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ON THE STRUCTURE OF CHARACTERISTIC SURFACES RELATED WITH PARTIAL DIFFERENTIAL EQUATIONS OF FIRST AND HIGHER ORDERS. Pt 2

ПРО СТРУКТУРУ ХАРАКТЕРИСТИЧНИХ ПОВЕРХОНЬ, ПОВ'ЯЗАНИХ ІЗ ДИФЕРЕНЦІАЛЬНИМИ РІВНЯННЯМИ З ЧАСТИННИМИ ПОХІДНИМИ ПЕРШОГО ТА ВИЩИХ ПОРЯДКІВ. Ч. 2

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The generalized characteristics method is developed in the framework of the geometric Monge picture. The Hopf – Lax-type extremality solutions to a wide class of Cauchy problem for nonlinear partial differential equations of first and higher orders are derived. The special Hamilton – Jacobi-type case is analized separately. The exact extremality Hopf – Lax-type solution for Cauchy problem to the nonlinear Burgers equation is received, its linearization to the Hopf – Cole expression and to the related Airy-type linear partial differential equation is found and discussed.

Розвинуто узагальнений метод характеристик у рамках геометричного підходу Монжа. Отримано екстремальні розв'язки типу Хопфа – Лакса широкого класу задач Коші для нелінійних диференціальних рівнянь з частинними похідними першого та вищих порядків. Окремо досліджено спеціальний випадок типу Гамільтона – Якобі. Отримано точний екстремальний розв'язок типу Хопфа – Лакса задачі Коші для нелінійного рівняння Бюргерса. Знайдено та проаналізовано його лінеаризацію у вигляді виразу Хопфа – Коула та пов'язаного з ним лінійного диференціального рівняння з частинними похідними типу Ейрі.

1. Introduction. It is well known [1, 2] that solutions to linear partial differential equations can be studied effectively by making use of many different approaches, such as the Fourier method, the spectral theory and the Green function method. Nevertheless, all of them, regrettably, can not be applied for analysing solution manifolds of general nonlinear partial differential equations even of the first and second orders. Since the classical Cauchy works on the problem, by now there exist [2 – 4] only a few approaches to treating such equations, among which the famous characteristics method that appears to be the most effective and fruitful. During the last century this method was further developed by many mathematicians, amongst whom are P. Lax, H. Hopf, O. A. Oleinik, S. N. Kruzhkov, V. Maslov, P. Lions, L. Evans, Blackmore [2, 4 – 10] and others. Still long ago it was observed that there is a deep connection between the characteristics method and the Hamiltonian analysis, reducing the problem to studying some systems of ordinary differential equations. This aspect had become prevailing in works of H. Hopf, P. Lax, and O. Oleinik (see [2, 9, 11]), who described, doing this way, a wide class of so-called generalized solutions to first order nonlinear partial differential equations. The most known result within this field is attributed to H. Hopf and P. Lax, who have found for the first time a very interesting variational representation for solutions of first order nonlinear partial differential equations called a Hopf – Lax-type representation. Since these results were

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strongly based on some geometrical notions, it was natural to analyze the Cauchy characteristics method from the differential-geometrical point of view, initiated still in the classical works of G. Monge and E. Cartan. Within the framework of the Monge geometric approach to studying solutions of partial differential equations we proposed in Part 1 [12] a generalization of the Cauchy characteristic method for equations of first and higher orders, making use of the special tensor fields intimately related with them. These tensor fields appear very naturally within the developed Monge approach as some geometric objects, generalizing the classical Hamilton-type equations for characteristic vector fields. Moreover, this geometric approach jointly with some Cartan's compatibility considerations is naturally extended to a wide class of nonlinear partial diffrerential equations of the second and higher orders. Namely, if for instance a first order differential equation is given as

$$H(x; u, u_x) = 0, \tag{1.1}$$

where $x \in \mathbb{R}^n$, $H \in C^1(\mathbb{R}^n \times \mathbb{R}^{n+1}; \mathbb{R})$, $\|H_{u_x}\| \neq 0$, the characteristics vector fields on the related Monge hypersurface

$$S_H := \left\{(x; u, p) \in \mathbb{R}^n \times \mathbb{R}^{n+1} : H(x; u, p) = 0\right\} \tag{1.2}$$

are represented [12] as follows:

$$\frac{dx}{d\tau} = \mu^{(1|1)}\frac{\partial H}{\partial x}, \quad \frac{du}{d\tau} = \left\langle p, \mu^{(1|1)}\partial H/\partial p\right\rangle, \quad \frac{dp}{d\tau} = -\mu^{(1|1),*}\left(\frac{\partial H}{\partial x} + p\frac{\partial H}{\partial u}\right). \tag{1.3}$$

Here $\mu^{(1|1)} \in C^1(\mathbb{R}^{n+1} \times \mathbb{R}^n; \mathbb{R}^n \otimes \mathbb{R}^n)$ are some smooth tensor fields on S_H and $\tau \in \mathbb{R}$ is an evolution parameter. Vector fields (1.3) ensure [12] the tangency to the hypersurface $S_H \subset \subset \mathbb{R}^n \times \mathbb{R}^{n+1}$ and the projection compatibility condition with the dual Monge cone K^* upon the corresponding solution hypersurface $\bar{S}_H \subset \mathbb{R}^{n+1}$ (see Fig. 1), generated by the characteristic strips $\Sigma_H \subset S_H$ through smoothly imbedded sets $\Sigma \subset S_H$ consisting of points carrying the solutions to our problem (1.1). Similar results were obtained in [12] also for both partial differential equations of higher orders and systems.

In general, the problem (1.1) is endowed with some boundary condition on a smooth hypersurface $\Gamma_\varphi \subset \mathbb{R}^n$ like

$$u|_{\Gamma_\varphi} = u_0, \tag{1.4}$$

where $u_0 \in C^1(\Gamma_\varphi; \mathbb{R})$ is a given function. The hypersurface $\Gamma_\varphi \subset \mathbb{R}^n$ can be, for simplicity, defined as

$$\Gamma_\varphi := \{x \in \mathbb{R}^n : \varphi(x) = 0\}, \tag{1.5}$$

where $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a smooth mapping endowed with some local coordinates $s(x) \in \mathbb{R}^{n-1}$ in the corresponding open neighborhoods $O_\varepsilon(x) \subset \Gamma_\varphi$ of all points $x \in \Gamma_\varphi$ for some $\varepsilon > 0$. Thus, we are interested in constructing analytical solutions to the boundary problem (1.1), (1.4), and (1.5) and studying their properties. This and related aspects of this problem will be discussed in detail below.

