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EXISTENCE AND ASYMPTOTIC BEHAVIOR OF THE POSITIVE SOLUTIONS OF NEUTRAL IMPULSIVE DIFFERENTIAL EQUATIONS

ІСНУВАННЯ ТА АСИМПТОТИЧНА ПОВЕДІНКА ДОДАТНИХ РОЗВ'ЯЗКІВ НЕЙТРАЛЬНИХ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ З ІМПУЛЬСНОЮ ДІЄЮ

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In this article we consider first order neutral impulsive differential equations with constant coefficients and constant delays. We study the asymptotic behavior of the eventually positive solutions of these equations and establish necessary and sufficient conditions for the existence of such solutions.

Розглянуто нейтральні диференціальні рівняння першого порядку з імпульсною дією, сталими коефіцієнтами та сталими запізненнями. Вивчено асимптотичну поведінку евентуально додатних розв'язків цих рівнянь та знайдено достатні умови їх існування.

1. Introduction. Impulsive differential equations with deviating arguments (IDEDA) are adequate mathematical models for simulating processes that depend on their history and are subject to short-time disturbances. Such processes occur in the theory of optimal control, theoretical physics, population dynamics, biotechnology, industrial robotics, etc. In contrast to the theory of impulsive differential equations (see [1–4]) and differential equations with deviating arguments (see [5–9]), the theory of IDEDA, due to theoretical and practical difficulties, is developing rather slowly. We note here that [10] is the first work where IDEDA were considered. For more results, concerning IDEDA, we choose to refer to [11–16]. Much less we know about the neutral impulsive differential equations, i.e., the equations in which the highest-order derivative of the unknown function appears in the equation with the argument t (the present state of the system), as well as with one or more retarded and/or advanced arguments (the past and/or the future state of the system). Note that equations of this type appear in networks, containing lossless transmission lines. Such networks arise , for example, in high speed computers, where lossless transmission lines are used to interconnect switching circuits (see [17, 18]).

As it is known (see [5]), the appearance of the neutral term in a differential equation can cause or destroy properties of its solutions. Moreover, the study of neutral differential equations, in general, presents complications which are unfamiliar for nonneutral differential equations. As for a discussion on some more applications and some drastic differences in behavior of the solution of neutral differential equations see, for example, [19–21].

2. Preliminaries. Consider the first order neutral impulsive linear differential equation of the form

$$
\begin{aligned}
&[y(t) - cy(t-h)]' + ry(t) + qy(t-\sigma) = 0, \quad t \neq \tau_k, \quad k \in N, \\
&\Delta[y(\tau_k) - cy(\tau_k - h)] + py(\tau_k - \sigma) = 0, \quad k \in N,
\end{aligned}
\tag{1}
$$

where $h, \sigma \in (0, +\infty)$, $c \in (0,1)$ and r, q, p are constants, while τ_k, $k \in N$, with $t_0 = \tau_0 < < \tau_1 < \tau_2 < \ldots < \tau_k < \ldots$ and $\lim\limits_{k \to +\infty} \tau_k = +\infty$ are fixed moments of impulsive effect with the property $\max\{\tau_{k+1} - \tau_k\} < +\infty$, $k \in N$.

Denote by $i(\tau_0, t)$ the number of impulses τ_k with $\tau_k \in (\tau_0, t)$, $k \in N$, and by $A_\tau C(R, R)$ the set of all functions $u : R \to R$, which are absolutely continuous in every interval (τ_k, τ_{k+1}), $k \in N$. The functions u are continuous from the left as $t \to \tau_k - 0$, i.e., $u(\tau_k - 0) = u(\tau_k)$, $k \in N$, and may have discontinuity of the first kind at the points $\tau_k \in R_+$, $k \in N$.

Let $\rho = \max\{\sigma, h\}$. We will say that $y(t)$ is a solution of Eq. (1), if there exists a number $T_0 \in R$ such that $y \in A_\tau C([T_0 - \rho, +\infty], R)$, the function $z(t) = y(t) - cy(t-h)$ is continuously differentiable for $t \geq T_0$, $t \neq \tau_k$, $k \in N$, and $y(t)$ satisfies Eq. (1) for all $t \geq T_0$.

In what follows we will assume that every solution $y(t)$ of the equation (1) is *regular,* i.e., $y(t)$ is defined on $[T_y, +\infty)$ for some $T_y \geq T_0$ and $\sup\{|y(t)| : t \geq T\} > 0$ for each $T \geq T_y$.

A regular solution $y(t)$ of Eq. (1) is said to be *eventually positive* (*eventually negative*), if there exists a number $T_1 > 0$ such that $y(t) > 0$ ($y(t) < 0$) for every $t \geq T_1$. Also, note that a regular solution $y(t)$ of Eq. (1) is called *nonoscillatory,* if there exists a number $t_0 \geq 0$ such that $y(t)$ is of constant sign for every $t \geq t_0$. Otherwise, it is called *oscillatory*.

In this article we will study the asymptotic behavior of the eventually positive solutions of Eq. (1) and we will establish necessary and sufficient conditions for the existence of such solutions only in the case where Eq. (1) is a **neutral impulsive differential equation**, that is, when Eq. (1) is **neutral** ($h \neq 0$ for $c \neq 0$) and **impulsive** ($p \neq 0$ and $\tau_{k+1} - \tau_k = h, k \in N$).

So, in the sequel we will assume that

$$
c \in (0,1), \quad p \neq 0, \quad \tau_{k+1} - \tau_k = h, \quad k \in N \quad \text{and} \quad i(t-\sigma, t) = K = \text{const.} \tag{H}
$$

In order to obtain our results, we use an adaption of a special case of Lemma 3 of [6] (see also [22]), which describes the asymptotic behavior of the function $z(t) = y(t) - cy(t-h)$ and the asymptotic behavior of the eventually positive (resp. eventually negative) solutions $y(t)$ of Eq. (1).

