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FRICTIONLESS PLANAR MOTION OF A COUPLE OF HINGED RODS*

РУХ ПАРИ З'ЄДНАНИХ СТЕРЖНІВ У ПЛОЩИНІ ПРИ ВІДСУТНОСТІ ТЕРТЯ

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In this paper we treat the planar frictionless motion induced by a starting pulse on a two-body, four degrees of freedom system, consisting of two equal rods hinged together. A full discussion of all the possible planar forceless motions is given, and the hyperelliptic functions are found to be necessary. A particular case, namely the asymptotic one, in its two kinematic variants (open/closed) is faced. It is ruled by the nonlinear differential equation

$$\dot{\varphi} = \operatorname{sign} \dot{\varphi}_0 \, \frac{\sqrt{3A}\, \sin\varphi}{\sqrt{(1+3\cos^2\varphi)\left(1+3\sin^2\varphi\right)}}, \quad \varphi(0) = \varphi_0,$$

whose integration provides a link between the time and the Lagrangian coordinate φ by means of elliptic integrals of the I, II, III kinds.

The other (angle) coordinate θ, has been drawn to the quadratures by knowing φ.

Розглянуто плоский рух двох тіл, що утворюють систему з чотирма ступенями волі і складаються з двох однакових з'єднаних між собою стержнів, індукований початковим імпульсом. Розглянуто всі можливі рухи системи при відсутності впливу сил та виявлено необхідність використання гіпереліптичних функцій. Зокрема, розглянуто асимптотику у двох кінематичних випадках (відкритому та замкненому), що описується диференціальним рівнянням

$$\dot{\varphi} = \operatorname{sign} \dot{\varphi}_0 \, \frac{\sqrt{3A}\, \sin\varphi}{\sqrt{(1+3\cos^2\varphi)\left(1+3\sin^2\varphi\right)}}, \quad \varphi(0) = \varphi_0,$$

інтегрування якого визначає зв'язок між часом та лагранжевою координатою φ за допомогою еліптичних інтегралів I, II та III типів. Інтегрування іншої (кутової) координати θ зводиться до квадратур після знаходження φ.

1. Introduction: our problem. Euler and Lagrange have established a mathematically sound foundation of the Newtonian Mechanics. After them, Jacobi has imported (1837) from Optics to Classical Mechanics the Hamilton's formalism based on the least action, and arrived to a new formulation of it, now referred to as the *Hamilton – Jacobi theory*.

Nevertheless the number of cases in which the ordinary differential equation (ODE) of the analytical Dynamics can be integrated in closed form, or even reduced to quadratures, is quite limited. It was then made a crucial step in arriving at the concept of "integrability", developed by Liouville, whose definition (1855) of integrable n-Hamiltonian system in the

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unknown functions $x_j(t)$, $j = 1, \ldots, n$, is based on the so called *"first integrals"*, namely some special (supposed known) functions $\Phi(x_1, x_2, \ldots, x_n)$ of the n variables x_j, $j = 1, \ldots, n$, that are constants when the Lagrangian coordinates $x_1, x_2, \ldots, x_n$ satisfy the motion's ODE.

A n-system is then considered (fully) integrable if one succeeds in knowing n first integrals of it.

Liouville's definition of integrable systems covered many classical examples. Among them we find the Keplerian motion, the armonic oscillators solved by trigonometric functions, the rigid bodies (*spinning tops*) of the Euler – Poinsot, and of the Lagrange type, needing the elliptic functions. Among the other famous items of the Classical Mechanics we recall the pendulum, which many special functions are employed for. Namely, the elliptic ones for treating its large oscillations (and its complete revolutions!), or for the centrifugal governor (see [1]); the Bessel functions for the variable length pendulum; and finally, the Mathieu functions for the pendulum oscillating from a vibrating base.

At a higher level we can put the mentioned geodesic motions over an ellipsoid, or the one-dimensional heat conduction through a radiating wire, or the Kovalevskaya's case on the rotation of a rigid body about a fixed point, all needing the hyperelliptic functions.

Our article treats a problem that is *direct*, namely: given nothing but the acting forces (which could be a statically equilibrate system) and the initial conditions of position and speed, the motion is required.

Following our own recent researchs [1 – 3], we consider a two-body mechanical system S, without (dry) friction and in vacuum (no air resistance), four degrees of freedom. We will perform its closed form integration with the help of some special function of the Mathematical Physics. The recent progress in their evaluation by computer algebra allows to easily calculate the motions (see our sample problems).