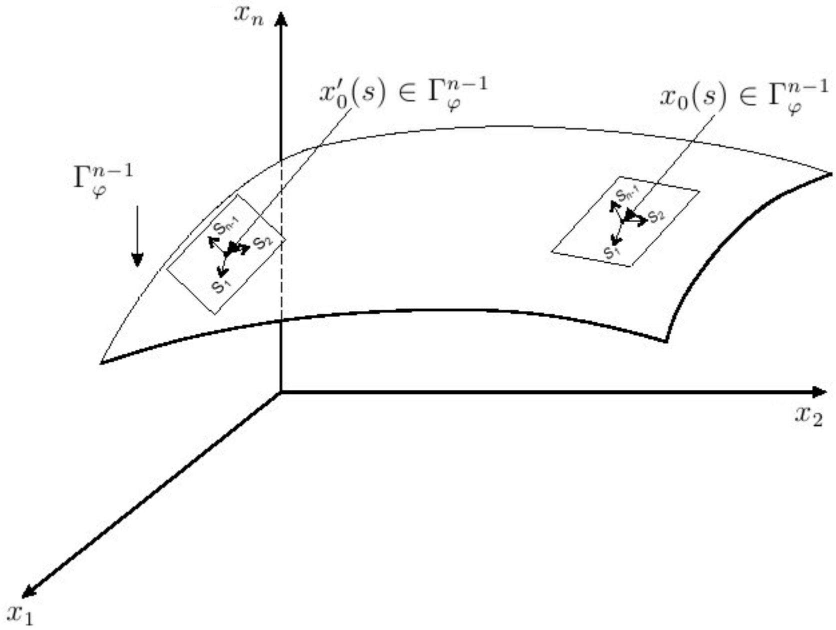

Fig. 1. The boundary $\Gamma_\varphi^{n-1} = \{x_0 \in \mathbb{R}^n : \varphi(x_0) = 0\}$, $x_0(s) \in \Gamma_\varphi^{n-1}$, $s \in \mathbb{R}^{n-1}$ are local coordinates.

2. Boundary problem analysis. Consider the set of characteristic equations (1.3) on the hypersurface $S_H \subset \mathbb{R} \times \mathbb{R}^{n+1}$, which start at points $(x_0; u_0, p_0) \in \Gamma_\varphi$ under the additional condition that the corresponding projection $\Sigma \to \overline{\Sigma}$ upon the subspace $\mathbb{R}^{n+1}$ (see Fig. 2) coincides with the boundary set $(\Gamma_\varphi; u_0) \subset \mathbb{R}^{n+1}$, that is,

$$\overline{\Sigma} := (\Gamma_\varphi; u_0), \tag{2.1}$$

where $u_0 \in C^1(\Gamma_\varphi; \mathbb{R})$ is our boundary condition. The condition (2.1) assumes evidently that the set $\Sigma \subset S_H$ can be defined as follows:

$$\Sigma = \left(\overline{\Sigma}; p_0\right) \tag{2.2}$$

with $p_0 \in C^1(\Gamma_\varphi; \mathbb{R}^n)$ being yet unknown smooth mappings. For it to be determined we need to ensure for all points $\Sigma \subset S_H$ the mentioned above compatibility condition, that is the condition

$$du|_\Sigma = \langle p, dx \rangle |_\Sigma, \tag{2.3}$$

where $\Sigma \subset S_H$ is given by (2.2). As a result of (2.3) one finds easily that

$$\begin{gathered} \frac{\partial u_0(s)}{\partial s} - \left\langle p_0(s), \frac{\partial x_0(s)}{\partial s} \right\rangle = 0, \\ H(x_0(s); u_0(s), p_0(s)) = 0 \end{gathered} \tag{2.4}$$

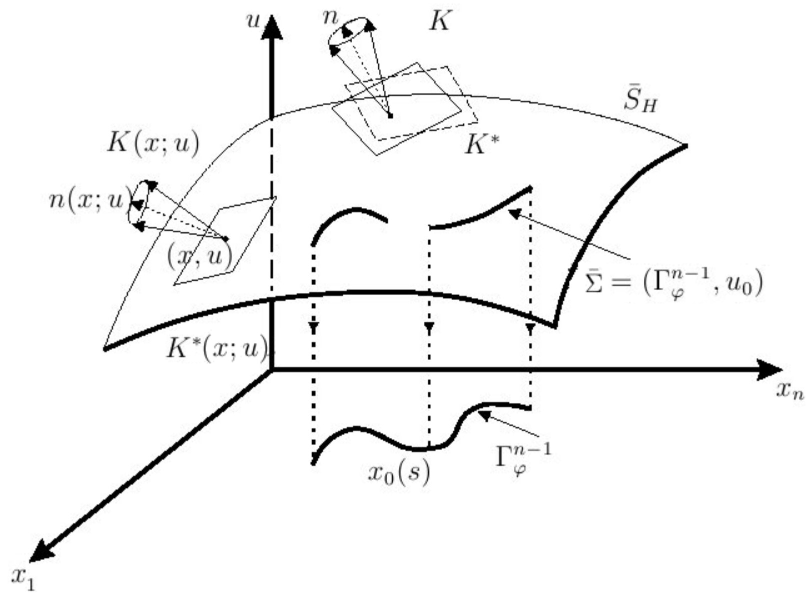

Fig. 2. Geometric Monge method. The boundary conditions: $\bar{\Sigma} = \left(\Gamma_\varphi^{n-1}, u_0\right) \subset \subset \bar{S}_H, u_0 \in C^1(\Gamma_\varphi^{n-1};\mathbb{R}), \bar{S}_H := \left\{(x,u) \in \mathbb{R}^{n+1} : \ u = \psi(x)\right\}$ is the boundary problem solution hypersurface.

for all points $x_0 = x_0(s) \in \Gamma_\varphi$, $s \in \mathbb{R}^{n-1}$. Here we took into account that any point $x \in \Gamma_\varphi$ is parametrized by means of the corresponding local coordinates $s = s(x_0) \in \mathbb{R}^{n-1}$, defined in the corresponding ε-vicinities $O_\varepsilon(x) \subset \Gamma_\varphi$, $\varepsilon > 0$.