Lemma 1. *Let $y(t)$ be an eventually positive solution of Eq. (1), where $r, p, q, h, \sigma \in (0, +\infty)$ and $c \in (0,1)$. Then:*

(a) $z(t) > 0$ for all large t with $\lim\limits_{t \to +\infty} z(t) = 0$ and $\lim\limits_{\tau_k \to +\infty} |\Delta z(\tau_k)| = 0$;

(b) $\lim\limits_{t \to +\infty} y(t) = 0$ and $\lim\limits_{\tau_k \to +\infty} |\Delta y(\tau_k)| = 0$.

Proof. (a) Since both $y(t)$ and $-y(t)$ are solutions to Eq. (1), in the sequel we will consider only eventually positive solutions $y(t)$ of Eq. (1). That is, in what follows we will assume that there is a number $t_0 > 0$ such that the solution $y(t)$ of Eq. (1) is defined on the interval $[t_0, +\infty)$ and $y(t) > 0$ for every $t \geq t_0$.

Set $T_0 = t_0 + \rho$, where $\rho = \max\{\sigma, h\}$. Then for every $t \geq T_0$ and $\tau_k \geq T_0$, $k \in N$, we have

$$[z(t)]' < 0, \quad t \neq \tau_k, \quad k \in N,$$
$$\Delta[z(\tau_k)] < 0, \quad k \in N. \tag{2}$$

Observe that, in view of (2), it follows that $z(t)$ is a strictly decreasing function, and so $\lim\limits_{t\to+\infty} z(t)$ exists in the set $R \cup \{-\infty\}$. Therefore, we have exactly one of the following three cases:

(i) $\lim\limits_{t\to+\infty} z(t) = -\infty$, or $\lim\limits_{t\to+\infty} z(t) = L$, where $L < 0$ is a constant,

(ii) $\lim\limits_{t\to+\infty} z(t) = L$, where $L > 0$ is a constant, or

(iii) $\lim\limits_{t\to+\infty} z(t) = 0$.

Assume that (i) holds. Then there exists a $\delta > 0$ and a $t_1 \geq T_0$ such that $z(t) < -\delta$ for every $t \geq t_1$, $t \neq \tau_k$, $k \in N$, that is,

$$y(t) - cy(t-h) < -\delta, \quad t \geq t_1, \quad t \neq \tau_k, \quad k \in N. \tag{3}$$

This implies that there is an index ν, $\nu \in N$, and $\delta_\nu > 0$, such that $z(\tau_k) < -\delta_\nu$ for every $\tau_k \geq \tau_\nu \geq t_1$, $k \geq \nu$.

Since $\Delta z(\tau_k) < 0, \tau_k \geq \tau_\nu \geq t_1$, $k \geq \nu$, we see that $z(\tau_k + 0) < z(\tau_k) < -\delta_\nu$ for every $\tau_k \geq \tau_\nu \geq t_1$, $k \geq \nu$. Hence, we derive $y(\tau_k + 0) - cy(\tau_k + 0 - h) < -\delta_\nu$, $\tau_k \geq \tau_\nu \geq t_1$, $k \geq \nu$ or equivalently $y(\tau_k + 0) < cy(\tau_k + 0 - h) - \delta_\nu$, $\tau_k \geq \tau_\nu \geq t_1$, $k \geq \nu$ and finally

$$y(\tau_k + 0) < cy(\tau_k + 0 - h), \quad \tau_k \geq \tau_\nu \geq t_1, \quad k \geq \nu. \tag{4}$$

In view of (4), we conclude that $\{y(\tau_k + 0)\}$, $k \in N$, for $\tau_k \geq \tau_\nu \geq t_1$, $k \geq \nu$, $\nu \in N$, is a decreasing sequence of positive real numbers and so from (3), by consecutive iterations, we find that

$$y(t) < -\delta + cy(t-h) < -\delta + c[-\delta + cy(t-2h)] < \ldots$$
$$\ldots < -\delta(1 + c + c^2 + \ldots + c^n) + c^n y(t - nh), \quad n = 1, 2, \ldots,$$

for every $t = t_1 + nh$, $n \in N$. Hence, we see that $n = \dfrac{t - t_1}{h}$ and $n \to +\infty$ as $t \to +\infty$. Since $c \in (0, 1)$, we conclude that in the last expression $c^n \to 0$, $y(t - nh) = y(t_1) > 0$ and, since $\sum\limits_{k=0}^{+\infty} c^k = \dfrac{1}{1-c}$, it follows that

$$\lim_{t\to+\infty} y(t) < -\frac{1}{1-c}\delta, \quad t > t_1. \tag{5}$$

Since $\{y(\tau_k + 0)\}$, $k \in N$, for $\tau_k \geq \tau_\nu \geq t_1$, $k \geq \nu$, $\nu \in N$, is a decreasing sequence of positive numbers and since $y(\tau_k) < cy(\tau_k + 0 - h) - \delta_\nu$, $k \in N$, for $\tau_k \geq \tau_\nu \geq t_1$, $k \geq \nu$, $\nu \in N$, a similar to (5) result follows for all large enough τ_k, $k \in N$, $k \geq \nu$, $\nu \in N$. Namely, for $M_\tau = \sup_{\tau_k > t_1}\{y(\tau_k + 0)\}$, we obtain

$$y(\tau_k + 0) < -\frac{1}{1-c}\delta_\nu + c^n y(\tau_k + 0 - nh), \quad k \in N, \quad \text{for} \quad \tau_k \geq \tau_\nu \geq t_1, \quad k \geq \nu, \quad \nu \in N,$$

or equivalently

$$y(\tau_k+0) < -\frac{1}{1-c}\delta_\nu + c^n M_\tau, \quad k \in N, \quad \text{for} \quad \tau_k \geq \tau_\nu \geq t_1, \quad k \geq \nu, \quad \nu \in N,$$

and finally

$$y(\tau_k+0) < -\frac{1}{1-c}\delta_\nu, \quad k \in N, \quad \text{for} \quad \tau_k \geq \tau_\nu \geq t_1, \quad k \geq \nu, \quad \nu \in N. \tag{6}$$

Now, in view of (5) and (6), we see that $y(t) < -\dfrac{1}{1-c}\max\{\delta, \delta_\nu\} < 0$, $\nu \in N$, for all large t. But this contradicts the positivity of the solution $y(t)$ and shows that the case (i) is impossible.