2. System description. Our system consists of a couple of homogeneous rods, each having a mass m, frictionlessly hinged at the point O. They are launched *in vacuum* on a polished horizontal plane: the weights and the relevant reactions are all directed normally to the motion plane, and other forces are not considered capable to affect the movement, which therefore means that no force is acting, and it is pulsed by the initial conditions only.

The S's degrees of freedom are four: in fact it would be always possible to determine its configuration knowing the three degrees of the first rod, plus the angle of the second with the first one.

Nevertheless a different practical parameters' choice $(\xi, \eta, \theta, \varphi)$ has been deemed more to fit (see Fig. 1). The system centre of mass, say G, coincides with the middle point of the straight segment connecting the central points M and N of the rods OA and OB. Its position during the time will be known by its Cartesian coordinates $\xi_G(t)$ and $\eta_G(t)$. Afterwards, for defining the whole system position with respect to G, the angle θ formed by GO with the axis $G\xi'$, always parallel to $\Omega\xi$, will be given. Finally, as a fourth variable, the half-angle φ in O is assumed.

We will start by computing the system kinetic energy T, adding the amounts of the single rods. To each rod of length 2ℓ-whose barycentric moment of inertia is $\frac{1}{3}m\ell^2$, we will apply the König theorem, obtaining

$$T = m\left(\dot{\xi}_G^2 + \dot{\eta}_G^2\right) + \frac{1}{3}m\ell^2\left[\dot{\varphi}^2\left(1 + 3\cos^2\varphi\right) + \dot{\theta}^2(1 + 3\sin^2\varphi)\right] \qquad (2.1)$$

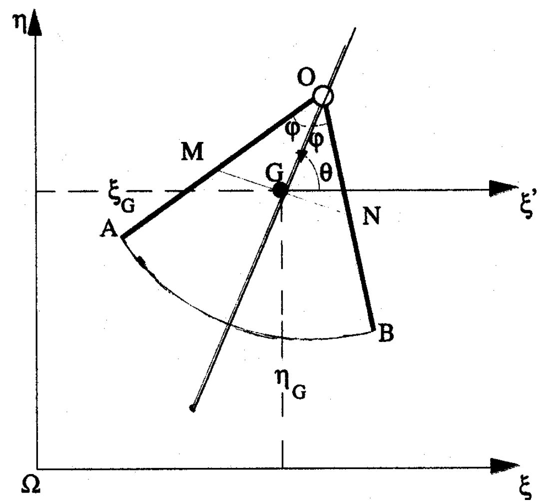

Fig. 1. Sketch and geometrical elements of S.

(see Fig. 1, and for a different approach [4, p. 340], where our problem is proposed without an explicit solution).

Of the four Lagrangian parameters, *only* φ appears in the kinetic energy, which is algebraically depending on the time derivatives of *all* of them.

Not being there any conservative force field, the Lagrange function will then coincide with

$$T = T\left(\varphi; \dot{\xi}_G, \dot{\eta}_G, \dot{\varphi}, \dot{\theta}\right).$$

3. System motion equations *in vacuo*. As a consequence of above, the Lagrange equations will be

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot{\xi}}\right) = 0, \\ \dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot{\eta}}\right) = 0, \end{cases} \qquad \begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot{\theta}}\right) = 0, \\ \dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot{\varphi}}\right) - \dfrac{\partial T}{\partial \varphi} = 0. \end{cases} \tag{3.1}$$

They have to be solved for the four unknown functions with eight initial conditions prescribing the starting values for them and for their first derivatives,

$$\begin{cases} \xi(0) = \xi_0, \\ \eta(0) = \eta_0, \\ \varphi(0) = \varphi_0, \\ \theta(0) = \theta_0, \end{cases} \qquad \begin{cases} \dot{\xi}(0) = \dot{\xi}_0, \\ \dot{\eta}(0) = \dot{\eta}_0, \\ \dot{\varphi}(0) = \dot{\varphi}_0, \\ \dot{\theta}(0) = \dot{\theta}_0. \end{cases}$$

Let us settle the ranges and the initial values for ξ_G, η_G, θ, φ.

The motion can, without any restriction, be located all on the first quadrant of $\xi\Omega\eta$, and then the barycentre coordinates ξ_G, η_G will be always greater than zero during the time, possibly starting by the values $\xi_{G_0} = 0$, $\eta_{G_0} = 0$.

Let us go to the angle coordinates.

