty} (v_k - P_n v_k, u) = 0$. From (4.1) it follows that for a.a. $t \in S$ $(I_n^* f)(t) = P_n f(t)$.

Now we prove the second part of this proposition. Let condition (γ) be true, $f \in X^*$ and $n \geq 1$ be arbitrary and fixed. Then from Remark 4.2 and condition (γ) it follows that for each $f_0 \in L_{q_0}(S; H)$ and $f_i \in L_{q_i}(S; V_i^*)$ such that $f = f_0 + f_1 + f_2$, we have

$$\|I_n^* f_0\|_{L_{q_0}(S;H)} + \|I_n^* f_1\|_{L_{q_1}(S;V^*)} + \|I_n^* f_2\|_{L_{q_2}(S;V_2^*)} =$$

$$= \left(\int\limits_S \|P_n f_0(\tau)\|_H^{q_0} d\tau \right)^{\frac{1}{q_0}} + \left(\int\limits_S \|P_n f_1(\tau)\|_{V_1^*}^{q_1} d\tau \right)^{\frac{1}{q_1}} + \left(\int\limits_S \|P_n f_2(\tau)\|_{V_2^*}^{q_2} d\tau \right)^{\frac{1}{q_2}} \leq$$

$$\leq \left(\int\limits_S \|f_0(\tau)\|_H^{q_0} d\tau \right)^{\frac{1}{q_0}} + C_1 \left(\int\limits_S \|f_1(\tau)\|_{V_1^*}^{q_1} d\tau \right)^{\frac{1}{q_1}} + C_2 \left(\int\limits_S \|f_2(\tau)\|_{V_2^*}^{q_2} d\tau \right)^{\frac{1}{q_2}} \leq$$

$$\leq \max\{C_1, C_2\} \left(\|f_0\|_{L_{q_0}(S;H)} + \|f_1\|_{L_{q_1}(S;V_1^*)} + \|f_2\|_{L_{q_2}(S;V_2^*)} \right),$$

as $C_1, C_2 \geq 1$. Hence, from the definition of $\|f\|_{X^*}$ it follows that

$$\|I_n^* f\|_{X^*} \leq \max\{C_1, C_2\} \|f\|_{X^*}.$$

The proposition is proved.

From the last two propositions and properties of I_n^*, we immediately obtain the following corollary.

Corollary 4.1. *For all* $n \geq 1$, $X_n^* = P_n X^* = I_n^* X$, *i.e.,*

$$X_n^* = \{P_n f(\cdot) \mid f(\cdot) \in X^*\} = \{I_n^* f \mid f \in X^*\}.$$

Proposition 4.3. *The set* $\bigcup\limits_{n\geq 1} X_n$ *is dense in* $(X, \|\cdot\|_X)$.

Proof. 1. At first we prove that the set $L_\infty(S; V)$ is dense in the space $(X, \|\cdot\|_X)$. Let $x \in X$ be arbitrary and fixed. Then for each $n \geq 1$ we consider

$$x_n(t) := \begin{cases} x(t), & \|x(t)\|_V \leq n, \\ 0, & \text{otherwise}. \end{cases} \tag{4.2}$$

Obviously for all $n \geq 1$, $x_n \in L_\infty(S;V)$. From (2.2) it follows that there exists $\gamma > 0$ such that, according to (4.2), as $i = 1,2$ and for a.a. $t \in S$, we get

$$\|x_n(t) - x(t)\|_H \leq \gamma \|x_n(t) - x(t)\|_V \to 0,$$

$$\|x_n(t) - x(t)\|_{V_i} \leq \|x_n(t) - x(t)\|_V \to 0 \quad \text{as} \quad n \to \infty, \tag{4.3}$$

$$\|x_n(t)\|_H \leq \|x(t)\|_H, \ \ \|x_n(t)\|_{V_i} \leq \|x(t)\|_{V_i}. \tag{4.4}$$

Further let

$$\phi_H^n(t) = \|x_n(t) - x(t)\|_H^{p_0}, \quad \phi_{V_i}^n(t) = \|x_n(t) - x(t)\|_{V_i}^{p_i}.$$

Hence from (4.3) and (4.4) we obtain

$$\phi_H^n(t) \to 0, \quad \phi_{V_i}^n(t) \to 0 \ \text{ as } n \to \infty, \tag{4.5}$$

$$|\phi_H^n(t)| \leq 2^{p_0} \|x(t)\|_H^{p_0} =: \phi_H(t), \quad |\phi_{V_i}^n(t)| \leq 2^{p_i} \|x(t)\|_{V_i}^{p_i} =: \phi_{V_i}(t) \tag{4.6}$$

for a.a. $t \in S$. As $x \in X$, we have ϕ_H, ϕ_{V_1}, $\phi_{V_2} \in L_1(S)$. So, because of (4.5) and (4.6), we can apply the Lebesgue theorem with integrable majorants ϕ_H, ϕ_{V_1} and ϕ_{V_2}, respectively (see [10]). Hence it follows that $\phi_H^n \to \bar{0}$ and $\phi_{V_i}^n \to \bar{0}$ in $L_1(S)$ as $i = 1,2$. Consequently, $\|x_n - x\|_X \to 0$ as $n \to \infty$.

2. Further, let for some linear variety L from V

$$\Upsilon(L) := \{y \in (S \to L) \mid y \text{ is a simple function}\}$$

(see [1, p. 152]). Let us prove that the set $\Upsilon(V)$ is dense in the normed space $(L_\infty(S,V), \|\cdot\|_X)$. Every arbitrary fixed element $x \in L_\infty(S,V)$ is measurable, according to Bochner, as a function from the class $(S \to V)$. So, there exists a sequence $\{x_n\}_{n\geq 1} \subset \Upsilon(V)$ such that

$$x_n(t) \to x(t) \text{ in } V \quad \text{as } n \to \infty \quad \text{for a.a. } t \in S. \tag{4.7}$$

Since $x \in L_\infty(S,V)$ it follows that $\operatorname*{ess\,sup}_{t\in S} \|x(t)\|_V =: c < +\infty$. For each $n \geq 1$ we introduce

$$y_n(t) := \begin{cases} x_n(t), & \|x_n(t)\|_V \leq 2c, \\ \bar{0}, & \text{otherwise}. \end{cases} \tag{4.8}$$

From (4.7) and (4.8) it follows that $y_n \in \Upsilon(V)$ as $n \geq 1$ and, moreover,

$$y_n(t) \to x(t) \text{ in } V \quad \text{as } n \to \infty \quad \text{for a.a. } t \in S.$$

Hence, taking into account (2.2), we obtain that as $i = 1,2$ and for a.a. $t \in S$

$$y_n(t) \to x(t) \text{ in } H, \quad y_n(t) \to x(t) \text{ in } V_1, \quad y_n(t) \to x(t) \text{ in } V_2 \quad \text{as } n \to \infty.$$

As in item 1, assuming

$$\phi_H \equiv \phi_{V_1} \equiv \phi_{V_2} \equiv \max\{(3c)^{p_1}, (3c)^{p_2}, (3c\gamma)^{p_0}\} \in L_1(S)),$$

we obtain that $y_n \to x$ in X as $n \to \infty$. So, $\Upsilon(V)$ is dense in $(L_\infty(S,V), \|\cdot\|_X)$.

3. Since the set $\operatorname{span}\{h_n\}_{n\geq 1} = \bigcup_{n\geq 1} H_n$ is dense in $(V, \|\cdot\|_V)$ and (2.2) holds, it is clear that the set $\Upsilon(\bigcup_{n\geq 1} H_n) = \bigcup_{n\geq 1} \Upsilon(H_n)$ is dense in $(\Upsilon(V), \|\cdot\|_X)$.