Consider now the case (ii). In this case, integrating Eq. (1) from T_0 to t, $t \geq T_0$, we find that

$$\int_{T_0}^{t} z'(s)ds + \int_{T_0}^{t} [ry(s) + qy(s-\sigma)]ds = 0$$

and hence

$$z(t) - z(T_0) - \sum_{T_0 \leq \tau_k < t} \Delta z(\tau_k) + \int_{T_0}^{t} [ry(s) + qy(s-\sigma)]ds = 0.$$

Since $\Delta z(\tau_k) = -py(\tau_k - \sigma)$, $\tau_k \geq T_0$, $k \in N$, we obtain

$$z(t) = z(T_0) - \sum_{T_0 \leq \tau_k < t} py(\tau_k - \sigma) - \int_{T_0}^{t} [ry(s) + qy(s-\sigma)]ds. \tag{7}$$

Because of the fact that $z(t) = y(t) - cy(t-h) > L > 0$ for all large t, we conclude that $0 < L < z(t) < y(t)$ for all large t. Using this result in (7), we obtain

$$z(t) \leq z(T_0) - L\left[\sum_{T_0 \leq \tau_k < t} p + \int_{T_0}^{t} [r+q]ds\right],$$

which implies $\lim\limits_{t\to+\infty} z(t) = -\infty$ and, as in the case (i), leads to a contradiction. Thus, the case (ii) is also impossible.

From the above observation, we see that the only possible case is that of (iii), i.e., the case, where $\lim\limits_{t\to+\infty} z(t) = 0$. Note that, since $z(t)$, $t \geq T_0$, is a decreasing function, it follows that $z(t) > 0$ for all large t and $\lim\limits_{k\to+\infty} |\Delta z(\tau_k)| = 0$, $k \in N$. This completes the proof of (a).

To prove (b), remark that, by (a), we have $\lim\limits_{t\to+\infty} [y(t) - cy(t-h)] = 0$, where $y(t) - cy(t-h) > > 0$ for all large t. Furthermore, $\lim\limits_{\tau_k\to+\infty} |\Delta z(\tau_k)| = 0$ implies $\lim\limits_{\tau_k\to+\infty} |\Delta y(\tau_k) - c\Delta y(\tau_k - h)| = 0$.

Next, as in (a), we prove (2), by which $z'(t) < 0$ for every $t \geq T_0$ and hence $y'(t) < cy'(t-h)$ for every $t \geq T_0$. If we assume that $y'(t) > 0$ for every $t \geq T_0$, then obviously $y'(t) < y'(t-h)$

for every $t \geq T_0$ and so y' is a strictly decreasing function on the interval $[T_0, +\infty)$. This implies that $y''(t) < 0$ for every $t \geq T_0$, and $y(t)$ has to be increasing and concave on $[T_0, +\infty)$. But this is impossible, because $y(t)$ is an eventually positive function and $\lim\limits_{t\to+\infty}[y(t) - cy(t-h)] = 0$. Therefore, we see that it must be $y'(t) < 0$ for every $t \geq T_0$. This means that $y(t)$ is a positive decreasing piecewise continuous function in every interval of the form $(\tau_k, \tau_{k+1}]$ for every $\tau_k \geq T_0$, $k \in N$, and changes its values at every moment τ_k, $k \in N$, of impulsive effect, where $\lim\limits_{\tau_k\to+\infty} |\Delta y(\tau_k) - c\Delta y(\tau_k - h)| = 0$. It is clear now that $\lim\limits_{t\to+\infty} y(t)$ exists in R. Moreover, since $y(t) - cy(t-h) > 0$ for all large t and since $y(t)$ is decreasing as $t \to +\infty$, it follows that $\lim\limits_{t\to+\infty} y(t)$ is also a finite number.

Thus, in view of the above observation, we conclude that

$$\liminf_{t\to+\infty} y(t) - c\liminf_{t\to+\infty} y(t-h) \leq \liminf_{t\to+\infty}[y(t) - cy(t-h)] = 0$$

and hence

$$(1-c)\liminf_{t\to+\infty} y(t) \leq 0,$$

which leads to $\liminf\limits_{t\to+\infty} y(t) = 0$ and finally to the conclusion that

$$\liminf_{t\to+\infty} y(t) = \limsup_{t\to+\infty} y(t) = \lim_{t\to+\infty} y(t) = 0$$

and $\lim\limits_{\tau_k\to+\infty} |\Delta y(\tau_k)| = 0$, which completes the proof of (b).

The proof of the lemma is complete.

3. Main results. Analyzing the forms of the solutions of the equation (1), when $c = 0$ or $h = 0$, i.e., when Eq. (1) is a **nonneutral equation**, we assume that the solutions of the **neutral impulsive differential equation** (1) must be in an appropriate *exponential impulsive form*. So, we will say that a function $u(t)$ has the form of *impulsive exponent*, if there are real numbers A and λ such that $u(t)$ can be expressed in the form

$$u(t) = e^{-\lambda t} A^{i(\tau_0, t)}, \tag{8}$$

where $i(\tau_0, t)$ is the number of impulses τ_k with $\tau_k \in (\tau_0, t)$, $k \in N$. In the following the constant A will be referred as *"pulsatile constant"* of the impulsive exponent.

Remark 1. Note that, in the above formula (8), $i(\tau_0, t) = -1$ for every $t \in (\tau_0 - h, \tau_0)$.