The system of relationships (2.4) must be solvable for a mapping $p_0 : \Gamma_\varphi \to \mathbb{R}^n$ at all points $x_0 \in \Gamma_\varphi$, what gives rise to the determinant condition

$$\det\left(\frac{\partial x_0(s)}{\partial s}; \frac{\partial H}{\partial p_0}(x_0(s), u_0(x_0(s)), p_0(x_0(s)))\right) \neq 0, \tag{2.5}$$

owing to the implicit function theorem [13]. If the condition (2.5) is satisfied at points $\left(x_0; u_0, p_0^{(j)}\right) \in S_H$, where $j = \overline{1,N}$ for some $N \in \mathbb{Z}_+$ and all points $(x_0; u_0) \in \overline{\Sigma}$, the system of equations (2.4) possesses exactly $N \in \mathbb{Z}_+$ different smooth solution $p_0^{(j)} \in C^1(\Gamma_\varphi; \mathbb{R}^n)$, $j = \overline{1,N}$, thereby determining the corresponding Cauchy data (2.2) for the characteristic vector fields (1.3). It is clear enough that our boundary problem (1.1), (1.4), and (1.5) possesses, in general, many solutions of different functional classes, depending on the kind of chosen boundary conditions. For instance, as it was studied and analyzed in [2, 14] this boundary problem can possess also so-called generalized solutions, which allow at some additional conditions the so-called Hopf–Lax inf-type extremality form, being often very useful for studying their asymptotic and other properties.

Concerning this Hopf–Lax-type extremality solution problem for our equation (1.1) under the boundary conditions (1.4) and (1.5) the meaning of the involved before in [12] tensor fields $\mu^{(1|1)} \in C^1(\mathbb{R}^n \times \mathbb{R}^{n+1}; \mathbb{R}^n \otimes \mathbb{R}^n)$ becomes more understandable. Namely, it consists in finding solutions to (1.1), which will satisfy both the imposed boundary conditions and the related Hopf – Lax inf-type extremality representation [2, 11, 14].

3. The Hopf – Lax inf-type extremality representation. Assume now that $p_0 \in C^1(\Gamma_\varphi; \mathbb{R}^n)$ is a smooth solution to the system (2.4), thereby defining completely the sought Cauchy data $\Sigma \subset S_H$ for the characteristic vector fields (1.3). Thus, making use of suitable if any methods for solving these ordinary differential equations in the space $\mathbb{R}^n \times \mathbb{R}^{n+1}$ at some appropriately chosen tensor field $\mu^{(1|1)} \in C^1(\mathbb{R}^n \times \mathbb{R}^{n+1}; \mathbb{R}^n \otimes \mathbb{R}^n)$ one can find, in particular, the function $u \in C^2(\mathbb{R}^n; \mathbb{R})$ for each reachable point $x = x(t) \in \mathbb{R}^n$ in the form

$$u(x(t)) = u(x(0)) + \int_0^t \left\langle p(\tau), \mu^{(1|1)} \frac{\partial H}{\partial p}(\tau) \right\rangle d\tau \tag{3.1}$$

at any moment of "time" $t \in \mathbb{R}$. As, by definition, $x(0) := x_0(s) \in \Gamma_\varphi$ and $u(x(0)) := u_0(x_0(s))$, $s \in \mathbb{R}^{n-1}$, the solution (3.1) can be rewritten

$$u(x(t)) = u_0(x_0(s)) + \int_0^t \left\langle p(\tau), s^{(1|1)} \frac{\partial H}{\partial p}(\tau) \right\rangle d\tau \tag{3.2}$$

for any $t \in \mathbb{R}$, where the integrand function in (3.2) is assumed to be known.

Pose now the following "inverse" vector field problem for the equation

$$\frac{dx}{d\tau} = \mu^{(1|1)} \frac{\partial H}{\partial p} \tag{3.3}$$

with the following "inverse" Cauchy data

$$x|_{\tau=t(x)} = x \in \mathbb{R}^n, \quad x|_{\tau=0} = x_0(s(x)) \in \Gamma_\varphi \tag{3.4}$$

for some $s(x) \in \mathbb{R}^{n-1}$ at the moment of "time" $t(x) \in \mathbb{R}$ corresponding to an arbitrary reachable point $x \in \mathbb{R}^n$ as it is shown on Fig. 3. Respectively, for each found above point $x_0(s(x)) \in \Gamma_\varphi$, $x \in \mathbb{R}^n$, one can suitably determine the unique point $p_0(s(x)) \in \mathbb{R}^n$, $x \in \mathbb{R}^n$, making use of the system (3.2). As a result, one can write down owing to the conditions (3.4) the following expression:

$$u(x) = u_0(x_0(s(x))) + \int_0^{t(x)} \mathcal{L}(\tau | x_0(s(x)); x) d\tau, \tag{3.5}$$

where $\mathcal{L}$ is the so-called "Lagrangian" function:

$$\mathcal{L}(\tau | x_0(s(x)); x) := \left\langle p(\tau), \mu^{(1|1)} \frac{\partial H}{\partial p}(\tau) \right\rangle, \tag{3.6}$$

being defined by solutions to the equations (3.3), (3.4) and to the equation

$$\frac{dp}{d\tau} = -\mu^{(1|1)*} \left(\frac{\partial H}{\partial x} + \frac{\partial H}{\partial u} \right) \tag{3.7}$$

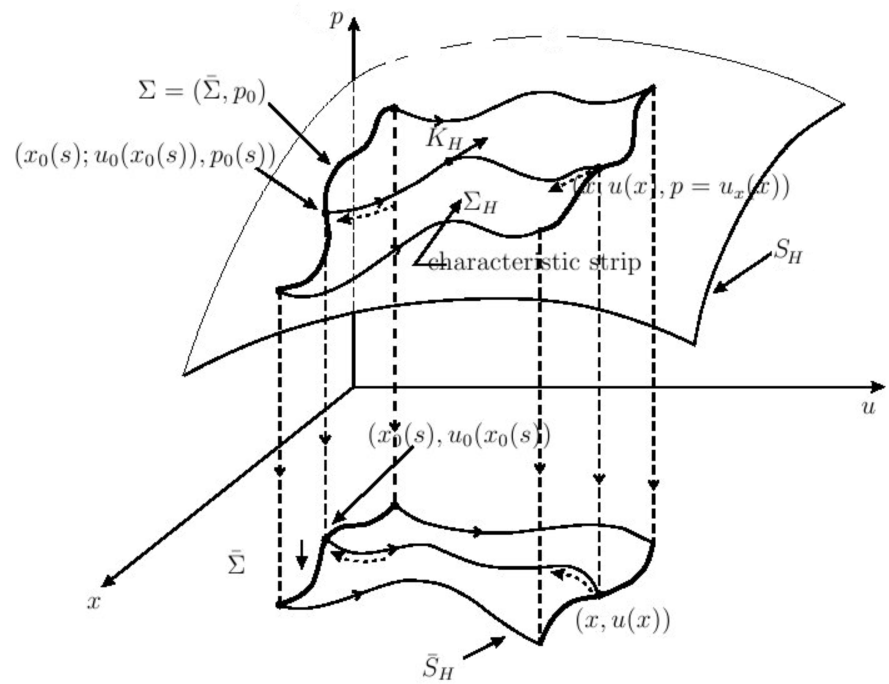

Fig.3. The geometric Monge method. The characteristic surface: $S_H = \{(x; u, p) \in \in \mathbb{R}^{2n+1} : H(x; u, p) = 0\}$ and intial conditions for the vector field $K_H : S_H \to \to T(S_H)$, satisfying the Cartan's compatibility conditions: $du - < p, dx >_{|K_H, \Gamma_\varphi^{n-1}} = = 0$ iff $\bar{S}_H || K^*$ and there exist data $\Sigma = (\bar{\Sigma}, p_0)$ defining the characteristic strip Σ_H.

under the corresponding "inverse" Cauchy data

$$p|_{\tau=0} = p_0(s(x)) \in \mathbb{R}^n \tag{3.8}$$

for any reachable point $x \in \mathbb{R}^n$.