We establish a positive orientation on the straight line from the centre of mass G to the hinge O, and evaluate positively the counterclockwise angle θ formed by GO and the oriented axis $G\xi'$, always parallel to $\Omega\xi$. Then θ has the meaning of polar anomaly of the compasses' axis, and $0 \le \theta \le 2\pi$. For the initial value, there will be no problem in assuming that S will start up with its own axis superimposed on $\Omega\xi$ then $\theta_0 = 0$. To this end, another initial condition on the derivative shall be added, like (e. g.) $\dot{\theta}_0 = +1$ (leaning up) or $\dot{\theta}_0 = -1$ (leaning down).

During the motion, we select as compasses' "opening" the *convex* angle formed by the rods, always $\le \pi$. Then for the half-opening φ we will have $0 \le \varphi \le \pi/2$.

Among the ∞^2 possible starting conditions on the opening, we will examine the following remarkable kinematic variants:

i) *start with rods tendency to joining*, namely $\dot{\varphi}_0 < 0$,

ii) *start with rods tendency to broadening*, namely $\dot{\varphi}_0 > 0$.

Of course the φ_0 value is not critical, because each possible value could be taken as a starting one without any impliance on the next behaviour, but in order to avoid tedious classification of the goniometric subcases, due to the inversion of the sine, in the first occurrence we take $0 < \varphi_0 < \dfrac{\pi}{2}$, while in the second $\dfrac{\pi}{2} < \varphi_0 < \pi$. Then we could assume $\dot{\varphi}_0 = -1$ for i) and +1 for ii); for both cases any φ_0 could be taken such that $\sin^2\varphi_0 > 0$.

In order to obtain the Lagrange equations, recalling the kinetic energy (2.1), by (3.1), we have

$$\dot{\xi} = C_1, \qquad \dot{\eta} = C_2, \qquad \dot{\theta}^2 = \frac{C}{1 + 3\sin^2\varphi}, \tag{3.2}$$

with C_1, C_2, C being three constant values depending on the initial conditions. By the first two formulas of (3.2) we see that

$$\frac{d\eta_G}{d\xi_G} = \text{const},$$

then a straight line will be the trajectory of the centre of mass, whose motion will be uniform, according to the first two of (3.2).

On the contrary, the angle coordinates are coupled since by the third of (3.2), of course, we need another equation.

The last Lagrange one could be obtained by injecting the T expression (2.1) in the fourth of (3.1), and handling it. But one arrives much sooner at the same conclusion by seeing that, due to the problem nature and to the links, the kinetic energy must be invariant with respect to the time (the fourth *first integral*, in the sense of Liouville). Then, by setting the energy (3.2) a constant, remembering the first two of (3.2), we obtain the fourth ODE, namely, a further relationship between the angles and their time-derivatives,

$$\dot{\varphi}^2\left(1 + 3\cos^2\varphi\right) + \dot{\theta}^2\left(1 + 3\sin^2\varphi\right) = \text{const.} \tag{3.3}$$

To the above constant, surely greater than zero, the form A^2 can be given, namely,

$$A^2 = \dot{\varphi}_0^2\left(1 + 3\cos^2\varphi_0\right) + \dot{\theta}_0^2\left(1 + 3\sin^2\varphi_0\right).$$

By the third of (3.2), we have

$$C = \dot{\theta}_0^2 \left(1 + 3\sin^2 \varphi_0\right) > 0.$$

Dividing the third of (3.2) to the (3.3) and putting

$$\nu^2 = \frac{\dot{\varphi}_0^2}{\dot{\theta}_0^2}, \ \sin^2 \varphi_0 = x_0,$$

we obtain

$$\frac{4 - 3x_0}{(1 + 3x_0)^2}\nu^2 + \frac{1}{1 + 3x_0} = \frac{A^2}{C^2}, \tag{3.4}$$

useful for the next discussion, and showing that *the initial condition on* θ_0 *has no influence on the motion nature*, because θ_0 does appear neither in A^2 nor in C^2.