Consider Eq. (1) when $r = 0$ and the hypothesis (H) holds. That is, consider the equation

$$\begin{aligned} &[y(t) - cy(t-h)]' + qy(t-\sigma) = 0, \qquad t \neq \tau_k, \quad k \in N, \\ &\Delta[y(\tau_k) - cy(\tau_k - h)] + py(\tau_k - \sigma) = 0, \qquad k \in N, \end{aligned} \tag{1_1}$$

subject to the hypothesis (H).

Our first result provides a necessary and sufficient condition for existence of an eventually positive solution to Eq. (1_1) in the mentioned above *form (8) of impulsive exponent.*

Theorem 1. *Eq. (1_1) admits an eventually positive solution $y(t)$ in the form (8) if and only if the algebraic equation*

$$-\lambda\left(1-\frac{p}{q}\lambda\right)^{K}+c\lambda e^{\lambda,h}\left(1-\frac{p}{q}\lambda\right)^{K-1}+qe^{\lambda\sigma}=0 \qquad (9)$$

has at least one real root λ with $\lambda<\dfrac{q}{p}$ for $pq>0$ and $\lambda>\dfrac{q}{p}$ for $pq<0$.

Proof. Let $y(t)$ be an eventually positive solution to Eq. (1_1) defined by the formula (8). Substituting $y(t)$ into Eq. (1_1), from the differential equation we obtain

$$-\lambda e^{-\lambda t}A^{i(\tau_0,t)}+c\lambda e^{\lambda h}A^{i(\tau_0,t-h)}e^{-\lambda t}+qe^{\lambda\sigma}A^{i(\tau_0,t-\sigma)}e^{-\lambda t}=0,$$

while from the impulsive conditions we have

$$(A-1)A^{i(\tau_0,\tau_k)}e^{-\lambda\tau_k}-c(A-1)e^{\lambda h}A^{i(\tau_0,\tau_k-h)}e^{-\lambda\tau_k}+pe^{\lambda\sigma}A^{i(\tau_0,\tau_k-\sigma)}e^{-\lambda\tau_k}=0.$$

Having in mind the hypothesis (H), we find

$$i(t-\sigma,t)=K \quad \text{and} \quad i(\tau_0,t)=n, \quad \text{when} \quad t\in(\tau_0+nh,\tau_0+(n+1)h), \quad n\in N,$$

which leads to the system

$$-\lambda A^{K}+c\lambda e^{\lambda h}A^{K-1}+qe^{\lambda\sigma}=0,$$

$$(A-1)A^{K}-c(A-1)e^{\lambda h}A^{K-1}+pe^{\lambda\sigma}=0.$$

Since for $A=1-\dfrac{p}{q}\lambda$ the second equation of this system coincides with the first one, it is enough to consider the system

$$A=1-\frac{p}{q}\lambda,$$

$$\lambda A^{K}-c\lambda e^{\lambda h}A^{K-1}-qe^{\lambda\sigma}=0,$$

which is equivalent to the algebraic equation (9). In the sequel Eq. (9) will be referred to as the characteristic equation of Eq. (1_1). Obviously, if $y(t)$ is an eventually positive solution of Eq. (1_1) in the form (8) with the positive "pulsatile constant" $A=1-\dfrac{p}{q}\lambda$, where $\lambda<\dfrac{q}{p}$ if $pq>0$ and $\lambda>\dfrac{q}{p}$ if $pq<0$, then λ satisfies the characteristic equation (9). Conversely, consider the characteristic equation (9) and assume that it has a real root $\lambda_*<\dfrac{q}{p}$ if $pq>0$ and $\lambda_*>\dfrac{q}{p}$ if $pq<0$. Then Eq. (1_1) admits an eventually positive solution $y(t)$ in the form (8), i.e., $y(t)=e^{-\lambda_* t}A^{i(\tau_0,t)}$, with the positive "pulsatile constant" $A=1-\dfrac{p}{q}\lambda_*$.

The proof of the theorem is complete.

Next, consider Eq. (1) when it depends on a single deviation $h = \sigma$. That is, consider the equation

$$\begin{aligned} &[y(t) - cy(t-h)]' + ry(t) + qy(t-h) = 0, \qquad t \neq \tau_k, \quad k \in N, \\ &\Delta[y(\tau_k) - cy(\tau_k - h)] + py(\tau_k - h) = 0, \qquad k \in N. \end{aligned} \tag{1_2}$$

The following theorem provides a necessary and sufficient condition for the existence of an eventually positive solution for Eq. (1_2) in the *form (8)*.

Theorem 2. *Eq. (1_2) admits an eventually positive solution $y(t)$ in the form (8) if and only if the algebraic equation*

$$(r-\lambda)\left(1 - \frac{p(\lambda - r)}{q + cr}\right) + (c\lambda + q)e^{\lambda,h} = 0 \tag{10}$$

has at least one real root λ with $\lambda < r + \dfrac{q+cr}{p}$ for $p(q+cr) > 0$ and $\lambda > r + \dfrac{q+cr}{p}$ for $p(q+cr) > 0$.