In order to end the proof we remark that for each $n \geq 1$, $\Upsilon(H_n) \subset X_n$.

The proposition is proved.

Proposition 4.4. *Let $L : D(L) \subset X \to X^*$ be a linear maximal monotone operator. Then the normed space $D(L)$ with the graph norm $\|y\|_{D(L)} = \|y\|_X + \|Ly\|_{X^*}$ is complete (hence, it is weakly complete).*

Proof. Let $\{y_n\}_{n\geq 1} \subset D(L)$ be a Cauchy sequence. Since X is a Banach space, there exists $y \in X$ such that

$$y_n \to y \quad \text{in } X \quad \text{as } n \to \infty. \tag{4.9}$$

Analogously there exists $\chi \in X^*$ such that

$$Ly_n \to \chi \quad \text{in } X^* \quad \text{as } n \to \infty. \tag{4.10}$$

Now we prove that

$$\langle \chi - Lu, y - u\rangle \geq 0 \qquad \text{for each } u \in D(L).$$

Let $u \in D(L)$ be arbitrary and fixed. In virtue of (4.9), (4.10) and since L is monotone on $D(L)$, it follows that for each $n \geq 1$

$$0 \leq \langle Ly_n - Lu, y_n - u\rangle \to \langle \chi - Lu, y - u\rangle \quad \text{as } n \to \infty.$$

Consequently, from the maximal monotony of L, the required statement follows.

The proposition is proved.

5. Faedo – Galerkin method. For each $n \geq 1$ let us set $L_n := I_n^* L I_n : D(L_n) = D(L) \cap \cap X_n \subset X_n \to X_n^*$, $A_n := I_n^* A I_n : X_n \to X_n^*$, $B_n := I_n^* B I_n : X_n \to X_n^*$, $f_n := I_n^* f \in X_n^*$.

Remark 5.1. We will also denote by I_n^* the operators conjugate to the canonical embeddings of X_n in X_1 and of X_n in X_2, because these operators coincide with I_n^* on $X_1^* \cap X_2^*$ which is dense in X_1^*, X_2^*, X^*.

Now we consider $D(L)$ as a normed space with the graph norm $\|y\|_{D(L)} = \|y\|_X + \|Ly\|_{X^*}$ for each $y \in D(L)$. We remark that if the linear operator L is closed and densely defined, then $(D(L), \|\cdot\|_{D(L)})$ is a Banach space continuously embedded in X.

In addition to the problem (2.3), (2.4) we consider the following class of problems:

$$L_n y_n + A_n(y_n) + B_n(y_n) = f_n, \tag*{$(5.1)_n$}$$

$$y_n \in D(L_n). \tag*{$(5.2)_n$}$$

Remark 5.2. We consider on $D(L_n)$ the graph norm $\|y_n\|_{D(L_n)} = \|y_n\|_{X_n} + \|L_n y_n\|_{X_n^*}$ for each $y_n \in D(L_n)$.

Definition 5.1. *We say that a solution to (2.3), (2.4) $y \in D(L)$ is obtained via the Faedo–Galerkin method, if y is the weak limit of a subsequence $\{y_{n_k}\}_{k\geq 1}$ from $\{y_n\}_{n\geq 1}$ in $D(L)$, where for each $n \geq 1$, y_n is a solution to the problem (5.1)$_n$, (5.2)$_n$.*

6. Choice of the basis. As it is well-known from [11] there exist separable Banach spaces that do not have a Schauder basis. Hence we need to introduce some constructions of a basis to satisfy condition (γ).

Definition 6.1. *We say that a system of vectors $\{h_i\}_{i\geq 1}$ of a separable Hilbert space $(V; (\cdot,\cdot)_V)$, continuously and densely embedded in a Hilbert space $(H; (\cdot,\cdot)_H)$, is a special basis for the pair of spaces $(V; H)$ if it satisfies the following conditions:*
$\{h_i\}_{i\geq 1}$ is orthonormal in $(H, (\cdot,\cdot)_H)$;
$\{h_i\}_{i\geq 1}$ is orthogonal in $(V, (\cdot,\cdot)_V)$;
$\forall i \geq 1$ $(h_i, v)_V = \lambda_i (h_i, v)_H$ $\forall v \in V$, where $0 \leq \lambda_1 \leq \lambda_2, \ldots, \lambda_j \longrightarrow \infty$ as $j \longrightarrow \infty$.

Lemma 6.1. *If V is a Hilbert space, compactly and densely embedded in a Hilbert space H, then there exists a special basis $\{h_i\}_{i\geq 1}$ for $(V; H)$. Moreover, for an arbitrary such system, the triple $(\{h_i\}_{i\geq 1}; V; H)$ satisfies condition (γ) with the constant $C = 1$.*

Proof. From [12, p. 54–58], under these assumptions, it is well-known that there exists a special basis $\{h_i\}_{i\geq 1}$ for the pair $(V; H)$. So, in order to complete the proof it is enough to show that the triple $(\{h_i\}_{i\geq 1}; V; H)$ satisfies condition (γ) with the constant $C = 1$ for an arbitrary special basis $\{h_i\}_{i\geq 1}$ for $(V; H)$. Therefore, let $H_n = \operatorname{span}\{h_i\}_{i=1}^n$ and let us denote by P_n the operator of orthogonal projection from H to H_n. Obviously, $P_m \in \mathcal{L}(V; V)$.

Further let us prove that for all $n \geq 1$

$$\|P_n h\|_V \leq \|h\|_V \qquad \forall h \in \bigcup_{m\geq 1} H_m. \tag{6.1}$$

Let $n \geq 1$ be fixed. Then $h \in \bigcup\limits_{m\geq 1} H_m \Rightarrow \exists m_0 \geq n+1 : h \in H_{m_0}$. Whence, since $\{h_i\}_{i\geq 1}$ is orthonormal in H, we have $h = \sum\limits_{i=1}^{m_0} (h, h_i)_H h_i$, $P_n h = \sum\limits_{i=1}^{n} (h, h_i)_H h_i$. In order to obtain (6.1) it is necessary to show that $P_n h$ is orthogonal to $(h - P_n h)$ in V. In fact, $(P_n h, h - P_n h)_V = \left(\sum\limits_{i=1}^{n} (h, h_i)_H h_i, \sum\limits_{j=n+1}^{m_0} (h, h_j)_H h_j \right)_V = \sum\limits_{i=1}^{n} \sum\limits_{j=n+1}^{m} (h, h_i)_H (h, h_j)_H (h_i, h_j)_V = 0$, since $\{h_i\}_{i\geq 1}$ is orthogonal in V. So, from continuity of P_n on V we have that for all $n \geq 1$ and $v \in V$ $\|P_n v\|_V \leq \|v\|_V$.

The lemma is proved.

Now let us make the same for Banach spaces. We consider that I is a subset of $\mathbb{R}$.
Let $\{Z_\alpha\}_{\alpha \in I}$ be a family of Banach spaces such that
for all $\alpha_1, \alpha_2 \in I$, $\alpha_1 < \alpha_2$, $Z_{\alpha_2} \subset Z_{\alpha_1}$ with a continuous embedding;
there exists a set Φ such that for all $\alpha \in I$, Φ is dense in Z_α;
for all $\alpha_0 \in I$ and $x \in \Phi$, $\|x\|_{Z_\alpha} \to \|x\|_{Z_{\alpha_0}}$ as $\alpha \to \alpha_0$, $\alpha \in I$.
We also consider a Banach space H such that $Z_\alpha \subset H$ with continuous embeddings for all $\alpha \in I$ and the set Φ is dense in H.