Furthermore, if the third of (3.2) is replaced in (3.3), we have a highly nonlinear ODE in the unknown function $\varphi(t)$,

$$\begin{aligned} \left(\frac{d\varphi(t)}{dt}\right)^2 &= \frac{3A^2 \sin^2 \varphi(t) + A^2 - C^2}{(1 + 3\cos^2 \varphi(t)) \left(1 + 3\sin^2 \varphi(t)\right)}, \\ \varphi(0) &= \varphi_0. \end{aligned} \tag{3.5}$$

Separating the variables in (3.5), we get

$$t = \pm \frac{1}{A\sqrt{3}} \int\limits_{\varphi_0}^{\varphi} \sqrt{(1 + 3\cos^2 \phi) \left(1 + 3\sin^2 \phi\right) \left(\delta + \sin^2 \phi\right)^{-1}}\, d\phi,$$

where

$$\delta = \frac{A^2 - C^2}{3A^2} \lesseqgtr 0,$$

and the $\pm$ sign means that when $\dot{\varphi}_0$ is positive, we will take the + sign, while the minus sign has to be taken when $\dot{\varphi}_0 < 0$. From now on, we will suppose $\dot{\varphi}_0 > 0$ (it is easy to switch to the case $\dot{\varphi}_0 < 0$).

We make the change $\sin^2 \phi = x > 0$ and so, if $0 < \varphi_0, \varphi < \dfrac{\pi}{2}$,

$$d\phi = \frac{1}{2\sqrt{x(1 - x)}}\, dx, \tag{3.6}$$

while, of course, the sign has to be changed when $\dfrac{\pi}{2} < \varphi_0, \varphi < \pi$. Therefore we obtain

$$t = \frac{1}{2A\sqrt{3}} \int\limits_{\sin^2 \varphi_0}^{\sin^2 \varphi} \frac{\sqrt{(4 - 3\,x)\,(1 + 3\,x)}}{\sqrt{x\,(1 - x)\,(x + \delta)}}\, dx. \tag{3.7}$$

Now it is convenient to rewrite the integrand in (3.7) (all the factors are nonnegative since $0 \le x \le 1$), as

$$\frac{\sqrt{(4-3\,x)\ (1+3\,x)}}{\sqrt{x\,(1-x)\,(x+\delta)}} = \sqrt{\frac{(4-3\,x)^2\ (1+3\,x)^2}{x\,(1-x)\,(x+\delta)\,(4-3\,x)\ (1+3\,x)}} =$$

$$= \frac{(4-3x)(1+3x)}{3\sqrt{x\,(1-x)\,(x+\delta)\left(\frac{4}{3}-\,x\right)\ \left(\frac{1}{3}+\,x\right)}}, \tag{3.8}$$

for brevity the polynomial under the square root in the last expression of (3.8) will be denoted by:

$$P_5(x) = x\left(x-\frac{4}{3}\right)\left(x+\frac{1}{3}\right)(x-1)\,(x+\delta)$$

to stress the fact that $P_5(x)$ is a quintic. Then

$$t = \frac{1}{6A\sqrt{3}}\int\limits_{\sin^2\varphi_0}^{\sin^2\varphi}\left(\frac{4}{\sqrt{P_5(x)}}+\frac{9x}{\sqrt{P_5(x)}}-\frac{9x^2}{\sqrt{P_5(x)}}\right)dx. \tag{3.9}$$

In such a way the integrand will have four *fixed* poles located at $-\dfrac{1}{3}$, 0, 1 and $\dfrac{4}{3}$. The remaining *moving* pole will be described by δ, and namely via the inequality

$$A^2 \lesseqgtr C^2.$$

When $\delta \neq 0$, namely $A^2 \lessgtr C^2$, by (3.9) we have the time as a sum of three hyperelliptic integrals, what will be a matter of a next paper. In this article we will restrict to a comparative analysis of the three motions $A^2 \lesseqgtr C^2$, and to solve the case $A^2 = C^2$.

4. Three possible motions: $\boldsymbol{A^2 \lesseqgtr C^2}$. First of all, following [4], we analyze the different motions taking place in the three cases, i.e.,

$$\text{(i)}\qquad \frac{A^2}{C^2} > 1, \qquad\quad \text{(ii)}\qquad \frac{A^2}{C^2} < 1, \qquad\quad \text{(iii)}\qquad \frac{A^2}{C^2} = 1.$$

(i) Equation (3.5) then shows that $\dot\varphi^2$ is never zero, and so $\dot\varphi$ will be either positive, or negative. Then φ will either always increase, or always decrease, without inversions; this means that the rods will be becoming closer (if $\dot\varphi_0 < 0$) or farther (if $\dot\varphi_0 > 0$): the motion is then *aperiodic*. By (3.4) we obtain the inequality

$$F(x_0) = 9\,x_0^2 + 3\,x_0(2+\nu^2) - 4\,\nu^2 < 0.$$

Its quadratic resolvant has one variation, and then only one positive and acceptable root (the variable x_0 must always be positive), say x_0^*:

$$x_0^* = \frac{-\left(2+\nu^2\right)+\sqrt{\left(2+\nu^2\right)^2+144\,\nu^2}}{18}.$$

The condition

$$A^2 > C^2$$

will be then satisfied if $F(x_0) < 0$, and then $0 < x_0 < x_0^*$. This last condition then links the initial values φ_0, $\dot{\varphi}_0$, $\dot{\theta}_0$ for having an aperiodic motion.