Proof. Let $y(t)$ be an eventually positive solution to Eq. (1_2) defined by the formula (8). Substituting $y(t)$ into Eq. (1_2), from the differential equation we obtain

$$-\lambda e^{-\lambda t} A^{i(\tau_0,t)} + c\lambda e^{\lambda h} A^{i(\tau_0,t-h)} e^{-\lambda t} + r e^{-\lambda t} A^{i(\tau_0,t)} + q e^{\lambda h} A^{i(\tau_0,t-h)} e^{-\lambda t} = 0$$

and from the impulsive conditions we get

$$(A-1)A^{i(\tau_0,\tau_k)} e^{-\lambda\tau_k} - c(A-1)e^{\lambda h} A^{i(\tau_0,\tau_k-h)} e^{-\lambda\tau_k} + p e^{\lambda h} A^{i(\tau_0,\tau_k-h)} e^{-\lambda\tau_k} = 0.$$

Set, for convenience, $i(\tau_0, \tau_k) = n$, $n \in N$. Then we obtain the following system:

$$-\lambda A^n + c\lambda e^{\lambda h} A^{n-1} + rA^n + qe^{\lambda h} A^{n-1} = 0, \quad n \in N,$$

$$(A-1)A^n - c(A-1)e^{\lambda h} A^{n-1} + pe^{\lambda h} A^{n-1} = 0, \quad n \in N,$$

which after simplifications becomes

$$(r-\lambda)A + (c\lambda + q)e^{\lambda h} = 0,$$

$$(A-1)A + (p - c(A-1))e^{\lambda h} = 0.$$

If we choose $A = 1 - \dfrac{p(\lambda - r)}{q + cr}$, then it is easy to see that the second equation of the last system coincides with the first one. So, it is enough to consider the following system:

$$A = 1 - \frac{p(\lambda - r)}{q + cr},$$

$$(r-\lambda)A + (c\lambda + q)e^{\lambda h} = 0$$

which is equivalent to the algebraic equation (10). In the following, Eq. (10) will be referred to as the characteristic equation of Eq. (1_2). Obviously, if $y(t)$ is an eventually positive solution of Eq. (1_2) in the form (8) with the positive "pulsatile constant" $A = 1 - \dfrac{p(\lambda - r)}{q + cr}$, where $\lambda < r + \dfrac{q+cr}{p}$ if $p(q+cr) > 0$ and $\lambda > r + \dfrac{q+cr}{p}$ if $p(q+cr) < 0$, then λ satisfies the characteristic equation (10). Convesely, consider the characteristic equation (10) and assume that it has a real root $\lambda_* < r + \dfrac{q+cr}{p}$ if $p(q+cr) > 0$ and $\lambda_* > r + \dfrac{q+cr}{p}$ if $p(q+cr) < 0$. Then Eq. (1_2) admits an eventually positive solution $y(t)$ in the form (8), i.e., in the form $y(t) = e^{-\lambda_* t} A^{i(\tau_0, t)}$, with the positive "pulsatile constant" $A = 1 - \dfrac{p(\lambda_* - r)}{q + cr}$. The proof of the theorem is complete.

Consider Eq. (1_2) when $r = 0$. That is, consider the equation

$$\begin{aligned} &[y(t) - cy(t-h)]' + qy(t-h) = 0, \quad t \neq \tau_k, \quad k \in N, \\ &\Delta[y(\tau_k) - cy(\tau_k - h)] + py(\tau_k - h) = 0, \quad k \in N, \end{aligned} \tag{1_3}$$

which is a special case of Eq. (1_1).

In the case of Eq. (1_3), as an immediate consequence of the last theorem, we have the following.

Corollary 1. *Equation* (1_3), *admits an eventually positive solution* $y(t)$ *in the form (8) if and only if the algebraic equation*

$$-\lambda\left(1 - \frac{p}{q}\lambda\right) + (c\lambda + q)e^{\lambda, h} = 0 \tag{11}$$

has at least one real root λ *with* $\lambda < \dfrac{q}{p}$ *for* $pq > 0$ *and* $\lambda > \dfrac{q}{p}$ *for* $pq < 0$.

As an illustration of the above result, we give the following example.

Example 1. The neutral impulsive differential equation

$$[y(t) - 0,14711y(t-1)]' + 0,06306365y(t-1) = 0, \quad t \neq \tau_k, \quad k \in N,$$

$$\Delta[y(\tau_k) - 0,14711y(\tau_k - 1)] + 0,1438y(\tau_k - 1) = 0, \quad k \in N,$$

for every $t > \tau_0$ satisfies the assumptions of Corollary 1 and, therefore, admits the eventually positive solution $y(t) = e^{-\lambda_* t} A^{-i[\tau_0, t)}$ with the initial function

$$\varphi(t) = Ae^{-\lambda_* t}, \quad t \in (\tau_0 - 1, \tau_0],$$

where $\lambda_* = 0,1592 < \dfrac{q}{p} = 0,438551$ and $A = 1 - \dfrac{p}{q}\lambda_* = 0,636986$.

Remark 2. It is easy to see that, according to Lemma 1, this solution has the properties

$$\lim_{t \to +\infty} y(t) = 0, \quad \lim_{\tau_k \to +\infty} |\Delta y(\tau_k)| = 0.$$

Example 2. The neutral impulsive differential equation

$$[y(t) - 0,2y(t-1)]' + 0,02y(t-1) = 0, \quad t \neq \tau_k, \quad k \in N,$$

$$\Delta[y(\tau_k) - 0,2y(\tau_k - 1)] - 0,03y(\tau_k - 1) = 0, \quad k \in N,$$

for every $t > \tau_0$ satisfies the assumptions of Corollary 1 and so admits the eventually positive solution $y(t) = e^{-\lambda_* t} A^{-i[\tau_0, t)}$ with the initial function

$$\varphi(t) = Ae^{-\lambda_* t}, \quad t \in (\tau_0 - 1, \tau_0],$$

where $\lambda_* = 0,493 > \dfrac{q}{p} = -0,666667$ and $A = 1 - \dfrac{p}{q}\lambda_* = 0,091$.

Remark 3. Observe that this solution has the properties

$$\lim_{t \to +\infty} y(t) = 0, \quad \lim_{\tau_k \to +\infty} |\Delta y(\tau_k)| = 0,$$

even though the assumptions of Lemma 1 are not fulfilled.