By integrating the expressing (3.5) one finds the following solution to the boundary problem (1.1), (1.4), and (1.5):

$$u(x) = u_0\left(x_0(s(x))\right) + \mathcal{P}\left(x_0(s(x)); x\right), \tag{3.9}$$

where $x_0(s(x)) \in \Gamma_\varphi$, $x \in \mathbb{R}^n$, is, as above, any reachable by the vector field (3.3) point, and, by definition, the "kernel"

$$\mathcal{P}(x_0(s(x)); x) := \int_0^{t(x)} \mathcal{L}(\tau | x_0(s(x)); x) d\tau. \tag{3.10}$$

The obtained solution (3.9) allows an additional interpretation strongly motivated by the previous Hopf – Lax-type results [2, 11, 14]. Namely, consider the expression (3.5) with the Lagrangian function given by the expression (3.6) and the following extremality problem:

$$\delta u = \delta \left[(u_0(x_0(x))) + \int_0^t \mathcal{L}_0 \left(x(\tau); u(\tau), \dot{x}(\tau) \right) \, d\tau \right] = 0 \tag{3.11}$$

under a fixed ending point $x \in \mathbb{R}^n$ and varying both a moment of "time" $t \in \mathbb{R}_+$ and a point $x_0(s) \in \Gamma_\varphi$. Here we put, by definition, $\dot{x} := dx/d\tau$ and

$$\mathcal{L}_0 \left(\tau | x_0(s(x)); x \right) |_{\dot{x} = \mu^{(1|1)} \frac{\partial H}{\partial p}} := \left\langle p(\tau), \mu^{(1|1)} \frac{\partial H}{\partial p} \right\rangle \tag{3.12}$$

for all $(x, p) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\tau \in \mathbb{R}$.

By means of standard variational analysis calculations one gets easily that the condition (3.11) is realized exactly upon solutions to the vector field (1.3) under the Cauchy data at points in $\Sigma \subset S_H$, defined by (2.2) and (2.3). Thus, one can formulate at some natural conditions the following important theorem.

Theorem 3.1. *The expression (3.9) at any reachable point $x \in \mathbb{R}^n$, for suitable points $x_0(s) \in \Gamma_\varphi$ defined by the "inverse" Cauchy problem (3.3) and (3.4), solves the boundary problem (1.4), (1.5) for the partial differential equation (1.1) and allows, under some natural conditions on the tensor field $\mu^{(1|1)} \in C^1(\mathbb{R}^n \times \mathbb{R}^{n+1}; \mathbb{R}^n \otimes \mathbb{R}^n)$, the following Hopf–Lax-type extremality representation:*

$$u(x) = \inf_{y \in \Gamma_\varphi} \left\{ u_0(y) + \mathcal{P}(y; x) \right\}, \tag{3.13}$$

where the "kernel" $\mathcal{P} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is given by the analytical expression (3.10).

Proof. For the proof we need only to consider the extremum conditions (3.11), (3.12) and ensure that the function $[u_0(\cdot) + \mathcal{P}(\cdot; x)] : \Gamma_\varphi \to \mathbb{R}$ attains its finite infimum at some point $y = x_0(s(x)) \in \Gamma_\varphi$ for all reachable points $x \in \mathbb{R}^n$. The latter depends, in particular, on the functional properties of the boundary conditions $(\Gamma_\varphi; u_0) := \bar{\Sigma} \subset \mathbb{R}^{n+1}$ and on a choice of the tensor field $\mu^{(1|1)} \in C^1(\mathbb{R}^n \times \mathbb{R}^{n+1}; \mathbb{R}^n \otimes \mathbb{R}^n)$, defining our vector field (1.3), describing correspondingly the set of reachable points $x \in \mathbb{R}^n$. Having assumed these natural conditions, we find right away that the infimum (3.13) is attained exactly at the point $x_0(s(x)) \in \Gamma_\varphi$ and at the moment of "time" $t(x) \in \mathbb{R}_+$, satisfying the conditions (3.3) and (3.4), giving rise to the found before solution (3.9) of the boundary problem (1.1), (1.4), and (1.5), that ends the proof.

The Hopf–Lax-type extremality property of the solution to boundary problem (1.1), (1.4), and (1.5) appears to have a very interesting and important for applications form in the case of Cauchy problems for generalized Hamilton–Jacobi-type equations, which will be discussed in the section below.

4. The Hopf–Lax-type extremality solutions to generalized Hamilton–Jacobi equations. Assume we are given the following generalized scalar Hamilton–Jacobi equation

$$u_t + H(x, t; u, u_x) = 0 \tag{4.1}$$

with a Hamiltonian function $H \in C^2(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}; \mathbb{R})$ and the related Cauchy data

$$u|_{t=t_0} = u_0, \tag{4.2}$$

where $u_0 \in C^1(\mathbb{R}^n; \mathbb{R})$ and $t_0 \in \mathbb{R}$.