Let $\{h_j\}_{j\geq 1} \subset \Phi$ be a system of vectors.

Proposition 6.1. *Let the above assumptions be true. If for some $\alpha_0 \in I$ the triple $(\{h_j\}_{j\geq 1}; Z_{\alpha_0}; H)$ satisfies condition (γ) with the constant $C \geq 1$, then the set of $\alpha \in I$ for which the triple $(\{h_j\}_{j\geq 1}; Z_\alpha; H)$ satisfies the same condition with the same constant is closed in I.*

Proof. For an arbitrary $\alpha \in I$, statement $G(\alpha)$ means that the triple $(\{h_j\}_{j\geq 1}; Z_\alpha; H)$ satisfies condition (γ) with the constant C.

We denote

$$I_+ = \{\alpha \in I \mid G(\alpha) \text{ is true}\} \quad \text{and} \quad I_- = I \setminus I_+.$$

Let $\alpha \in I$ be an arbitrary cluster point of I_+. Then there exists $\{\alpha_n\}_{n\geq 1} \subset I_+$ such that $\alpha_n \to \alpha$. For each fixed element $x \in \Phi$, by using the definition of I_+,

$$\forall m \geq 1 \quad \forall x \in \Phi \quad \forall n \geq 1 \;: \qquad \|P_n x\|_{Z_{\alpha_m}} \leq C\|x\|_{Z_{\alpha_m}}$$

and passing to the limit as $m \to +\infty$ in the last inequality, we obtain

$$\|P_n x\|_{Z_\alpha} \leq C\|x\|_{Z_\alpha} \qquad \forall x \in \Phi \quad \forall n \geq 1.$$

Then from density of Φ in Z_α and continuity, P_n on Z_α, statement $G(\alpha)$ follows. So, $\alpha \in I_+$, i.e., the statement $G(\alpha)$ is true.

The proposition is proved.

Now we consider one application of the above proposition. But at first we need to give some definitions from the interpolation theory.

For an interpolation pair A_0, A_1 (i.e., Banach spaces A_0 and A_1 that are continuously embedded in some linear topological space) let us consider the functional

$$K(t,x) = \inf_{x=x_0+x_1:\; x_0\in A_0,\; x_1\in A_1} \left(\|x_0\|_{A_0} + t\|x_1\|_{A_1}\right), \qquad t > 0,\; x \in A_0 + A_1.$$

For fixed $x \in A_0 + A_1$, this map is a monotone increasing, continuous, concave function of the variable $t > 0$ (see [9], Lemma 1.3.1).

For $\theta \in (0,1)$ and $1 < p < +\infty$ let us consider the following space:

$$(A_0, A_1)_{\theta,p} = \left\{ x \in A_0 + A_1 \;\Big|\; \int_0^{+\infty} \left[t^{-\theta} K(t,x)\right]^p \frac{dt}{t} < +\infty \right\}; \tag{6.2}$$

$(A_0, A_1)_{\theta,p}$ with $\|x\|_{\theta,p} = \left(\int_0^{+\infty} [t^{-\theta}K(t,x)]^p \frac{dt}{t}\right)^{\frac{1}{p}}$ it is a Banach space (for more details see [9]) and this results in (see [9], Theorem 1.3.3):

$$A_0 \cap A_1 \subset (A_0, A_1)_{\theta,p} \subset A_0 + A_1 \qquad \forall\theta \in (0,1) \quad \forall 1 < p < +\infty \tag{6.3}$$

with dense and continuous embeddings.

Definition 6.2. *Let $1 \leq r < 2$. We say that a filter of Banach spaces $\{Z_p\}_{p\geq r}$, a Hilbert space H, and a system of vectors, $\{h_i\}_{i\geq 1}$, complete in Z_p $\forall p \geq r$ satisfy the main conditions, if* *a)* $p_2 > p_1 > r$ $Z_{p_2} \subset Z_{p_1} \subset H$ *with continuous and dense embeddings;*

b) $p_2 > p > p_1 > r$ $(Z_{p_1}, Z_{p_2})_{\theta,p} = Z_p$, *where* $\theta = \theta(p) \in (0,1) : \dfrac{1}{p} = \dfrac{1-\theta}{p_1} + \dfrac{\theta}{p_2}$;

c) Z_2 *is a Hilbert space;*

d) for some $C \geq 1$ *the triple* $(\{h_i\}_{i\geq 1}; Z_2; H)$ *satisfies condition* (γ) *with a constant* C, *the set* $I_+(C) = \{p \geq 2 \mid$ *the triple* $(\{h_i\}_{i\geq 1}; Z_p; H)$ *does not satisfy condition* (γ) *with the constant* $C\}$ *(if it is not empty) contains its minimal element and the set* $I_-(C) = \{p \in [r; 2] \mid$ *the triple* $(\{h_i\}_{i\geq 1}; Z_p; H)$ *does not satisfy condition* (γ) *with the constant* $C\}$ *(if it is not empty) contains its maximal element.*

Lemma 6.2. *Let* $1 \leq r < 2$, $\{Z_p\}_{p\geq r}$, *be a filter of Banach spaces,* H *a Hilbert space* H, *and a system of vectors,* $\{h_i\}_{i\geq 1}$, *complete in* Z_p $\forall p \geq r$ *satisfy the main conditions. Then, for all* $p > r$ *the triple* $(\{h_i\}_{i\geq 1}; Z_p; H)$ *satisfies condition* (γ).

Remark 6.1. In the case $Z_2 \subset H$ with compact a embedding, due to Lemma 6.1, as the vector system $\{h_i\}_{i\geq 1}$ we can choose a special basis for the pair $(Z_2; H)$. In particular, the above result means that the special basis for $(Z_2; H)$ is a Schauder basis for an arbitrary space Z_p as $r < p \leq 2$.

Proof. Let $N > 2$ and $M \in (r, 2)$ be arbitrary fixed numbers. Now we apply Proposition 6.1 with $I = (M, N)$, $\alpha_0 = 2$, $\Phi = Z_N$. In order to do this, it is sufficient to prove that

$$\|x\|_{Z_q} \to \|x\|_{Z_p} \quad \text{as} \quad q \to p\,(q \in I) \quad \forall p \in I\, \forall x \in Z_N. \tag{6.4}$$

Let p be an arbitrary element of I (hence there exists δ such that $[p-\delta, p+\delta] \subset I$), x be a fixed element of the space Z_N. From (6.2) and the main condition b) for $\{Z_p\}_{p\geq r}$ and H, for all $q \in [p-\delta, p+\delta]$ it results in

$$\|x\|_{Z_q} = \|x\|_{(Z_M, Z_N)_{\theta,q}} = \left(\int\limits_0^{+\infty} \left[t^{-\theta} K(t,x)\right]^q \frac{dt}{t} \right)^{\frac{1}{q}}, \tag{6.5}$$

where $1/q = (1-\theta(q))/M + \theta(q)/N$, i.e.,

$$\theta(q) = \frac{\frac{1}{M} - \frac{1}{q}}{\frac{1}{M} - \frac{1}{N}} \in [\theta(p-\delta), \theta(p+\delta)] = \left[\frac{\frac{1}{M} - \frac{1}{p-\delta}}{\frac{1}{M} - \frac{1}{N}}, \frac{\frac{1}{M} - \frac{1}{p+\delta}}{\frac{1}{M} - \frac{1}{N}} \right] \subset (0,1).$$

The following relations prove (6.4). Denote

$$f(t,q) = \left[t^{-\theta(q)} K(t,x)\right]^q \frac{1}{t} \qquad \forall (t,q) \in (0,+\infty) \times [p-\delta, p+\delta].$$