The positiveness of x_0^* gives a further condition,

$$\varepsilon^4 - \varepsilon^2 + 144\,\varepsilon - 288 > 0, \tag{4.1}$$

being

$$\varepsilon = 2 + \nu^2 = 2 + \frac{\dot{\varphi}_0^2}{\dot{\theta}_0^2} > 0,$$

which involves the quadratic ratio of the starting angular speeds.

(ii) In this case we can put

$$C^2 - A^2 = 3A^2 \sin^2 \alpha;$$

then the right-hand side of (3.3) takes the form

$$3A^2(\sin^2 \varphi - \sin^2 \alpha).$$

The motion will be possible if and only if $\sin^2 \varphi > \sin^2 \alpha$; therefore φ shall move between α and $\pi - \alpha$. Then we are faced with an *oscillatory motion* of each rod as to OG. This will happen if $x_0 > x_0^*$, whilst (4.1) keeps its validity.

(iii) In this case we have

$$t = \frac{1}{A\sqrt{3}} \int\limits_{\varphi_0}^{\varphi} \sqrt{\left(1 + 3\cos^2 \phi\right)\left(1 + 3\sin^2 \phi\right)}\,\frac{d\phi}{\sin \phi}.$$

If φ goes to zero (or to π), $t \to +\infty$, namely: varying φ always in the same sense, the rods tend to be superimposed, but without joining. This condition then concerns an *asymptotic motion*.

It will happen if $x_0 = x_0^*$, whilst (4.1) keeps its validity.

5. The asymptotic case: $A^2 = C^2$. In this particular (*asymptotic*) case, by (3.9), assuming $0 < \varphi_0 < \dfrac{\pi}{2}$ and $\dot{\varphi}_0 < 0$ (tendency to joining) we see that

$$-6A\sqrt{3}t = \int\limits_{\sin^2 \varphi_0}^{\sin^2 \varphi} \left(\frac{4}{x\sqrt{P_3(x)}} + \frac{9}{\sqrt{P_3(x)}} - \frac{9x}{\sqrt{P_3(x)}} \right) dx,$$

with

$$P_3(x) = \left(x - \frac{4}{3}\right)\left(x + \frac{1}{3}\right)(x - 1)\,.$$

Then we have to consider the following integrals:

$$\mathcal{I}_1 = \int_{\sin^2\varphi_0}^{\sin^2\varphi} \frac{dx}{x\sqrt{P_3(x)}}, \ \mathcal{I}_2 = \int_{\sin^2\varphi_0}^{\sin^2\varphi} \frac{dx}{\sqrt{P_3(x)}}, \ \mathcal{I}_3 = \int_{\sin^2\varphi_0}^{\sin^2\varphi} \frac{x\,dx}{\sqrt{P_3(x)}}.$$

All the integrals $\mathcal{I}_j$ can be brought to a combination of Legendre elliptic standard integrals of I, II, III kinds. Taking into account (e. g.) the *tendency to joining*, there is no loss of generality to assume

$$\sin^2\varphi_0 > 0, \ \sin^2\varphi = y \leq 1.$$

In these hypotheses we can develop a deeply symmetrical treatment of all the $\mathcal{I}_j$, which will enable us to provide a link between the time and the Lagrangian coordinate φ by means of I, II, III kinds elliptic integrals.

(a) The simpler integral $\mathcal{I}_2$ can be drawn back to the standard integral

$$\int_c^y \frac{dz}{\sqrt{(a-z)(b-z)(z-c)}},$$

where

$$c < y_0, y \leq b < a$$

(we have $c = -1/3, b = 1, a = 4/3$). Then

$$\mathcal{I}_2 = \int_{y_0}^{y} \frac{dz}{\sqrt{(a-z)(b-z)(z-c)}} =$$

$$= \int_c^y \frac{dz}{\sqrt{(a-z)(b-z)(z-c)}} - \int_c^{y_0} \frac{dz}{\sqrt{(a-z)(b-z)(z-c)}}.$$