Having applied the results obtained above, one gets easily that the corresponding characteristic vector fields on S_H are defined as

$$\begin{gathered} \frac{dx}{d\tau} = \mu^{(1|1)} \frac{\partial H}{\partial p}, \quad \frac{dp}{d\tau} = -\mu^{(1|1)} \left(\frac{\partial H}{\partial x} + p \frac{\partial H}{\partial u} \right), \\ \frac{du}{d\tau} = \left\langle p, \mu^{(1|1)} \frac{\partial H}{\partial p} \right\rangle - H, \quad \frac{dt}{d\tau} = 1, \end{gathered} \tag{4.3}$$

where $\mu^{(1|1)} \in C^1(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}; \mathbb{R}^n \otimes \mathbb{R}^n)$ is some suitable tensor field, allowing to solve the following "inverse" Cauchy problem:

$$\frac{dx}{d\tau} = \mu^{(1|1)} \frac{\partial H}{\partial p}, \quad t = \tau, \quad x|_{\tau=t} = x, \quad x|_{\tau=t_0} = x_0(x,t), \tag{4.4}$$

where $\tau \in [t_0, t] \subset \mathbb{R}$, $x \in \mathbb{R}^n$, is any reachable point of the vector field (4.4) in $\mathbb{R}^n$ and $x_0(x, t_0) \in \mathbb{R}^n$ is the corresponding initial point at which our orbit $x : [t_0, t] \to \mathbb{R}^n$ starts. Thus, we can now write down a solution to the Cauchy problem (4.1), (4.2), making use of the previous results (3.9) and (3.10),

$$u(x,t) = u_0(x_0(x,t)) + \mathcal{P}(x_0(x,t), t_0; x, t), \tag{4.5}$$

where, by definition, the "kernel"

$$\mathcal{P}(x_0(x,t), t_0; x, t) := \int_{t_0}^{t} \left[\left\langle p(\tau), \mu^{(1|1)} \frac{\partial H}{\partial x}(\tau) \right\rangle - H(x(\tau), t; u(\tau), p(\tau) \right] d\tau \tag{4.6}$$

is defined for any reachable point $x \in \mathbb{R}^n$ and $t_0 \le t \in \mathbb{R}$. As the expression (4.5) solves the Cauchy problem (4.1), (4.2) for some mapping $x_0 : \mathbb{R}^{n+1} \to \mathbb{R}^n$ defined by the "inverse" Cauchy data, we obtain the following theorem.

Theorem 4.1. *The Hopf – Lax-type extremality expression*

$$u(x,t) = \inf_{y \in \mathbb{R}^n} \{u_0(y) + \mathcal{P}(y, t_0; x, t)\} \tag{4.7}$$

solves the Cauchy problem (4.1), (4.2), where the "kernel" $\mathcal{P} : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}$ is suitably defined by (4.6) for any reachable point $x \in \mathbb{R}^n$ and $t \ge t_0 \in \mathbb{R}$ in such a way that the infimum for the mapping $[u_0(\cdot) + \mathcal{P}(\cdot, t_0; x, t)] : \mathbb{R}^n \to \mathbb{R}$ is attainable and finite.

It is to be mentioned here that the "resolving kernel" $\mathcal{P} : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}$, defined by the expression (4.6), depends strongly on the choice of a tensor field $\mu^{(1|1)} \in C^1(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}; \mathbb{R}^n \otimes \otimes \mathbb{R}^n)$, ensuring both the effective solvability of the ordinary differential equations (4.3) and the corresponding existence of the infimum of the expression (4.7). Concerning these aspects, the related Cauchy data for (4.3) at $\tau = t_0 \in \mathbb{R}$ are given, owing to (4.4) and (2.4), as follows:

$$x|_{\tau=t_0} = x_0(x,t), \quad p|_{\tau=t_0} = p_0(x,t), \tag{4.8}$$

where the point $p_0(x,t) \in \mathbb{R}^n$ satisfies the compatibility condition

$$\frac{\partial u_0(x_0,t)}{\partial x} - \left\langle p_0(x,t), \frac{\partial x_0(x,t)}{\partial x} \right\rangle = 0 \tag{4.9}$$

for all reachable points $x \in \mathbb{R}^n$ and moments of time $t \in \mathbb{R}$. For the system (4.9) to be solvable, the natural condition

$$\det\left(\frac{\partial x_0(x,t)}{\partial x}\right) \neq 0 \tag{4.10}$$

must be satisfied at all points $(x,t) \in \mathbb{R}^n \times \mathbb{R}$.

As the existence of the infimum (4.7) depends implicitly also on the Cauchy data $u_0 : \mathbb{R}^n \to \to \mathbb{R}$, it looks very suggestive to represent a wide class of Cauchy problems for the Hamilton–Jacobi equation (4.1) in the Hopf–Lax-type form by choosing suitable tensor fields $\mu^{(1|1)} \in C^1(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}; \mathbb{R}^n \times \mathbb{R}^n)$. Concerning this aspect a very important yet too complicated problem of finding the related relationships between the Cauchy data and suitable tensor fields $\mu^{(1|1)} \in C^1(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}; \mathbb{R}^n \otimes \mathbb{R}^n)$ at which the problem (4.7) is reasonably posed, remains up to now to be unsolved. We can mention here also that results similar to those obtained above hold also for boundary problems posed for nonlinear partial differential equations of higher orders and suitable systems.

5. An example: the Burgers nonlinear differential equation of the second order. Consider the following Cauchy problem for the well known [15] Burgers differential equation

$$u_t + u_{xx} + uu_x = 0, \quad u|_{t=0^+} = u_0, \tag{5.1}$$

where $(x,t) \in \mathbb{R}^2$ and it is assumed that $u \in C^{(3,2)}(\mathbb{R} \times \mathbb{R}; \mathbb{R})$, $u_0 \in C^2(\mathbb{R}; \mathbb{R})$. The corresponding Monge surface S_H is defined as

$$S_H := (x,t;u,p) \in \left(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^6\right) : H(x,t;u,p) = 0, \tag{5.2}$$

where

$$p := \left(p_{(1,0)}, p_{(0,1)}, p_{(1,0)}, p_{(1,1)}, p_{(2,0)}, p_{(0,2)}\right) \in \mathbb{R}^6,$$

$$H(x,t;u,p) := p_{(0,1)} + p_{(2,0)} + up_{(1,0)}. \tag{5.3}$$

The projection upon the surface $\overline{S}_H \subset (\mathbb{R} \times \mathbb{R}_+) \times \mathbb{R}$ of solutions to (5.1) is given [12] by the following Cartan compatibility conditions:

$$du = p_{(1,0)}dx + p_{(0,1)}dt, \quad dp_{(1,0)} = p_{(2,0)}dx + p_{(1,1)}dt, \quad dp_{(0,1)} = p_{(1,1)}dx + p_{(0,2)}dt, \tag{5.4}$$

for all points $(x,t) \in (\mathbb{R} \times \mathbb{R}_+)$. Put now

$$\begin{aligned} \frac{dx}{d\tau} &= a_H(x,t:u,p), \quad \frac{dt}{d\tau} = \overline{a}_H(x,t;u,p), \\ \frac{dp_{(j)}}{d\tau} &= b_{(j)}(x,t;u,p), \quad \frac{du}{d\tau} = c_H(x,t;u,p), \end{aligned} \tag{5.5}$$