From (6.2) and (6.5) it follows that for each $q \in [p-\delta, p+\delta]$, we have $f(\cdot, q) \in L_1[0,+\infty)$; moreover, for each $t \in (0,+\infty)$, $f(t,\cdot) \in C([p-\delta, p+\delta])$. Furthermore, noticing that for each $t > 0$ and $q \in [p-\delta, p+\delta]$,

$$\left[t^{-\theta(q)} K(t,x)\right]^q \frac{1}{t} \leq \max\left\{ \left[t^{-\theta(p-\delta)} K(t,x)\right]^{p-\delta}, \left[t^{-\theta(p-\delta)} K(t,x)\right]^{p+\delta}, \right.$$

$$\left[t^{-\theta(p+\delta)}K(t,x)\right]^{p-\delta}, \left[t^{-\theta(p+\delta)}K(t,x)\right]^{p+\delta}\Big\}\frac{1}{t} =: g(t),$$

and having in mind that (6.3) holds and $x \in Z_N = Z_M \cap Z_N$, we have

$$\int\limits_0^{+\infty} g(t)dt \le 4\max\left\{\int\limits_0^{+\infty}\left[t^{-\theta(p-\delta)}K(t,x)\right]^{p-\delta}\frac{dt}{t},\right.$$

$$\int\limits_0^{+\infty}\left[t^{-\theta(p-\delta)}K(t,x)\right]^{p+\delta}\frac{dt}{t}, \int\limits_0^{+\infty}\left[t^{-\theta(p+\delta)}K(t,x)\right]^{p-\delta}\frac{dt}{t},$$

$$\left.\int\limits_0^{+\infty}\left[t^{-\theta(p+\delta)}K(t,x)\right]^{p+\delta}\frac{dt}{t}\right\} = 4\max\left\{\|x\|^{p-\delta}_{(Z_M,Z_N)_{\theta(p-\delta),p-\delta}}, \|x\|^{p+\delta}_{(Z_M,Z_N)_{\theta(p-\delta),p+\delta}},\right.$$

$$\left.\|x\|^{p-\delta}_{(Z_M,Z_N)_{\theta(p+\delta),p-\delta}}, \|x\|^{p+\delta}_{(Z_M,Z_N)_{\theta(p+\delta),p+\delta}}\right\}.$$

Thus, the theorem of continuous dependence of the Lebesgue integral on a parameter [13] (Theorem 8.1.1) assures the convergence (6.4).

To finish the proof we remark that the set $\{p \ge 2 \,|$ the triple $(\{h_i\}_{i\ge1}; Z_p; H)$ does not satisfy condition (γ) with the constant $C\}$ contains its minimal element (respectively, the set $I_-(C) = \{p \in [r;2] \mid$ the triple $(\{h_i\}_{i\ge1}; Z_p; H)$ does not satisfy condition (γ) with the constant $C\}$ contains its maximal element), which contradicts Proposition 6.1.

Corollary 6.1. *Let V_1, V_2 be Banach spaces, continuously embedded in the Hilbert space H. Let us assume that for some filter of Banach spaces $\{Z^i_p\}_{p\ge r_i}$, $r_i \in [1;2)$, $i = 1,2$, there exists $p_i > r_i$ such that $V_i = Z^i_{p_i}$, within to equivalent norms. Moreover, there exists a Hilbert space $Z \subset V_1 \cap V_2$, compactly and densely embedded in H, such that for a special basis $\{h_j\}_{j\ge1}$ for the pair $(Z;H)$ with $\{h_j\}_{j\ge1} \subset \cap_{p>r_i} Z^i_p$ for some $0 \le \mu_1 \le \mu_2, ..., \mu_j \longrightarrow \infty$ as $j \longrightarrow \infty$ and $s_i > 0$, $i = 1,2$,*

$$Z^i_2 = \left\{u \in H \,\Big|\, \sum_{j=1}^{\infty}\mu_j^{s_i}(u,h_j)^2 < +\infty\right\}$$

is a Hilbert space with the inner product

$$(u,v)_{Z^i_2} = \sum_{j=1}^{\infty}\mu_j^{s_i}(u,h_j)(v,h_j);$$

$\{Z^i_p\}_{p\ge r_i}$ together with H and the system of vectors $\{h_i\}_{i\ge1}$ satisfies the main conditions. Then the triple $(\{h_j\}_{j\ge1}; V_i; H)$ satisfies condition (γ).

Proof. Having in mind the proof of Lemmas 6.1 and 6.2, it is enough to show that $\{h_j\}_{j\ge1}$ is a special basis for $(Z^i_2;H)$. In fact, since $\{h_j\}_{j\ge1}$ is orthogonal in H we have

$$\forall r,s \ge 1: \qquad (h_r,h_s)_{Z^i_2} = \sum_{j=1}^{\infty}\mu_j^{s_i}(h_r,h_j)(h_s,h_j) = \mu_r^{s_i}\begin{cases}1, & r = s,\\ 0, & r \ne s,\end{cases}$$

$$\forall r \geq 1: \qquad (h_r, v)_{Z_2^i} = \sum_{j=1}^{\infty} \mu_j^{s_i}(h_r, h_j)(v, h_j) = \mu_r^{s_i}(h_r, v) \quad \forall v \in Z_2^i.$$

The corollary is proved.

Remark 6.2. In what follows we may assume that the triple of spaces V_1, V_2 and H satisfies the conditions of Corollary 6.1!

7. The main resolvability theorem.

Theorem 7.1. *Let $L : D(L) \subset X \to X^*$, $A : X_1 \to X_1^*$, and $B : X_2 \to X_2^*$ be maps such that*

1) L is linear maximal monotone and satisfies the following conditions:

(L_1) for each $n \geq 1$ and $x_n \in D(L_n) = X_n \cap D(L)$, $Lx_n \in X_n^*$;

(L_2) for each $n \geq 1$, the set $D(L_n)$ is dense in X_n;

(L_3) for each $n \geq 1$, L_n is a maximal monotone operator;

2) there exist Banach spaces W_1 and W_2 such that $W_1 \subset X_1$, $W_2 \subset X_2$, and $D(L) \subset W_1 \cap W_2$ with a continuous embedding;

3) A is λ_0-pseudomonotone on W_1 and satisfies condition (Π);

4) B is λ_0-pseudomonotone on W_2 and satisfies condition (Π);

5) the pair $(A; B)$ is s-mutually bounded and the sum $C = A + B : X \to X^*$ is finite-dimensionally locally bounded and weakly coercive.

Furthermore, let $\{h_j\}_{j\geq 1} \subset V$ be a complete system of vectors in V_1, V_2, H such that for $i = 1, 2$ the triple $(\{h_j\}_{j\geq 1}; V_i; H)$ satisfies condition (γ).

Then for each $f \in X^*$ the set

$$K_H(f) := \big\{ y \in D(L) \;\big|\; y \text{ is a solution to (2.3), (2.4), obtained via the Faedo – Galerkin method} \big\}$$

is nonempty and we have the representation

$$K_H(f) = \bigcap_{n\geq 1} \Big[\bigcup_{m\geq n} K_m(f_m) \Big]_{X_w}, \tag{7.1}$$

where for each $n \geq 1$,

$$K_n(f_n) = \{ y_n \in D(L_n) \mid y_n \text{ is a solution of } (5.1)_n, (5.2)_n \}$$

and $[\,\cdot\,]_{X_w}$ is the closure of an operator in the space X with respect to the weak topology.

Moreover, if the operator $A + B : X \to X^*$ is coercive, then $K_H(f)$ is weakly compact.