Let us recall [5, p. 72] (formula 233.00)

$$\int_c^y \frac{dz}{\sqrt{(a-z)(b-z)(z-c)}} = gF\left(\psi(y), k\right) = g\,u_1(y),$$

with

$$g = \frac{2}{\sqrt{a-c}}, \ \ k = \sqrt{\frac{b-c}{a-c}}, \ \sin\psi(y) = \sqrt{\frac{y-c}{b-c}}$$

where $F\left(\psi, k\right)$ is the incomplete elliptic integral of first kind of amplitude ψ, and modulus k,

$$u_1 = F\left(\psi, k\right) = \int_0^{\psi} \frac{d\zeta}{\sqrt{1-k^2\sin^2\zeta}}.$$

Therefore,

$$\mathcal{I}_2 = g\left[F(\psi(y),k) - F(\psi(y_0),k)\right] =$$

$$= \frac{2}{\sqrt{a-c}}\left[F\left(\arcsin\sqrt{\frac{y-c}{b-c}}, \sqrt{\frac{b-c}{a-c}}\right) - F\left(\arcsin\sqrt{\frac{y_0-c}{b-c}}, \sqrt{\frac{b-c}{a-c}}\right)\right] =$$

$$= \frac{2\sqrt{15}}{5}\left\{F\left(\arcsin\frac{\sqrt{1+3\sin^2\varphi}}{2}, \frac{2\sqrt{5}}{5}\right) - F\left(\arcsin\frac{\sqrt{1+3\sin^2\varphi_0}}{2}, \frac{2\sqrt{5}}{5}\right)\right\}. \quad (5.1)$$

(b) With the same meaning for all the symbols, we can now evaluate both $\mathcal{I}_1$ and $\mathcal{I}_3$. We start with $\mathcal{I}_1$:

$$\mathcal{I}_1 = \int_{y_0}^{y} \frac{dz}{z\sqrt{(a-z)(b-z)(z-c)}} =$$

$$= \int_{c}^{y} \frac{dz}{z\sqrt{(a-z)(b-z)(z-c)}} - \int_{c}^{y_0} \frac{dz}{z\sqrt{(a-z)(b-z)(z-c)}}. \quad (5.2)$$

Now we apply formula 233.18 of [5, p. 74],

$$\int_{c}^{y} \frac{dz}{z\sqrt{(a-z)(b-z)(z-c)}} = \frac{g}{c}\int_{0}^{u_1(y)} \frac{du}{1-\dfrac{c-b}{c}\operatorname{sn}^2 u}, \quad (5.3)$$

where, as usually $\operatorname{sn} u$ denotes the Jacobi elliptic function, sine amplitude,

$$\sin\varphi = \sin\operatorname{am}(u,k) = \operatorname{sn} u.$$

Finally, we recall formula 336.01 of [5, p. 201]

$$\int_{0}^{u_1(y)} \frac{du}{1-\dfrac{c-b}{c}\operatorname{sn}^2 u} = \Pi\left(u_1(y), \frac{c-b}{c}, k\right), \quad (5.4)$$

where $\Pi(\psi,\alpha,k)$ denotes the third kind incomplete elliptic integral of amplitude ψ, parameter α, and modulus k,

$$\Pi(\psi,\alpha,k) = \int_{0}^{\psi} \frac{d\zeta}{\left(1-\alpha\sin^2\zeta\right)\sqrt{1-k^2\sin\zeta}}.$$

Now by (5.3) and (5.4), we can evaluate the elliptic integrals arising in (5.2),

$$\begin{aligned}\mathcal{I}_1 &= \frac{g}{c}\left[\Pi\left(u_1(y), \frac{c-b}{c}, k\right) - \Pi\left(u_1(y_0), \frac{c-b}{c}, k\right)\right] = \\ &= \frac{g}{c}\left[\Pi\left(\arcsin\sqrt{\frac{y-c}{b-c}}, \frac{c-b}{c}, \sqrt{\frac{b-c}{a-c}}\right) - \Pi\left(\arcsin\sqrt{\frac{y_0-c}{b-c}}, \frac{c-b}{c}, \sqrt{\frac{b-c}{a-c}}\right)\right] = \\ &= -\frac{6}{5}\sqrt{15}\left[\Pi\left(\arcsin\frac{\sqrt{1+3\sin^2\varphi}}{2}, 4, \frac{2\sqrt{5}}{5}\right) - \Pi\left(\arcsin\frac{\sqrt{1+3\sin^2\varphi_0}}{2}, 4, \frac{2\sqrt{5}}{5}\right)\right].\end{aligned} \tag{5.5}$$