where $\tau \in \mathbb{R}$, $(j) \in J := (j_1, j_2) \in \{\overline{0,2} \times \overline{0,2}\} \backslash (0,0)$, being the related characteristic fields on the Monge hypersurface S_H, subject to which it remains to be invariant. This means that

$$\frac{dH}{d\tau} = \frac{\partial H}{\partial x} a_H + \frac{\partial H}{\partial t} \overline{a}_H + \frac{\partial H}{\partial u} c_H + \sum_{(j)} \frac{\partial H}{\partial p_{(j)}} b_{(j)} = 0 \tag{5.6}$$

for all $\tau \in \mathbb{R}$. Owing to the Cartan compatibility conditions (5.3) one gets that, along the vector field (5.5), the equalities

$$c_H = p_{(1,0)} a_H + p_{(0,1)}, \quad b_{(0,1)} = p_{(1,1)} a_H + p_{(0,2)} \overline{a}_H, \quad b_{(1,0)} = p_{(2,0)} + a_H + p_{(1,1)} \overline{a}_H \tag{5.7}$$

hold on S_H. Having substituted (5.7) into (5.6) one finds that the relationship

$$\begin{aligned} &\left(\frac{\partial H}{\partial x} + \frac{\partial H}{\partial u} p_{(1,0)} + \frac{\partial H}{\partial p_{(1,0)}} p_{(2,0)} + \frac{\partial H}{\partial p_{(1,1)}} p_{(1,1)} \right) a_H + \\ &\quad + \left(\frac{\partial H}{\partial t} + \frac{\partial H}{\partial u} p_{(0,1)} + \frac{\partial H}{\partial p_{(1,0)}} p_{(1,1)} + \frac{\partial H}{\partial p_{(0,1)}} p_{(0,2)} \right) \overline{a}_H + \\ &\quad + \frac{\partial H}{\partial p_{(1,1)}} b_{(1,1)} + \frac{\partial H}{\partial p_{(2,0)}} b_{(2,0)} + \frac{\partial H}{\partial p_{(0,2)}} b_{(0,2)} = 0 \end{aligned} \tag{5.8}$$

is satisfied on S_H. This means [12] that

$$\begin{aligned} a_H &= \mu^{(1|1)}_{(1,1)} \frac{\partial H}{\partial p_{(1,1)}} + \mu^{(1|1)}_{(2,0)} \frac{\partial H}{\partial p_{(2,0)}} + \mu^{(1|1)}_{(0,2)} \frac{\partial H}{\partial p_{(0,2)}}, \\ \bar{a}_H &= \bar{\mu}^{(1,1)}_{(1,1)} \frac{\partial H}{\partial p_{(1,1)}} + \bar{\mu}^{(1,1)}_{(2,0)} \frac{\partial H}{\partial p_{(2,0)}} + \bar{\mu}^{(1,1)}_{(0,2)} \frac{\partial H}{\partial p_{(0,2)}}, \\ b_{(j)} &= -\mu^{(1|1)}_{(j)} \left(\frac{\partial H}{\partial x} + \frac{\partial H}{\partial u} p_{(1,0)} + \frac{\partial H}{\partial p_{(1,0)}} p_{(2,0)} + \frac{\partial H}{\partial p_{(1,1)}} p_{(1,1)} \right) - \\ &\quad - \bar{\mu}^{(1|1)}_{(j)} \left(\frac{\partial H}{\partial t} + \frac{\partial H}{\partial u} p_{(0,1)} + \frac{\partial H}{\partial p_{(1,0)}} p_{(1,1)} + \frac{\partial H}{\partial p_{(0,1)}} p_{(0,2)} \right), \end{aligned} \tag{5.9}$$

where $j \in J$, $|j| = j_1 + j_2 = 2$. Taking into account now the expression (5.2), on gets that

$$
\begin{aligned}
a_H &= \mu^{(1|1)}_{(2,0)}, \quad \bar{a}_H = \bar{\mu}^{(1|1)}_{(2,0)},\\
b_{(0,1)} &= p_{(1,1)}\,\mu^{(1|1)}_{(2,0)} + p_{(0,2)}\,\bar{\mu}^{(1|1)}_{(2,0)},\\
b_{(1,0)} &= p_{(2,0)}\,\mu^{(1|1)}_{(2,0)} + p_{(1,1)}\,\bar{\mu}^{(1|1)}_{(2,0)},\\
c_H &= p_{(1,0)}\,\mu^{(1|1)}_{(2,0)} + p_{(0,1)}\,\bar{\mu}^{(1|1)}_{(2,0)},
\end{aligned}
\tag{5.10}
$$

$$
b_{(2,0)} = -\mu^{(1|1)}_{(2,0)}\left(+p^2_{(1,0)} + up_{(2,0)} + p_{(1,1)}\right) - \mu^{(1|1)}_{(2,0)}\left(+p_{(1,0)}\,p_{(0,1)} + up_{(1,1)} + p_{(0,2)}\right),
$$

$$
b_{(1,1)} = -\mu^{(1|1)}_{(1,1)}\left(+p^2_{(1,0)} + up_{(2,0)} + p_{(1,1)}\right),
$$

$$
b_{(0,2)} = -\mu^{(1|1)}_{(0,2)}\left(+p^2_{(1,0)} + up_{(1,0)} + p_{(1,1)}\right) - \bar{\mu}^{(1|1)}_{(0,2)}\left(+p_{(1,0)}p_{(0,1)} + up_{(1,1)} + p_{(0,2)}\right).
$$

For proceeding further put, for convenience,

$$
\mu^{(1|1)}_{(0,2)} = \alpha, \quad \bar{\mu}^{(1|1)}_{(2,0)} = 1, \quad \left|\mu^{(1|1)}_{(j)}\right|_{j\neq(0,2)} = 0 = \left|\mu^{(1|1)}_{(j)}\right|_{j\neq(2,0)}. \tag{5.11}
$$