Remark 7.1. A sufficient condition for getting the weak coercivity of $A + B$ is the following: A is coercive and satisfies condition (κ) on X_1, B is coercive and satisfies condition (κ) on X_2 (see Lemma 3.2).

Remark 7.2. From condition L_2 on the operator L and from Proposition 4.3, it follows that L is densely defined.

Proof. By Lemma 3.1 and Remark 3.2 we consider a λ_0-pseudomonotone on $W_1 \cap W_2$ (and hence on $D(L)$), finite-dimensionally locally bounded, weakly coercive map,

$$X \ni y \to C(y) := A(y) + B(y) \in X^*,$$

which satisfies condition (Π). Let $f \in X^*$ be fixed. Now we use the weak coercivity condition for C. There exists $R > 0$ such that

$$\langle C(y) - f, y \rangle \geq 0 \qquad \forall y \in X : \ \|y\|_X = R. \tag{7.2}$$

7.1. Resolvability of the approximating problems.

Lemma 7.1. *For all $n \geq 1$ there exists a solution of the problem (5.1)$_n$, (5.2)$_n$ $y_n \in D(L_n)$ such that $\|y_n\|_X \leq R$.*

Proof. In order to obtain this result we need to prove that for each $n \geq 1$ $C_n := A_n + B_n = I_n^*(A + B) : X_n \to X_n^*$ satisfies the following:

$i_1)$ C_n satisfies condition (Π);

$i_2)$ C_n is λ_0-pseudomonotone on $D(L_n)$, locally finite-dimensionally bounded;

$i_3)$ $\langle C_n(y_n) - f_n, y_n \rangle \geq 0 \qquad \forall y_n \in X_n : \ \|y_n\|_{X_n} = R.$

Let us consider i_1). Let $B \subset X_n$ be some nonempty bounded subset and $k > 0$ be a constant such that

$$\langle C_n(y), y \rangle \leq k \quad \text{for each } y \in B.$$

Since for each $y \in X_n$, $\langle C_n(y), y \rangle = \langle I_n^* C(y), y \rangle = \langle C(y), y \rangle$, we have

$$\langle C(y), y \rangle \leq k \quad \text{for each } y \in B.$$

Since C satisfies condition (Π), there exists $K > 0$ such that

$$\|C(y)\|_{X^*} \leq K \quad \text{for all } y \in B.$$

Consequently,

$$\sup_{y \in B} \|C_n(y)\|_{X^*} \leq K \|I_n^*\|_{\mathcal{L}(X^*; X_n^*)} < +\infty.$$

Now we consider i_2). Because of the boundedness of $I_n \in \mathcal{L}(X_n; X)$, $I_n^* \in \mathcal{L}(X^*; X_n^*)$ and the locally finite-dimensional boundedness of $C : X \to X^*$, it follows that C_n on X_n is locally finite-dimensional bounded.

Now we prove the λ_0-pseudomonotony of C_n on $D(L_n)$. Let $\{y_m\}_{m \geq 0} \subset D(L_n)$ be an arbitrary sequence such that $y_m \rightharpoonup y_0$ in $D(L_n)$, $C_n(y_m) \rightharpoonup d \in X_n^*$ as $m \to +\infty$ and inequality (3.1) hold. As $D(L_n) \subset D(L)$ with continuous embedding,

$$y_m \rightharpoonup y_0 \text{ in } D(L) \quad \text{as } m \to +\infty. \tag{7.3}$$

Since for all $m \geq 1$

$$\langle I_n^* C(y_m), y_m - y_0 \rangle = \langle C(y_m), y_m - y_0 \rangle,$$

we have

$$\overline{\lim_{m\to\infty}}\langle C(y_m), y_m - y_0\rangle = \overline{\lim_{m\to\infty}}\langle C_n(y_m), y_m - y_0\rangle \le 0. \tag{7.4}$$

Hence,

$$\overline{\lim_{m\to\infty}}\langle C(y_m), y_m\rangle \le \overline{\lim_{m\to\infty}}\langle C_n(y_m), y_m - y_0\rangle + \overline{\lim_{m\to\infty}}\langle C_n(y_m), y_0\rangle \le \langle d, y_0\rangle < +\infty.$$

Since C satisfies condition (Π), we have that the sequence $\{C(y_m)\}_{m\ge 1}$ is bounded in X^*. Hence, for a subsequence,

$$C(y_m) \rightharpoonup g \text{ in } X^* \text{ as } m \to \infty.$$

Consequently from (7.3) and (7.4), we get the existence of a subsequence $\{y_{m_k}\}_{k\ge 1} \subset \{y_m\}_{m\ge 1}$ such that for all $w \in X$

$$\underline{\lim_{k\to\infty}}\,\langle C(y_{m_k}), y_{m_k} - w\rangle \ \ge\ \langle C(y_0), y_0 - w\rangle.$$

This means that for each $w \in X_n$

$$\underline{\lim_{k\to\infty}}\,\langle C_n(y_{m_k}), y_{m_k} - w\rangle \ \ge\ \langle C_n(y_0), y_0 - w\rangle.$$

So, C_n is λ_0-pseudomonotone on $D(L_n)$.

Condition $i_3)$ holds thanks to (7.2).

Now let us continue the proof of the lemma. From [14] (Theorem 2.1) with $V = W = X = X_n$, $A = C_n$, $B \equiv \bar{0}$, $L = L_n$, $D(L) = D(L_n)$, $f = f_n$, $r = R$ and the properties $i_1)$–$i_3)$ for C_n, $L_2 - L_3$ for L_n, it follows that the problem $(5.1)_n$, $(5.2)_n$ has at least one solution $y_n \in D(L_n)$ such that $\|y_n\|_X \le R$.

The lemma is proved.

Let us remark that under condition (Π) imposed on C_n it is easy to find the next estimate (7.7) from which it is possible to use the λ_0-pseudomonotony for C on $D(L_n)$.

7.2. Passing to limit. Due to the Lemma 7.1 we have a sequence of Galerkin approximate solutions $\{y_n\}_{n\ge 1}$ that satisfies the next conditions:

$$\forall n \ge 1: \quad \|y_n\|_X \le R, \tag{7.5}$$

$$\forall n \ge 1: \quad y_n \in D(L_n) \subset D(L), \quad L_n y_n + C_n(y_n) = f_n. \tag{7.6}$$

In order to prove the above theorem we need to obtain an important result.

Lemma 7.2. *Let, for some subsequence $\{n_k\}_{k\ge 1}$ from the natural scale, the sequence $\{y_{n_k}\}_{k\ge 1}$ satisfy the next conditions:*

for all $k \ge 1$, $\quad y_{n_k} \in D(L_{n_k}) = D(L) \cap X_{n_k}$;

for all $k \ge 1$, $\quad L_{n_k} y_{n_k} + C_{n_k}(y_{n_k}) = f_{n_k}$;

$y_{n_k} \rightharpoonup y$ in X as $k \to \infty$ for some $y \in X$.

Then, $y \in K_H(f)$.