(c) To evaluate $\mathcal{I}_3$, we start by decomposing, as previously, the interval of integration (all the symbols once again keep their previous meanings):

$$\begin{aligned}\mathcal{I}_3 &= \int\limits_{y_0}^{y} \frac{z\,dz}{\sqrt{(a-z)(b-z)(z-c)}} = \\ &= \int\limits_{c}^{y} \frac{z\,dz}{\sqrt{(a-z)(b-z)(z-c)}} - \int\limits_{c}^{y_0} \frac{zdz}{\sqrt{(a-z)(b-z)(z-c)}}.\end{aligned}$$

By [5, p. 74], we use formula 233.17, and afterwards formula 331.01 (p. 201),

$$\begin{aligned}\int\limits_{c}^{y} \frac{z\,dz}{\sqrt{(a-z)(b-z)(z-c)}} &= g\,c \int\limits_{0}^{u_1(y)} \left(1 - \frac{c-b}{c}\operatorname{sn}^2 u\right) du = \\ &= \frac{g\,c}{k^2}\left[\left(k^2 - \frac{c-b}{c}\right) F\left(\psi(y), k\right) + \frac{c-b}{c} E\left(\psi(y), k\right)\right].\end{aligned}$$

Of course $E(\psi, k)$ is the second kind incomplete elliptic integral of amplitude ψ and modulus k,

$$E(\psi, k) = \int\limits_{0}^{\psi} \sqrt{1 - k^2 \sin\zeta}\, d\zeta.$$

Now we can easily evaluate $\mathcal{I}_3$; in fact we have

$$\mathcal{I}_3 = \frac{g\,c}{k^2}\left[\left(k^2-\frac{c-b}{c}\right)F\left(\psi(y),k\right)+\frac{c-b}{c}E\left(\psi(y),k\right)\right]-$$

$$-\frac{g\,c}{k^2}\left[\left(k^2-\frac{c-b}{c}\right)F\left(\psi(y_0),k\right)+\frac{c-b}{c}E\left(\psi(y_0),k\right)\right]=$$

$$=\frac{\sqrt{15}}{6}\left[4E\left(\arcsin\frac{\sqrt{1+3\sin^2\varphi_0}}{2},\frac{2\sqrt{5}}{5}\right)-\frac{16}{5}F\left(\arcsin\frac{\sqrt{1+3\sin^2\varphi_0}}{2},\frac{2\sqrt{5}}{5}\right)\right]-$$

$$-\frac{\sqrt{15}}{6}\left[4E\left(\arcsin\frac{\sqrt{1+3\sin^2\varphi}}{2},\frac{2\sqrt{5}}{5}\right)-\frac{16}{5}F\left(\arcsin\frac{\sqrt{1+3\sin^2\varphi}}{2},\frac{2\sqrt{5}}{5}\right)\right]. \quad (5.6)$$

The problem for the first angular varia ble φ is then solved, being the time (see (3.9)) known as a function of φ by means of the elliptic integrals $\mathcal{I}_j$,

$$t=t(\varphi)=-\frac{1}{6A\sqrt{3}}\left(4\mathcal{I}_1(\varphi)+9\mathcal{I}_2(\varphi)-9\mathcal{I}_3(\varphi)\right) \quad (5.7)$$

where, according to (5.5), (5.1), and (5.6), each $\mathcal{I}_j$ has been reduced to the Legendre reference integrals F, E, Π.

Notice, finally, that in the situation of broadening we have to take the minus sign in (5.7): in fact the first choice is +, due to $\dot\varphi_0>0$, but, being $\frac{\pi}{2}<\varphi_0,\varphi<\pi$, the change of variable introduced in (3.6) produces a further minus sign because the equation $\sin^2\varphi=x$, in this case, has the solution $\varphi=\pi-\arcsin\sqrt{x}$.

6. The angle θ. For completing the integration, the fourth Lagrangian parameter θ has to be found as a function of time.

If $\dot\theta_0>0$, $C>0$, and, by the third of (3.2), we see that θ is always growing. In any case, its initial derivative's sign is kept in the time. By the same formula we see that the time law for θ is put back to the quadratures,

$$\theta(t)=\theta_0+C\int_0^t\frac{d\tau}{1+3\sin^2\varphi(\tau)}, \quad (6.1)$$

with $\varphi(\tau)$ being the inverse of (5.7).