Next, we formulate and prove a corollary which provides more concrete sufficient condition for existence of an eventually positive solution to Eq. (1_3) with positive coefficients q and p in the *form (8) of impulsive exponent.*

Corollary 2. *Equation (1_3) with positive coefficients q and p admits an eventually positive solution in the form (8) if*

$$\frac{1}{4\alpha_2} - \frac{c}{2} \leq p \leq \frac{1}{4\alpha_1} - \frac{c}{2},$$

where α_1 and α_2 with $\alpha_1 \leq \alpha_2$ are real roots of the equation

$$\alpha = e^{\frac{qh(c+2p)}{p}\alpha}$$

under the assumption that $qh \leq \dfrac{p}{(c+2p)e}$.

Proof. By Corollary 1, Eq. (1_3) admits an eventually positive solution $y(t)$ if and only if its characteristic equation (11) has a real root $\lambda < \dfrac{q}{p}$ when $pq > 0$. We will try to express, in terms of the constant coefficients and the constant delays involved in Eq. (1_3), how the existence of an eventually positive solution depends on the conditions of the corollary. For this purpose consider the function

$$F(\lambda) = -\lambda\left(1 - \frac{p}{q}\lambda\right) + (c\lambda + q)e^{\lambda h}.$$

Clearly, $F(0) = q > 0$. Moreover, it is easy to check that $F(\lambda) > 0$ for every $\lambda < 0$. Indeed, for any $s > 0$, set $\lambda = -s < 0$. Then, because of the fact that $c \in (0,1)$, we obtain $F(-s) = s\left(1 + \dfrac{p}{q}s\right) + (-cs + q)e^{-sh} > s + \dfrac{p}{q}s^2 - sce^{-sh} + qe^{-sh} > s(1 - ce^{-sh}) > s(1 - e^{-sh}) > 0$ for every $s > 0$ and $h > 0$. Thus, we see that, if there exists a root of $F(\lambda)$, it must be a positive number, i.e., $F(\lambda) = 0$ is possible only for a number $\lambda > 0$. In order to find such a number, set

$g(\lambda) = \lambda\left(1 - \frac{p}{q}\lambda\right)$ and $f(\lambda) = (c\lambda + q)e^{\lambda h}$ and consider the graphics of the functions $g(\lambda)$ and $f(\lambda)$. Remark that $g(0) = g\left(\frac{q}{p}\right) = 0$, $f\left(\frac{-q}{c}\right) = 0$ and $f(0) = q$. Furthermore, the function $g(\lambda)$ has its maximum $g_{\max}(\lambda) = \frac{q}{4p}$ at the point $\lambda = \frac{q}{2p}$, while the function $f(\lambda)$ has its minimum $f_{\min}(\lambda) = -ce^{-(c+qh)/h}$ at the point $\lambda = -\frac{c+qh}{ch}$. In view of this observation, having in mind the monotonicity of the function $f(\lambda)$ and the parabolic function $g(\lambda)$, we will try to locate a root $\lambda > 0$ of (11) by asking for $\lambda = \frac{q}{2p}$ to satisfy the inequality

$$g_{\max}\left(\frac{q}{2p}\right) = \frac{q}{4p} \geq f\left(\frac{q}{2p}\right) = \left[\left(\frac{cq}{2p}\right) + q\right] e^{\frac{qh}{2p}}.$$

Hence, we conclude that

$$\frac{1}{2(c+2p)} \geq e^{\frac{qh}{2p}} = e^{\frac{qh(c+2p)}{p}\frac{1}{2(c+2p)}}. \tag{12}$$

Consider now the inequality (12) and the well-known equation

$$\alpha = e^{m\alpha},$$

where m is the coefficient of proportionality.

The inequality (12) becomes an equality if and only if $\frac{qh(c+2p)}{p} = \frac{1}{e}$ and $\frac{1}{2(c+2p)} = e$, or equivalently if and only if $qh = 2p$. Therefore, Eq. (11) admits two positive real roots λ_1 and λ_2, with $\lambda_2 = \frac{q}{2p}$, for which $F(\lambda_1) = F(\lambda_2) = 0$. Obviously, the function $y(t) = e^{-\frac{q}{2p}t}A^{-i[\tau_1,t)}$, which is of the form (8), is one of two solutions of Eq. (1_3).

Next, remark that the inequality (12) is satisfied if and only if $\alpha_1 < \frac{1}{2(c+2p)} < \alpha_2$, where α_1 and α_2 are two real roots of the equation $\alpha = e^{\frac{qh(c+2p)}{p}\alpha}$ under the assumption that $\frac{qh(c+2p)}{p} < \frac{1}{e}$. Respectively Eq. (11) must have two positive real roots, say λ_1 and λ_2, with $\lambda_1 < \frac{q}{2p} < \lambda_2$ and so Eq. (1_3) must have two eventually positive solutions of the form (8), i.e., of the form $y(t) = e^{-\lambda_i t}A^{-i(\tau_0,t)}$, $i = 1, 2$.

Generalizing the above consideration, we are led to the inequality $F\left(\frac{q}{2p}\right) \leq 0$ and conclude that Eq. (11) admits two positive real roots λ_1 and λ_2 with $\lambda_1 < \lambda_2$ if and only if

$$\alpha_1 \leq \frac{1}{2(c+2p)} \leq \alpha_2 \Leftrightarrow \frac{1}{4\alpha_2} - \frac{c}{2} \leq p \leq \frac{1}{4\alpha_1} - \frac{c}{2},$$

where α_1 and α_2 are the roots of the equation $\alpha = e^{\frac{qh(c+2p)\alpha}{p}}$ under the assumption that $qh \leq \frac{p}{(c+2p)e}$.

The proof of the corollary is complete.

The following example illustrates the above result.