Proof. From the definitions of L_{n_k}, C_{n_k} and f_{n_k} for each $k \geq 1$, it follows that

$$\langle C_{n_k}(y_{n_k}), y_{n_k}\rangle = \langle f_{n_k} - L_{n_k} y_{n_k}, y_{n_k}\rangle = \langle f - L y_{n_k}, y_{n_k}\rangle \leq$$

$$\leq \|f\|_{X^*} \sup_{k\geq 1} \|y_{n_k}\|_X =: K_1 < +\infty,$$

where K_1 is a constant which does not depend on $k \geq 1$. Hence, due to property (Π) of the operator C, it follows that there exists $K_2 > 0$ such that for each $k \geq 1$

$$\|C(y_{n_k})\|_{X^*} \leq K_2 < +\infty. \tag{7.7}$$

Using the condition L_1 for L and Proposition 4.2, that for all $k \geq 1$

$$\|L y_{n_k}\|_{X^*} = \|L_{n_k} y_{n_k}\|_{X^*} = \|I^*_{n_k}(f - C(y_{n_k}))\|_{X^*} \leq$$

$$\leq \max\{C_1, C_2\}\,(\|f\|_{X^*} + K_2) =: K_3 < +\infty, \tag{7.8}$$

where K_3 is a constant which does not depend on $k \geq 1$. Hence, for each $k \geq 1$,

$$\|y_{n_k}\|_{D(L)} = \|y_{n_k}\|_X + \|L y_{n_k}\|_{X^*} \leq \sup_{k\geq 1}\|y_{n_k}\|_X + K_3 =: K_4 < +\infty,$$

where K_4 is a constant that does not depend on $k \geq 1$. Consequently, due to (7.7), Proposition 4.4 and Banach – Alaoglu theorem, there exists a subsequence $\{y_m\}$ of $\{y_{n_k}\}$ such that for some $y \in D(L)$ and $d \in X^*$ the next convergence takes place:

$$y_m \rightharpoonup y \text{ in } D(L), \qquad C(y_m) \rightharpoonup d \text{ in } X^*. \tag{7.9}$$

1. Let us prove that

$$\lim_{m\to\infty} \langle L y_m + C(y_m), y_m - y\rangle = 0. \tag{7.10}$$

Since the set $\bigcup_{n\geq 1} X_n$ is dense in X, for each m there exists $u_m \in X_m$ (for example $u_m \in \in \operatorname{argmin}_{v_m \in X_m} \|y - v_m\|_X$) such that $u_m \to y$ in X. So, due to (7.8), (7.7) we obtain that for each m

$$|\langle L y_m + C(y_m), y_m - y\rangle| \leq |\langle L y_m + C(y_m), y_m - u_m\rangle| + |\langle L y_m + C(y_m), u_m - y\rangle| \leq$$

$$\leq |\langle f, y_m - u_m\rangle| + (K_3 + K_2)\cdot \|y - u_m\|_X \to |\langle f, y - y\rangle| = 0.$$

2. Now we obtain that

$$\overline{\lim_{m\to\infty}}\, \langle C(y_m), y_m - y\rangle \leq 0. \tag{7.11}$$

From (7.10), (7.9) and from monotonicity of L, we have

$$\overline{\lim_{m\to\infty}} \langle C(y_m), y_m - y\rangle = \lim_{m\to\infty} \langle Ly_m + C(y_m), y_m - y\rangle - \lim_{m\to\infty} \big(\langle Ly_m - Ly, y_m - y\rangle +$$

$$+ \langle Ly, y_m - y\rangle\big) \le 0 + \overline{\lim_{m\to\infty}} \big(- \langle Ly_m - Ly, y_m - y\rangle\big) + \overline{\lim_{m\to\infty}} \langle Ly, y - y_m\rangle \le 0.$$

It follows from (7.9) and (7.11) that we can use the λ_0-pseudomonotonicity of C on $D(L)$. Hence, there exists a subsequence $\{y_k\}_k$ from $\{y_m\}_m$ such that

$$\forall \omega \in X : \quad \underline{\lim_{k\to\infty}} \langle C(y_k), y_k - \omega\rangle \ge \langle C(y), y - \omega\rangle. \tag{7.12}$$

We remark that the last relation is true as implied by Proposition 4.3. In particular, from (7.11) and (7.12) it follows that

$$\lim_{k\to\infty} \langle C(y_k), y_k - y\rangle = 0.$$

3. Let us prove that

$$\forall u \in D(L) \bigcap \Big(\bigcup_{n\ge 1} X_n\Big) : \quad \langle f - d - Ly + Lu, u\rangle \ge 0. \tag{7.13}$$

In order to prove (7.13) it is necessary to obtain that

$$\forall u \in D(L) \bigcap \Big(\bigcup_{n\ge 1} X_n\Big) : \quad \underline{\lim_{k\to\infty}} \langle Ly_k - Ly + Lu, u\rangle \ge 0. \tag{7.14}$$

Since L is monotone and using (7.9), for each $u \in D(L) \bigcap \big(\bigcup_{n\ge 1} X_n\big)$ we have

$$\underline{\lim_{k\to\infty}} \langle Ly_k - Ly + Lu, u\rangle \ge \underline{\lim_{k\to\infty}} \langle Ly_k - Ly, u\rangle = 0.$$

Further let $u \in D(L) \bigcap \big(\bigcup_{n\ge 1} X_n\big)$ be arbitrary and fixed. Then there exists $n_0 \ge 1$ such that $u \in D(L) \cap X_{n_0}$ and, for each $k : k \ge n_0$,

$$\langle Ly_k, u\rangle = \langle L_k y_k, u\rangle = \langle I_k^*(f - C(y_k)), u\rangle = \langle f - C(y_k), u\rangle \to \langle f - d, u\rangle. \tag{7.15}$$

So, (7.13) directly follows from (7.14) and (7.15).

4. Now we prove that $Ly = f - d$. Let us use (7.13). We obtain that for each $t > 0$ and $u \in D(L) \bigcap \big(\bigcup_{n\ge 1} X_n\big)$,

$$\langle f - d - Ly, t \cdot u\rangle \;\; \ge -\langle t \cdot Lu, t \cdot u\rangle,$$

which is equivalent to

$$\langle f - d - Ly, u\rangle \;\; \ge -t \cdot \langle Lu, u\rangle.$$

Hence,

$$\forall u \in D(L)\bigcap\big(\bigcup_n X_n\big): \quad \langle f-d-Ly, u\rangle \geq 0,$$

and, by the Proposition 4.3, the last relation is equivalent to $Ly = f - d$.

5. In order to prove that $y \in D(L)$ is a solution to (2.3), (2.4) it is enough to show that $d = C(y)$. Because of (7.11), (7.12) and (7.9), it follows that for each $\omega \in X$

$$\langle C(y), y-\omega\rangle \leq \varliminf_{k\to\infty} \langle C(y_k), y_k-\omega\rangle \leq$$

$$\leq \varlimsup_{k\to\infty} \langle C(y_k), y_k-y\rangle + \lim_{k\to\infty} \langle C(y_k), y-\omega\rangle \leq \langle d, y-\omega\rangle,$$

which is equivalent to the required statement. So, $y \in K_H(f)$.

The lemma is proved.

Using (7.5), (7.6), Lemma 7.2, the Banach – Alaoglu theorem and the topological property of the upper limit [15] (Property 2.29.IV.8) we see that

$$\varnothing \neq \bigcap_{n\geq 1}\Big[\bigcup_{m\geq n} K_m(f_m)\Big]_{X_w} \subset K_H(f).$$

The converse inclusion is obvious; it follows from the same topological property of the upper limit and from $D(L) \subset X$ with a continuous embedding.