7. Two sample problems in the asymptotic case. We are now going to test our result by considering the solution of (3.5) with the following values: $\dot\theta_0=-1$, $\dot\varphi_0=-1$ and $\sin\varphi_0=\frac{1}{\sqrt{3}}$ (tendency to joining). Notice that in this case we have the value:

$$\arcsin\frac{\sqrt{1+3\sin^2\varphi_0}}{2}=\frac{\pi}{4},$$

for the elliptic amplitude of the constant elliptic integrals appearing in (5.5), (5.1), and (5.6). For $A = \sqrt{5}$, (3.5) becomes

$$\dot{\varphi} = -\frac{\sqrt{15}\,\sin\varphi}{\sqrt{(1+3\cos^2\varphi)\left(1+3\sin^2\varphi\right)}}, \qquad \varphi(0) = \arcsin\frac{1}{\sqrt{3}}. \tag{7.1}$$

The following plot $(t, \varphi(t))$ shows the behaviour of the exact solution provided by (5.7) with the test values for $\dot{\theta}_0$, $\dot{\varphi}_0$ and φ_0. It resulted perfectly superimposed with the numerical output of (7.1) independently obtained with the help of the VisualDSolve Mathematica® package written by [6] and the NDSolve function, implemented in Mathematica®.

For the sake of completeness, we also treat the broadening case, where φ grows tending to π as $t \to \infty$. Taking $\varphi_0 = \pi - \arcsin\sqrt{\frac{2}{3}}$, we get an elliptic amplitude

$$\arcsin\frac{\sqrt{1+3\sin^2\varphi_0}}{2} = \frac{\pi}{3},$$

for the constant elliptic integrals appearing in (5.5), (5.1), and (5.6).

With $\dot{\theta}_0 = -1$, $\dot{\varphi}_0 = 1$ we find the differential equation

$$\dot{\varphi} = \frac{\sqrt{15}\,\sin\varphi}{\sqrt{(1+3\cos^2\varphi)\left(1+3\sin^2\varphi\right)}}, \qquad \varphi(0) = \pi - \arcsin\frac{\sqrt{2}}{\sqrt{3}}, \tag{7.2}$$

whose exact solution, provided by (5.7) is plotted below, and agrees perfectly with the independent numerical output given by `NDSolve` and [6].

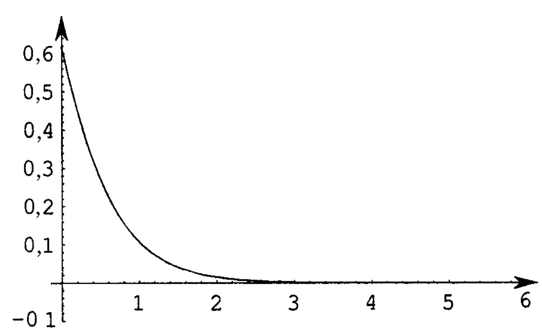

Fig. 2. Asymptotic tendency to joining.

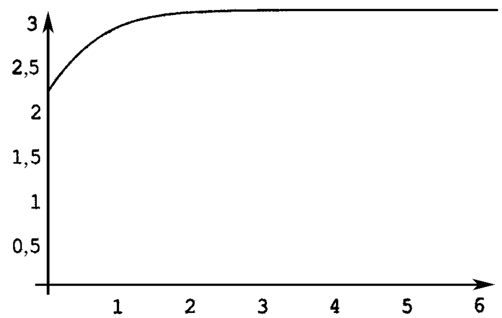

Fig. 3. Asymptotic tendency to broadening.

8. Conclusions. The barycentre of our four-degrees, two-bodies, planar system, not acted by any force, but pulsed by the initial conditions only, is (of course!) moving uniformly on a straight line.

On the contrary, the remaining Lagrangian parameters φ and θ are tightly tied each other.

The integration of the first ODE in the unknown $\varphi(t)$ needs the hyperelliptic functions for describing both system's motions, aperiodic revolutions and oscillations as well. On the contrary, the asymptotic motions are more tractable; they consist of the tendency-without getting either to the full opening or to the closing by the pair of compasses. Such a couple of motions can be treated using some elliptic integrals (see (5.7)), of course reducible to the Legendre reference standards F, E, Π.

The time law for the last Lagrangian parameter θ has been put back to the quadratures (see (6.1)): but the relevant integral cannot be evaluated in any closed form for the impossibility of knowing $\sin\varphi(t)$ entering the integrand structure.

A sample problem of the asymptotic case (in its two variants) has been treated obtaining $t = t(\varphi)$ through F, E, Π. Furthermore, by a purely computational work, we arrived at our last plots (Fig. 2 and 3) showing φ as function of t.

In both variants the asymptotic φ-behaviour has been found of absolute evidence.

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