Then from (5.10), (5.11), and (5.6) one gets the system

$$
\begin{gathered}
\frac{dx}{d\tau} = \alpha, \quad \frac{dt}{d\tau} = 1, \quad \frac{du}{d\tau} = \alpha\, p_{(1,0)} + p_{(0,1)},\\
\frac{dp_{(1,0)}}{d\tau} = \alpha\, p_{(2,0)} + p_{(1,1)}, \quad \frac{dp_{(0,1)}}{d\tau} = \alpha\, p_{(1,1)} + \bar{p}_{(0,2)},\\
\frac{dp_{(2,0)}}{d\tau} = -up_{(1,0)} - p_{(0,1)} + \bar{p}_{(2,0)}, \quad \frac{dp_{(1,0)}}{d\tau} = 0 = \frac{dp_{(0,2)}}{d\tau},\\
\frac{dp_{(1,0)}}{d\tau} = -\alpha\, up_{(1,0)} - \alpha\, p_{(0,1)} + \alpha\,\bar{p}_{(2,0)} + \bar{p}_{(1,1)},
\end{gathered}
\tag{5.12}
$$

which reduces to the following three equations:

$$
\begin{gathered}
\frac{dp_{(1,0)}}{d\tau} = -\alpha\, up_{(0,1)} - \alpha\, p_{(0,1)} + \alpha\, p_{(2,0)} + \bar{p}_{(1,1)},\\
\frac{dp_{(0,1)}}{d\tau} = ap_{(1,1)} + \bar{p}_{(2,0)}, \quad \frac{du}{d\tau} = ap_{(1,0)} + p_{(0,1)},
\end{gathered}
\tag{5.13}
$$

where $\alpha \in C^1(\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}^6;\mathbb{R})$, the quantities $p_{(1,1)} = \bar{p}_{(1,1)}, p_{(0,2)} = \bar{p}_{(0,2)}$ and $\bar{p}_{(2,0)} \in \mathbb{R}$ are real constants not depending on the evolution parameter $\tau \in \mathbb{R}$. For solving the system (5.13) we will make use of the ambiguous choice of the function $\alpha \in C^1(\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}^6;\mathbb{R})$. Namely, it is convenient to put here

$$
\alpha = \frac{c - \dfrac{dp_{(1,0)}}{d\tau}}{up_{(1,0)} + p_{(0,1)}} \tag{5.14}
$$

for all $\tau \in \mathbb{R}$ and some $c \in \mathbb{R}$. Then we easily deduce from (5.13) and (5.14) that simultaneously one gets two relationships,

$$\alpha = \frac{c - \bar{p}_{(1,1)}}{\bar{p}_{(2,0)}} := -k \tag{5.15}$$

where $k \in \mathbb{R}$ is some constant and

$$\frac{dp_{(1,0)}}{d\tau} = \left(up_{(1,0)} + p_{(0,1)}\right) k + c, \quad \frac{du}{d\tau} = -kp_{(1,0)} + p_{(0,1)}, \quad \frac{dp_{(0,1)}}{d\tau} = -k\bar{p}_{(1,1)} + \bar{p}_{(0,2)}. \tag{5.16}$$

From (5.16) we find that

$$\begin{gathered} p_{(0,1)} = \left(\bar{p}_{(0,2)} - k\bar{p}_{(1,1)}\right)\tau + \bar{p}_{(0,1)} := q\left(\tau\right), \quad \frac{du}{d\tau} = -kp_{(1,0)} + q(\tau), \\ \frac{dp_{(1,0)}}{d\tau} = (up_{(1,0)} + p_{(0,1)})k + c, \end{gathered} \tag{5.17}$$

where $q \in C^1(\mathbb{R};\mathbb{R})$ is some smooth function. The latter two equations can be reduced to the following one:

$$\frac{d}{d\tau}\left(u_\tau - \frac{ku^2}{2}\right) = q_\tau - k^2 q - kuq + kc, \tag{5.18}$$

which at the additional constraint $q(\tau) = 0$ for all $\tau \in \mathbb{R}$ gives rise to

$$u_\tau - \frac{ku^2}{2} = kc\tau + c_0, \tag{5.19}$$

where $c_0 \in \mathbb{R}$ is constant, and owing to (5.15), (5.17),

$$k = \frac{\bar{p}_{(0,2)}}{\bar{p}_{(1,1)}} = \frac{\bar{p}_{(1,1)} - c}{\bar{p}_{(2,0)}}, \quad \bar{p}_{(0,1)} = 0. \tag{5.20}$$

Put now, by definition,

$$u = -\frac{2}{k}\frac{d}{d\tau}\ln\psi, \tag{5.21}$$

and substitute it into (5.19). We then find easily the second order linear ordinary differential equation

$$\psi_{\tau\tau} + \frac{2}{k}\left(kc\tau + c_0\right)\psi = 0, \tag{5.22}$$

being the standard Airy differential equation, whose solutions allow the following [16] integral representation:

$$\psi = \bar{\psi}_0\,\mathcal{A}\,i_\pm\;(\tau|c_0, c, k) := \frac{\bar{\psi}_0}{2\pi}\int\limits_{\mathbb{R}} \exp\left[\pm\, i\left(\frac{c_0}{kc} + \tau\right)\lambda - \frac{2\lambda^3}{3k^2c}\right]d\lambda, \tag{5.23}$$

with the norming constant parameter $\bar{\psi}_0 \in \mathbb{C}$, and satisfying the boundary condition $\lim_{\tau\to+\infty} \psi(\tau) = 0$ if $\mp c > 0$.

Return now back to our vector fields (5.12), taking into account the results obtained above,

$$\begin{gathered}\frac{dx}{d\tau} = k, \quad x|_{\tau=0} = x_0(x,t), \quad x|_{\tau=t} = x, \quad k = [x - x_0(x,t)]/t, \\ u(x,t) = u_0(x_0(x,t),0) + \int_0^t \frac{du}{d\tau} d\tau = u_0(x_0(x,t),0) + \frac{2}{k}\int_0^t u^2(\tau)d\tau + \frac{kct^2}{2} + c_0 t,\end{gathered} \tag{5.24}$$

where the function $u_0 \in C^1(\mathbb{R};\mathbb{R})$ must satisfy the compatibility condition like (4.9),

$$\frac{\partial u_0(x_0(x,t))}{\partial x} - \left\langle \bar{p}_{(1,0)}(x_0(x,t)), \frac{\partial x_0(x,t)}{\partial x} \right\rangle = 0 \tag{5.25}$$

for all points $(x,t) \in \mathbb{R} \times \mathbb{R}_+$. Meanwhile, from (5.23) we get the general solution to (5.22),

$$\psi(\tau) = \mathcal{A}\, i_+(\tau|c_0,c,k) + \bar{\psi}_0\, \mathcal{A}\, i_-(\tau|c_0,c,k), \tag{5.26}$$

whence from (5.21),

$$u(\tau) = -\frac{2\left(\mathcal{A}\, i'_+(\tau|c_0,c,k) + \bar{\psi}_0\, \mathcal{A}\, i'_-(\tau|c_0,c,k)\right)}{k\left(\mathcal{A} i_+(\tau|c_0,c,k) + \bar{\psi}_0\, \mathcal{A}\, i_-(\tau|c_0,c,k)\right)} \tag{5.27}$$

for all $\tau \in \mathbb{R}$. As at $\tau = 0$ one has $u|_{\tau=0} = u_0(x_0(x,t))$, from (5.27) we derive the following three relationships:

$$\begin{aligned} u_0(x_0(x,t)) &= \frac{2t}{x_0(x,t)-x}\, \frac{\mathcal{A}i'_+\left(0|c_0,c,\dfrac{x-x_0(x,t)}{t}\right) + \bar{\Psi}_0 \mathcal{A}i'_-\left(0|c_0,c,\dfrac{x-x_0(x,t)}{t}\right)}{\mathcal{A}i_+\left(0|c_0,c,\dfrac{x-x_0(x,t)}{t}\right) + \bar{\Psi}_0 \mathcal{A}i_-\left(0|c_0,c,\dfrac{x-x_0(x,t)}{t}\right)}, \\ c_0(x,t|x_0) &= \frac{(x_0(x,t)-x)\, u_0^2(x_0(x,t))}{2t} - \frac{2t}{(x-x_0(x,t))}\frac{d^2}{d\tau^2} \ln\Psi(\tau)|_{\tau=0}, \\ c(x,t|x_0) &= \frac{2t^2}{(x_0(x,t)-x)^2}\frac{d^3}{dt^3} \ln\Psi(\tau)|_{\tau=0} + \frac{2u_0(x_0(x,t))}{(x-x_0(x,t))}\frac{d^2}{d\tau^2} \ln\Psi(\tau)|_{\tau=0}, \end{aligned} \tag{5.28}$$

supplying us with before undetermined three functional parameters $c_0 = c_0(x,t|x_0)$, $c = c(x,t|x_0)$ and $\bar{\psi}_0 = \bar{\psi}_0(x,t|x_0)$ at $x_0 = x_0(x,t) \in \mathbb{R}$ for all $(x,t) \in \mathbb{R} \times \mathbb{R}_+$. Thereby, we can substitute these functional parameters into the expression (5.27) having put $\tau = t \in \mathbb{R}_+$ and obtain the following expression:

$$u(x,t) = \frac{2t}{x_0 - x} \frac{\begin{array}{c}\mathcal{A}i'_+\Big(t|c_0(x,t|x_0), c(x,t|x_0), \dfrac{x-x_0}{t}\Big) + \\ +\bar{\psi}_0(x,t|x_0)\mathcal{A}i'_-\Big(t|c_0(x,t|x_0), c(x,t|x_0), \dfrac{x-x_0}{t}\Big)\end{array}}{\begin{array}{c}\Big(\mathcal{A}i_+\Big(t|c_0(x,t|x_0), c(x,t|x_0), \dfrac{x-x_0}{t}\Big) + \\ +\bar{\psi}_0(x,t|x_0)\mathcal{A}i_-\Big(t|c_0(x,t|x_0), c(x,t|x_0), \dfrac{x-x_0}{t}\Big)\Big)\end{array}}, \tag{5.29}$$

which is a solution of the Cauchy problem to the Burgers nonlinear differential equation (5.1) completely defined by the functional parameter $x_0 = x_0(x,t) \in \mathbb{R}$ for all $(x,t) \in \mathbb{R} \times \mathbb{R}_+$, solving the before discussed "inverse" Cauchy problem for the vector field (5.1). As this problem is, evidently, very hard and cumbersome, we can make use of the previously obtained results and state that the solution (5.29) owing to Theorem 4.1 allows the following Hopf–Lax-type extremality representation:

$$u(x,t) = \inf_{y \in \mathbb{R}} \left\{ \frac{2t}{y - x} \frac{\begin{array}{c}\Big(\mathcal{A}i'_+\Big(t|c_0(x,t|y), c(x,t|y), \dfrac{x-y}{t}\Big) + \\ +\bar{\psi}_0(x,t|y)\mathcal{A}\,i'_-\Big(t|c_0(x,t|y), c(x,t|y), \dfrac{x-y}{t}\Big)\Big)\end{array}}{\begin{array}{c}\Big(\mathcal{A}i_+\Big(t|c_0(x,t|y), c(x,t|y), \dfrac{x-y}{t}\Big) + \\ +\bar{\psi}_0(x,t|y)\mathcal{A}\,i_-\Big(t|c_0(x,t|y), c(x,t|y), \dfrac{x-y}{t}\Big)\Big)\end{array}} \right\}. \tag{5.30}$$

Thus, we can formulate the obtained result as the next final theorem.

Theorem 5.1. *The inf-type expression (5.30) is the Hopf–Lax-type extremality solution to the Cauchy problem (5.1) for the nonlinear Burgers equation.*

The method used for finding the extremality solution (5.30) to the nonlinear Burgers equation can be naturally applied to other nonlinear natural differential equations, including Korteweg–de Vries, nonlinear Schrödinger and other equations for which the problem of solving the Cauchy problem represents serious difficulties.

Remark 5.1. It is useful here to make a remark concerning the linearization result (5.22) for the solution (5.21) at $\tau = t \in \mathbb{R}_+$. Namely, this result means that by means of the mapping (5.21), written in the invariant form $u = \dfrac{d}{dx}\ln\Psi$, our nonlinear Burgers partial differential equation (5.1) transforms into the standard linear partial differential equation $\Psi_{xx} + \Psi_t = 0$, $(x,t) \in \mathbb{R} \times \mathbb{R}_+$, what is the classical Hopf–Cole result [15].

6. Conclusion. The results of the previous [12, 14] and this work convince us firmly that the geometrical Monge approach to studying solution of a wide class of nonlinear partial differential equations of first and higher order, based on our generalized characteristic method, is effective enough for many possible applications. The Hopf–Lax-type extremality representation of the corresponding solutions of both boundary and Cauchy problems gives rise to finding many new, in some sense, generalized solutions for a wide class of boundary and Cauchy data. Another still weakly investigated aspect of this approach is related with its application to analyzing suitable multidimensional symplectic reductions of boundary and Cauchy problems, giving rise [17–20] to new types of associated purely Hamiltonian nonlinear dynamical systems on functional manifolds of smaller spatial dimension. We plan to discuss this topic elsewhere.

Concerning the Burgers equation example (5.1), discussed in Section 5, we could see that the developed generalized characteristic method works well also for nonlinear partial differential equations of higher order. We proved the classical Hopf–Cole result [15] about the linearization of the Burgers equation. The exact Hopf–Lax-type extremality solution (5.30) was represented here through the classical Airy function. In particular, we obtained as a by product a little generalized linearization (5.21) and (5.22) of the Burgers equation (5.1), which can have some additional applications. Similar results are also valid for another nonlinear partial differential equations of first and higher orders like Korteweg — de Vries, nonlinear Schrödinger and other important nonlinear equations.

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