Now let us prove that $K_H(f)$ is weakly compact under the coercivity condition on the operator $C = A + B : X \to X^*$. Because of (7.1) it is enough to show that the given set is bounded. We obtain this statement arguing by contradiction. If $\{y_n\}_{n\geq 1} \subset K_H(f)$ is such that

$$\|y_n\|_X \to +\infty \quad \text{as} \quad n \to \infty,$$

we obtain the contradiction

$$+\infty \leftarrow \frac{1}{\|y_n\|_X}\langle C(y_n), y_n\rangle \leq \frac{1}{\|y_n\|_X}\langle Ly_n + C(y_n), y_n\rangle =$$

$$= \frac{1}{\|y_n\|_X}\langle f, y_n\rangle \leq \|f\|_{X^*} < +\infty.$$

8. An Application. ***8.1. On searching the periodic solutions of differential-operator equations via the Faedo – Galerkin method.*** Let $A : X_1 \to X_1^*$ and $B : X_2 \to X_2^*$ be single-valued maps. We consider the next problem:

$$y' + A(y) + B(y) = f, \tag{8.1}$$

$$y(0) = y(T) \tag{8.2}$$

in order to find solutions via the Faedo – Galerkin method in the class $W = \{y \in X \mid y' \in X^*\}$, where the derivative y' of an element $y \in X$ is considered in the sense of the scalar distributions space $D^*(S;V^*) = \mathcal{L}(D(S);V_w^*)$, with $V = V_1 \cap V_2$, V_w^* equal to V^* and the topology $\sigma(V^*,V)$ [16]. On W we consider the norm $\|y\|_W = \|y\|_X + \|y'\|_{X^*}$ for each $y \in W$. We also consider the spaces $W_i = \{y \in X_i \mid y' \in X^*\}$, $i = 1,2$.

Remark 8.1. It is clear that the space W is continuously embedded in $C(S;V^*)$. Hence, the condition (8.2) makes sense.

Together with the problem (8.1), (8.2) we consider the next class of problems in order to search for solutions in $W_n = \{y \in X_n \mid y' \in X_n^*\}$:

$$y_n' + A_n(y_n) + B_n(y_n) = f_n, \qquad (8.3)_n$$

$$y_n(0) = y_n(T), \qquad (8.4)_n$$

where the maps A_n, B_n, f_n were introduced in Section 5, the derivative y_n' of an element $y_n \in X_n$ is considered in the sense of $D^*(S;H_n)$.

Let $W_{\text{per}} := \{y \in W \mid y(0) = y(T)\}$, let us introduce a map $L : D(L) = W_{\text{per}} \subset X \to X^*$ in such way that $Ly = y'$ for each $y \in W_{\text{per}}$.

The main solvability theorem gives the next corollary.

Corollary 8.1. *Let $A : X_1 \to X_1^*$ and $B : X_2 \to X_2^*$ be maps such that*

1) A is λ_0-pseudomonotone on W_1 and satisfies condition (Π);

2) B is λ_0-pseudomonotone on W_2 and satisfies condition (Π);

3) the pair $(A;B)$ is s-mutually bounded and the sum $C = A + B : X \to X^*$ is finite-dimensionally locally bounded and weakly coercive.

Furthermore, let $\{h_j\}_{j\ge1} \subset V$ be a complete system of vectors in V_1, V_2, H such that, as $i = 1,2$, the triple $(\{h_j\}_{j\ge1};V_i;H)$ satisfies condition (γ).

Then for each $f \in X^*$ the set

$$K_H^{\text{per}}(f) := \big\{y \in W \;\big|\; y \text{ is a solution to (8.1), (8.2), obtained via the Faedo – Galerkin method}\big\}$$

is nonempty and we have the representation

$$K_H^{\text{per}}(f) = \bigcap_{n\ge1}\Big[\bigcup_{m\ge n} K_m^{\text{per}}(f_m)\Big]_{X_w},$$

where for each $n \ge 1$

$$K_n^{\text{per}}(f_n) = \{y_n \in W_n \mid y_n \text{ is a solution to } (8.3)_n, (8.4)_n\}.$$

Moreover, if the operator $A + B : X \to X^*$ is coercive, then $K_H^{\text{per}}(f)$ is weakly compact.

Proof. At first let us prove the maximal monotonicity of L on W_{per}. For $v \in X$, $w \in X^*$ such that, for each $u \in W_{\text{per}}$, $\langle w - Lu, v - u\rangle \ge 0$ is true, let us prove that $v \in W_{\text{per}}$ and $v' = w$.

If we take $u = h\varphi x \in W_{\rm per}$ with $\varphi \in \mathcal{D}(S)$, $x \in V$ and $h > 0$, we get

$$0 \leq \langle w - \varphi' hx, v - \varphi hx\rangle = \langle w, v\rangle -$$

$$- \left(\int_S (\varphi'(s)v(s) + \varphi(s)w(s))ds, hx \right) + \langle \varphi' hx, \varphi hx\rangle =$$

$$= \langle w, v\rangle + h\langle v'(\varphi) - w(\varphi), x\rangle,$$

where $v'(\varphi)$ and $w(\varphi)$ are values of the distributions v' and w on $\varphi \in \mathcal{D}(S)$. So, for each $\varphi \in \mathcal{D}(S)$ and $x \in V$, $\langle v'(\varphi) - w(\varphi), x\rangle \geq 0$ is true. Thus we obtain $v'(\varphi) = w(\varphi)$ for all $\varphi \in \mathcal{D}(S)$. It means that $v' = w \in X^*$. Now we prove $v(0) = v(T)$. If we use [1] with $u(t) \equiv v(T) \in W_{\rm per}$, we obtain that

$$0 \leq \langle v' - Lu, v - u\rangle = \langle v' - u', v - u\rangle =$$

$$= \frac{1}{2}\Big(\|v(T) - v(T)\|_H^2 - \|v(0) - v(T)\|_H^2 \Big) = -\frac{1}{2}\|v(0) - v(T)\|_H^2 \leq 0$$

and then $v(0) = v(T)$.

In order to prove this statement, it is enough to show that L satisfies the conditions $L_1 - L_3$. The condition L_1 follows from the next proposition.

Proposition 8.1. *For each $y \in X$ and $n \geq 1$, $P_n y' = (P_n y)'$, where the derivative of an element $x \in X$ has to be considered in the sense of $D^*(S; V^*)$.*

Proof. It is sufficient to show that for any $\varphi \in \mathcal{D}(S)$, $P_n y'(\varphi) = (P_n y)'(\varphi)$. In fact, from the definition of the derivative in the sense of $D^*(S; V^*)$ we have

$$P_n y'(\varphi) = -P_n y(\varphi') = -P_n \int_S y(\tau)\varphi'(\tau)d\tau =$$

$$= -\int_S P_n y(\tau)\varphi'(\tau)d\tau = -(P_n y)(\varphi') = (P_n y)'(\varphi) \quad \forall \varphi \in \mathcal{D}(S).$$

The proposition is proved.

Condition L_2 follows from [1] (Lemma VI.1.5) and from the fact that the set $C^1(S; H_n)$ is dense in $L_{p_0}(S, H_n) = X_n$.

Condition L_3 follows from [1] (Lemma VI.1.7) with $V = H = H_n$ and $X = X_n$.

The corollary is proved.

8.2. Example. Let us consider a bounded domain $\Omega \subset \mathbb{R}^n$ with a sufficiently smooth boundary $\partial\Omega$, $S = [0, T]$, $Q = \Omega \times (0; T)$, $\Gamma_T = \partial\Omega \times (0; T)$. Let, as $i = 1, 2$, $m_i \in \mathbb{N}$, N_1^i(respectively N_2^i) by the number of derivatives respect to the variable x of order $\leq m_i -$

-1 (respectively m_i) and $\left\{A^i_\alpha(x,t,\eta,\xi)\right\}_{|\alpha|\le m_i}$ be a family of real functions defined on $Q \times \times R^{N^i_1} \times R^{N^i_2}$. Let

$$D^k u = \{D^\beta u, |\beta| = k\} \text{ be the differentiations with respect to } x,$$

$$\delta_i u = \{u,\ Du,\ ...,\ D^{m_i-1}u\},$$

$$A^i_\alpha(x,t,\delta_i u, D^{m_i}v):\ \ x,t \to A^i_\alpha(x,t,\delta_i u(x,t), D^{m_i}v(x,t)).$$

Moreover, let $\psi : \mathbb{R} \to \mathbb{R}$ be a convex coercive function belonging to $C^1(\mathbb{R})$ with bounded derivative ψ'.