Example 3. The neutral impulsive differential equation

$$\left[y(t)-\frac{1}{4e}y(t-1)\right]'+\frac{1}{4e}y(t-1)=0,\quad t\neq\tau_k,\quad k\in N,$$

$$\Delta\left[y(\tau_k)-\frac{1}{4e}y(\tau_k-1)\right]+\frac{1}{8e}y(\tau_k-1)=0,\quad k\in N,$$

where $\tau_{k+1}-\tau_k=1$, $k\in N$, satisfies the assumptions of Corollary 2 and hence admits the eventually positive solution $y(t)=e^{-t}2^{-i(\tau_0,t)}$ for $t>\tau_0$ with the initial function

$$\varphi(t)=2e^{-t},\quad t\in(\tau_0-1,\tau_0].$$

This solution is of the form (8) and, according to Lemma 1, has the properties $\lim\limits_{t\to+\infty}y(t)=0$ and $\lim\limits_{\tau_k\to+\infty}|\Delta y(\tau_k)|=0$.

We conclude this section by considering Eq. (1_3) and the following neutral impulsive differential equation

$$\begin{aligned}&[y(t)-sy(t-h)]'+gy(t-h)=0,\quad t\neq\tau_k,\quad k\in N,\\&\Delta[y(\tau_k)-sy(\tau_k-h)]+dy(\tau_k-h)=0,\quad k\in N,\end{aligned}\qquad(1_4)$$

where g, d, $h\in(0,+\infty)$ and $s\in(0,1)$. The following comparison result shows that the existence of an eventually positive solution to Eq. (1_3) implies the existence of an eventually positive solution to Eq. (1_4).

Corollary 3. *Consider the equations (1_3) and (1_4). If $g\leq q$, $d\geq p$ and $c+2p\geq s+2d$, then, under the assumptions that $qh\leq\dfrac{p}{(c+2p)e}$ and $(c+2p)e^{\frac{qh}{2p}}\leq\dfrac{1}{2}$, the equation (1_4) admits an eventually positive solution.*

Proof. Remark that the assumptions $qh\leq\dfrac{p}{(c+2p)e}$ and $(c+2p)e^{\frac{qh}{2p}}\leq\dfrac{1}{2}$, according to Corollary 2, are sufficient ones for existence of an eventually positive solution to Eq. (1_3). Moreover, in view of our assumptions, it is easy to check that

$$\frac{1}{2(s+2d)}\geq\frac{1}{2(c+2p)}\geq e^{\frac{qh}{2p}}\geq e^{\frac{sh}{2d}}$$

and

$$gh\leq qh\leq\frac{p}{(c+2p)e}\leq\frac{d}{(s+2d)e}.$$

Therefore, we see that

$$\frac{1}{2(s+2d)}\geq e^{\frac{s\sigma}{2d}}$$

and

$$gh \le \frac{d}{(s+2d)e}.$$

But, in view of Corollary 2 and Theorem 2, the last two inequalities are sufficient for existence of an eventually positive solution to Eq. (1_4) and the proof of the corollary is complete.

4. Applications. Since the neutral differential equation (1_1) in the cases where: a) $p,\, h,\, \sigma \in (0, +\infty)$ and $q = 0$, and b) $q,\, h, \sigma \in (0, +\infty)$ and $p = 0$, has been studied in [22], in this section we will deal with the neutral differential equation (1_1) in all other cases. So, Theorem 1 and its corollaries, applied to Eq. (1_1) in the cases under consideration give the following results.

Theorem 3. *Eq. (1_1), where $\sigma = 0$, admits an eventually positive solution in form (8) if and only if the algebraic equation*

$$(-\lambda + q)\left(1 - \frac{p}{q}\lambda\right) + c\lambda e^{\lambda h} = 0$$

has at least one real root λ with $\lambda < \dfrac{q}{p}$ for $pq > 0$ and $\lambda > \dfrac{q}{p}$ for $pq < 0$.

Since this theorem is an immediate consequence of Theorem 1 and since its proof is similar to that of Theorem 1, we omit the details of the proof.

Corollary 4. *Suppose that the assumptions of Theorem 3 are satisfied. Then Eq. (1_1), where $p, q, h \in (0, +\infty)$ and $\sigma = 0$, admits an eventually positive solution in the form (8), if*

$$\alpha_1 \le \frac{(1-p)^2}{2c(1+p)} \le \alpha_2,$$

where $\alpha_1 \le \alpha_2$ are real roots of the equation

$$\alpha = e^{\frac{\alpha c q h (1+p)^2}{p(1-p)^2}},$$

under the assumption that $qh \le \dfrac{p(1-p)^2}{c(1+p)^2}\dfrac{1}{e}$.

The proof of this proposition is similar to that of Corollary 2 and therefore is omitted.

Example 4. The neutral impulsive differential equation

$$\left[y(t) - \frac{1}{6e}y(t-1)\right]' + \frac{\ln 2}{3}y(t) = 0, \quad t \ne \tau_k, \quad k \in N,$$

$$\Delta\left[y(\tau_k) - \frac{1}{6e}y(\tau_k - 1)\right] + \frac{1}{2}y(\tau_k) = 0, \quad k \in N,$$

where $\tau_{k+1} - \tau_k = 1$, $k \in N$, satisfies the assumptions of Theorem 3 and thus admits the eventually positive solutions $y(t) = e^{-\lambda_j t}\Big(1 - \dfrac{3}{2\ln 2}\lambda_j\Big)^{-i(\tau_0, t)}$, $j = 1, 2$, for $t > \tau_0$ with $\lambda_1 = 0,3086189$ and $\lambda_2 = 0,3329507$. Note that $\lambda_1 < \dfrac{q}{p}$ and $\lambda_2 < \dfrac{q}{p}$, where $\dfrac{q}{p} = 0,462$.

The next theorem allows us to obtain necessary and sufficient condition for existence of an eventually positive solutions to Eq. (1_1).

Theorem 4. *Consider Eq. (1_1), where $p,\, h \in (0,+\infty)$ and $q = \sigma = 0$. Assume also that $\max_{k\in N}\left\{\tau_k-\tau_{k-n}\right\} \le h < \min_{k\in N}\left\{\tau_k-\tau_{k-(n+1)}\right\}$, $n \in N$. Then Eq. (1_1) admits an eventually positive solution of the form (8) if and only if the algebraic equation*

$$(A-1+p)A^k - c(A-1)A^{k-n} = 0$$

has at least one positive real root $A \in (0,1)$.

Proof. Let $y(t) = e^{-\lambda t}A^{i(\tau_0,t)}$, where $A > 0$ is a real number and $i(\tau_0,t)$ is the number of impulses τ_k, $k \in N$, with $\tau_k \in [\tau_1,t)$, $k \in N$, be an eventually positive solution of Eq. (1_1).

Substituting $y(t)$ into Eq. (1_1), we obtain

$$\left[e^{-\lambda t}A^{i(\tau_0,t)} - ce^{-\lambda(t-h)}A^{i(\tau_0,t-h)}\right]' = 0,$$

$$e^{-\lambda\tau_k}A^{i(\tau_0,\tau_k)}(A-1) - ce^{-\lambda\tau_k}e^{\lambda h}(A-1)A^{i(\tau_0,\tau_k-h)} + pe^{-\lambda\tau_k}A^{i(\tau_0,\tau_k)} = 0$$

and, hence, we derive

$$\begin{gathered}-\lambda A^{i(\tau_0,t)} + c\lambda e^{\lambda h}A^{i(\tau_0,t-h)} = 0,\\ (A-1)A^{i(\tau_0,\tau_k)} - ce^{\lambda h}(A-1)A^{i(\tau_0,\tau_k-h)} + pA^{i(\tau_0,\tau_k)} = 0.\end{gathered} \tag{13}$$

Setting $\lambda = 0$ in (13), we see that the eventually positive solution of Eq. (1_1) can be defined by the equation

$$(A-1+p)A^k - c(A-1)A^{k-n} = 0. \tag{14}$$

Note that, as $p > 0$, we must have $A \neq 1$. Since the assumption $A > 1$ leads to the contradiction $A^k < cA^{k-n}$, we conclude that Eq. (1_1) has an eventually positive solution of the form $y(t) = A^{i(\tau_0,t)}$ if and only if Eq. (14) has a positive real root $A \in (0,1)$, which is in agreement with Lemma 1.

The proof of the theorem is complete.

Theorem 4 implies the following corollary.

Corollary 5. *Consider Eq. (1_1), where $p,\, h \in (0,+\infty)$ and $q = \sigma = 0$. Assume also that*

$$\tau_{k+1} - \tau_k = h, \quad k \in N.$$

Then Eq. (1) admits an eventually positive solution of the form (8) if and only if

$$p \le 1 + c - \sqrt{4c}.$$

Proof. As in the proof of Theorem 4, we assume the existence of an eventually positive solution in the form (8) with $A > 0$ and derive Eq. (14) which in our case becomes

$$(A-1)A^k - c(A-1)A^{k-1} + pA^k = 0.$$

Hence, after some elementary manipulations, we find

$$A^2 + (p - 1 - c)A + c = 0. \tag{15}$$

Observe that the two roots

$$A_1 = \frac{1 + c - p + \sqrt{(p - 1 - c)^2 - 4c}}{2} \quad \text{and} \quad A_2 = \frac{1 + c - p - \sqrt{(p - 1 - c)^2 - 4c}}{2}$$

of Eq. (15) are real numbers if and only if $p \leq 1 + c - \sqrt{4c}$, or $p \geq 1 + c + \sqrt{4c}$. Remark also that in the case where $p \geq 1 + c + \sqrt{4c}$, both of the roots A_1 and A_2 are negative numbers, which contradicts the fact that $A > 0$, and so the corresponding to them solutions are oscillatory. Therefore, in order to have eventually positive solutions of the form (8), at least one of the roots $A_1$and A_2 must be positive. But this is possible if and only if $p \leq 1 + c - \sqrt{4c}$. Thus, we see that Eq. (15) admits eventually positive solutions of the form $y(t) = A_j^{i(\tau_0,t)}$, $j = 1$ and/or $j = 2$ which are of the form (8), if and only if $p \leq 1 + c - \sqrt{4c}$.

The proof of the corollary is complete.

We conclude this article with the following result.

Theorem 5. *The differential equation (1), where* $h > 0$, $r = p = q = 0$, *always admits an eventually positive solution of the form* $y(t) = e^{-\lambda^* t}(1 - \lambda^*)^{i(\tau_0,t)}$, *where* λ^* *is a real root of the equation*

$$-(1 - \lambda) + ce^{\lambda h} = 0. \tag{16}$$

Proof. Following the proof of Theorem 1, we arrive at the system

$$-\lambda A^{i(\tau_0,t)} + c\lambda e^{\lambda h} A^{i(\tau_0,t-h)} = 0,$$

$$(A - 1)A^{i(\tau_0,\tau_k)} - c(A - 1)e^{\lambda h} A^{i(\tau_0,\tau_k - h)} = 0.$$

By setting $A = 1 - \lambda$, we are led to the equation (16), which always has a positive root $\lambda^* \in$ $\in (0, 1)$. Consequently, Eq. (1) under the conditions of Theorem 5, always admits an eventually positive solution of the form $y(t) = e^{-\lambda^* t}(1 - \lambda^*)^{i(\tau_0,t)}$.

The proof of the theorem is complete.

Example 5. The neutral impulsive differential equation

$$\left[y(t) - \frac{1}{6e}y(t - 1)\right]' = 0, \quad t \neq \tau_k, \quad k \in N,$$

$$\Delta\left[y(\tau_k) - \frac{1}{6e}y(\tau_k - 1)\right] = 0, \quad k \in N,$$

where $\tau_{k+1} - \tau_k = 1$, $k \in N$, satisfies all the assumptions of Theorem 5 and, there fore, admits the eventually positive solution $y(t) = e^{-\lambda t} A^{-i(\tau_0,t)}$, $t > \tau_0$, with $\lambda = 0,856$ and $A = 0,144$.

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