Let us consider the next problem with Dirichlet boundary conditions:

$$\frac{\partial y(x,t)}{\partial t} + \sum_{|\alpha|\le m_1} (-1)^{|\alpha|} D^\alpha(A^1_\alpha(x,t,\delta_1 y, D^{m_1}y)) + \sum_{|\alpha|\le m_2} (-1)^{|\alpha|} D^\alpha(A^2_\alpha(x,t,\delta_2 y, D^{m_2}y)) +$$

$$+\psi'(y(x,t)) = f(x,t) \qquad \text{in } Q, \tag{8.5}$$

$$y(x,0) = y(x,T) \qquad \text{in } \Omega, \tag{8.6}$$

$$D^\alpha y(x,t) = 0 \qquad \text{on } \Gamma_T \text{ as } |\alpha| \le m_i \text{ and } i = 1,2. \tag{8.7}$$

Let us assume that $H = L_2(\Omega)$ and $V_i = W^{m_i,p_i}_0(\Omega)$ with $p_i \in (1,2]$ such that $V_i \subset H$ with a continuous embedding. Under suitable conditions on the coefficients A^i_α, the given problem can be written as

$$y' + A_1(y) + A_2(y) + \varphi'_G(y) = f, \qquad y(0) = y(T), \tag{8.8}$$

where $f \in X^* = L_2(S; L_2(\Omega)) + L_{q_1}(S; W^{-m_1,q_1}(\Omega)) + L_{q_2}(S; W^{-m_2,q_2}(\Omega))$ $(p_i^{-1} + q_i^{-1} = 1)$, φ'_G is the Gateaux derivative of the functional $\varphi(y) = \int_Q \psi(y(x,t))dxdt$ in the space $L_2(S; L_2(\Omega))$. Each element $y \in W$ that satisfies (8.8) is called a generalized solution to the problem (8.5)–(8.7).

Choice of basis. Due to Corollary 6.1 and [9] (Theorem 4.3.1.2), under the main condition d), for the complete system of vectors in the spaces $W^{m_i,p_i}_0(\Omega)$, we can take the special basis for the pair $(H_0^{\max\{m_1;m_2\}+\varepsilon}(\Omega); L_2(\Omega))$ with a suitable $\varepsilon \ge 0$.

Definition of the operators A_i***.*** Let $A^i_\alpha(x,t,\eta,\xi)$, defined on $Q \times R^{N^i_1} \times R^{N^i_2}$, satisfy the next conditions:

for almost all $x,t \in$ Q, the map $\eta,\xi \to A^i_\alpha(x,t,\eta,\xi)$ is continuous on $R^{N^i_1} \times R^{N^i_2}$;

for all η,ξ, the map $x,t \to A^i_\alpha(x,t,\eta,\xi)$ is measurable on Q; (8.9)

for all $u,v \in L^{p_i}(0,T;V_i) =: \mathcal{V}_i$, $\quad A^i_\alpha(x,t,\delta_i u, D^{m_i}u) \in L^{q_i}(Q)$. (8.10)

Then for each $u \in \mathcal{V}_i$ the map

$$w \to a_i(u,w) = \sum_{|\alpha|\le m_i} \int_Q A^i_\alpha(x,t,\delta_i u, D^{m_i}u) D^\alpha w dxdt$$

is continuous on $\mathcal{V}_i$ and also

$$\text{there exists } A_i(u) \in \mathcal{V}_i^* \text{ such that } a_i(u,w) = \langle A_i(u), w\rangle. \tag{8.11}$$

Conditions on A_i. Similarly to [2] (Sections 2.2.5, 2.2.6, 3.2.1) we have

$$A_i(u) = A_i(u,u), \quad A_i(u,v) = A_{i1}(u,v) + A_{i2}(u),$$

where

$$\langle A_{i1}(u,v), w\rangle = \sum_{|\alpha|=m_i} \int\limits_Q A^i_\alpha(x,t,\delta_i u, D^{m_i} v) D^\alpha w dx dt,$$

$$\langle A_{i2}(u), w\rangle = \sum_{|\alpha|\le m_i-1} \int\limits_Q A^i_\alpha(x,t,\delta_i u, D^{m_i} u) D^\alpha w dx dt.$$

We add the next conditions:

$$\langle A_{i1}(u,u), u-v\rangle - \langle A_{i1}(u,v), u-v\rangle \ \ge 0 \ \forall u,v \in \mathcal{V}_i; \tag{8.12}$$

$$\text{if } u_j \rightharpoonup u \text{ in } \mathcal{V}_i, \ u_j' \rightharpoonup u' \text{ in } \mathcal{V}_i^* \text{ and if } \langle A_{i1}(u_j,u_j) - A_{i1}(u_j,u), u_j - u\rangle \to 0,$$

$$\text{then } A^i_\alpha(x,t,\delta u_j, D^{m_i} u_j) \rightharpoonup A^i_\alpha(x,t,\delta u, D^{m_i} u) \text{ in } L^{q_i}(Q); \tag{8.13}$$

$$\text{coercivity.} \tag{8.14}$$

Remark 8.2. Similarly to [2] (Theorem 2.2.8), sufficient conditions for getting (8.12), (8.13) are

$$\sum_{|\alpha|=m_i} A^i_\alpha(x,t,\eta,\xi)\xi_\alpha \frac{1}{|\xi| + |\xi|^{p_i-1}} \to +\infty \text{ as } |\xi| \to \infty$$

for almost all $x,t \in Q$ and $|\eta|$ bounded;

$$\sum_{|\alpha|=m_i} (A^i_\alpha(x,t,\eta,\xi) - A^i_\alpha(x,t,\eta,\xi^*))(\xi_\alpha - \xi^*_\alpha) > 0 \text{ as } \xi \ne \xi^*$$

for almost all $x,t \in Q$ and η.

The next condition allows for coercivity:

$$\sum_{|\alpha|=m_i} A^i_\alpha(x,t,\eta,\xi)\xi_\alpha \ge c|\xi|^{p_i} \text{ for sufficiently large } |\xi|.$$

A sufficient condition to get (8.10) (see [2, p. 332]) is

$$|A^i_\alpha(x,t,\eta,\xi)| \le c[|\eta|^{p_i-1} + |\xi|^{p_i-1} + k(x,t)], \quad k \in L_{q_i}(Q). \tag{8.15}$$

By analogy with the proof of [2] (Theorem 3.2.1 and Statement 2.2.6) we get the next proposition.

Proposition 8.2. *Let the operator $A_i : \mathcal{V}_i \to \mathcal{V}_i^*$, $i = 1, 2$, defined in (8.11), satisfy (8.9), (8.10), (8.12), (8.13) and (8.14). Then A_i is pseudomonotone on W_i. Moreover it is bounded if (8.15) holds.*

Due to the last statement and to Corollary 8.1, it follows that under the listed above conditions, for all $f \in X^*$ there exists $R > 0$ such that $K_H(f) := \{y \in W \mid y$ is a generalized solution to the problem (8.5)–(8.7), obtained via the Faedo–Galerkin method $\}$ is nonempty, weakly compact in the closed ball from the space X with the center in the origin and radius R, and also representation (7.1) holds.

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