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BEHAVIOR OF SOLUTIONS OF SECOND ORDER DIFFERENTIAL EQUATIONS WITH SUBLINEAR DAMPING

ПОВЕДІНКА РОЗВ'ЯЗКІВ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ ДРУГОГО ПОРЯДКУ З СУБЛІНІЙНИМ ЗГАСАННЯМ

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The authors investigate the asymptotic behavior of solutions of the damped nonlinear oscillator equation

$$x'' + a(t)|x'|^{\alpha}\mathrm{sgn}\,(x') + f(x) = 0,$$

where $uf(u) > 0$ for $u \neq 0$, $a(t) \geq 0$, and α is a positive constant with $0 < \alpha \leq 1$. The case $\alpha = 1$ has been investigated by a number of other authors. Here, it is shown that the behavior of solutions in the case of sublinear damping $(0 < \alpha < 1)$ is completely different from that in the case of linear damping $(\alpha = 1)$. Sufficient conditions for all nonoscillatory solutions to converge to zero and sufficient conditions for the existence of a nonoscillatory solution that does not converge to zero are given. They also give sufficient conditions for all solutions to be nonoscillatory. Some open problems for future research are also indicated.

Вивчається асимптотична поведінка розв'язків нелінійного рівняння з коливанням та згасанням

$$x'' + a(t)|x'|^{\alpha}\mathrm{sgn}\,(x') + f(x) = 0,$$

де $uf(u) > 0$ для $u \neq 0$, $a(t) \geq 0$ та α — додатна стала, що задовольняє умову $0 < \alpha \leq 1$. Випадок $\alpha = 1$ було розглянуто іншими авторами. Показано, що поведінка розв'язків у випадку сублінійного згасання, тобто для $0 < \alpha < 1$, повністю відрізняється від поведінки у випадку лінійного згасання $(\alpha = 1)$. Наведено достатні умови збіжності до нуля розв'язків, що не є коливними, а також достатні умови існування розв'язків, що не є коливними і не збігаються до нуля. Крім цього наведено достатні умови для того, щоб всі розв'язки були неколивними. Сформульовано кілька відкритих питань для подальшого дослідження.

1. Introduction. In this paper, we investigate the asymptotic behavior of the solutions of the damped nonautonomous nonlinear differential equation

$$x'' + a(t)|x'|^{\alpha}\mathrm{sgn}\,(x') + f(x) = 0, \tag{1.1}$$

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where $a : [0, \infty) \to [0, \infty)$ and $f : (-\infty, \infty) \to (-\infty, \infty)$ are continuous, $uf(u) > 0$ for $u \neq 0$, and α is a positive constant with $0 < \alpha \leq 1$. The literature on the attractivity behavior in the case of linear damping, i.e., for the equation

$$x'' + a(t)x' + f(x) = 0, \tag{1.2}$$

as well as for the related equation

$$x'' + q(t)f(x) = 0$$

is quite large since these are classical problems in the study of ordinary differential equations. There are a great number of generalizations and extensions to equations with different kinds of nonlinearities especially for $\alpha > 1$. Attractivity criteria in the case $\alpha > 1$ are often based on the fact that $|u|^\alpha < |u|$ for $\alpha > 1$ and $|u| < 1$. Such estimates do not hold if $0 < \alpha < 1$, and this partially explains why this case has not been as extensively studied.

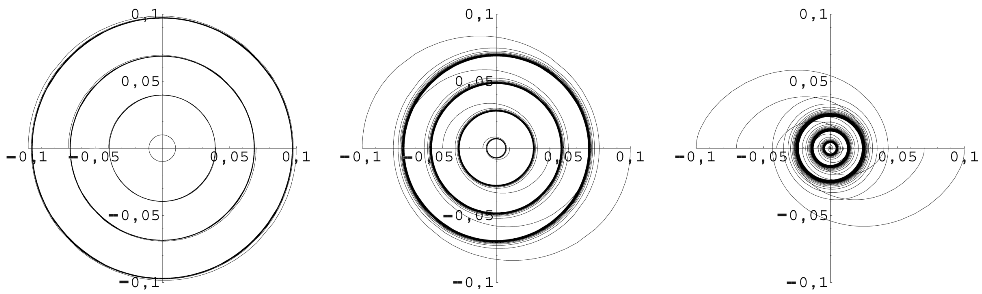
In [1, 2], the authors investigated autonomous equations with sublinear damping. As a special case of more general results, we obtained that every solution of (1.1) with $0 < \alpha < 1$ and $a(t) \equiv a$, i.e., the equation

$$x'' + a|x'|^\alpha \operatorname{sgn}(x') + f(x) = 0, \tag{1.3}$$

is nonoscillatory. On the other hand, it can easily be proved that for $\alpha > 1$, every solution of (1.3) is oscillatory. As we will see, if the equation is nonautonomous, the situation is more complicated and correspondingly more interesting.

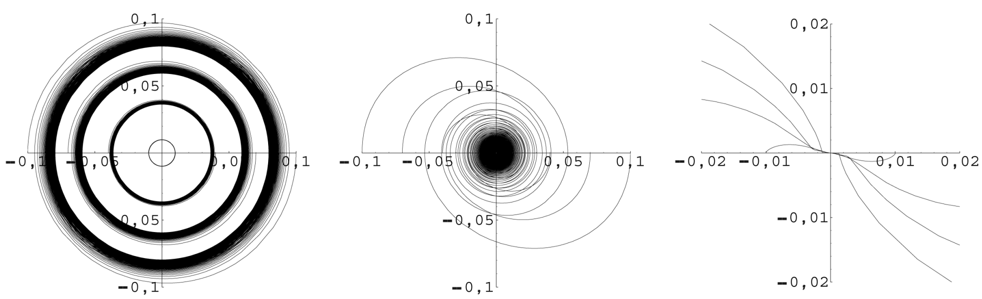
In the case of the nonlinear equation (1.1), the damping effect is a result of two factors, namely, the size of the coefficient $a(t)$ and the value of α. We will refer to the damping as being linear, sublinear, or superlinear depending on whether we have $\alpha = 1$, $0 < \alpha < 1$, or $\alpha > 1$, respectively. For equation (1.1) with a linear damping term, i.e., equation (1.2), it is known (see [3]; also see Lemma 1.1 below) that a solution either oscillates or is eventually monotonic. As a consequence, in this case, if the damping coefficient $a(t)$ is too small, then all solutions oscillate and there are some solutions that do not tend to zero as $t \to \infty$; this has been referred to as *under damping*. On the other hand, if $a(t)$ is too large, then all solutions are nonoscillatory and there are solutions that do not approach zero. This situation is known as *over damping*. Between these two extreme cases, we have what is called *small damping* if all solutions oscillate and tend to zero as $t \to \infty$, and *large damping* if the solutions are nonoscillatory and tend to zero as $t \to \infty$. Although this classification of damping phenomena was originally applied to the linear equation with linear damping, i.e., equation (1.2) with $f(x) = x$ (see [4]), it can also be applied to the nonlinear equation (1.1) as well. But it should be kept in mind that for equation (1.1), the total damping effect depends on both the size of $a(t)$ and the value of α.

To illustrate these various behaviors, consider the Figures 1 – 5 that represent the solutions of equation (1.1) in the case of under damping, small damping, constant damping, large damping, and over damping in the superlinear ($\alpha = 2$), linear ($\alpha = 1$), and sublinear ($\alpha = 0,5$) cases, respectively. All these simulations were produced taking $f(x) = x$.



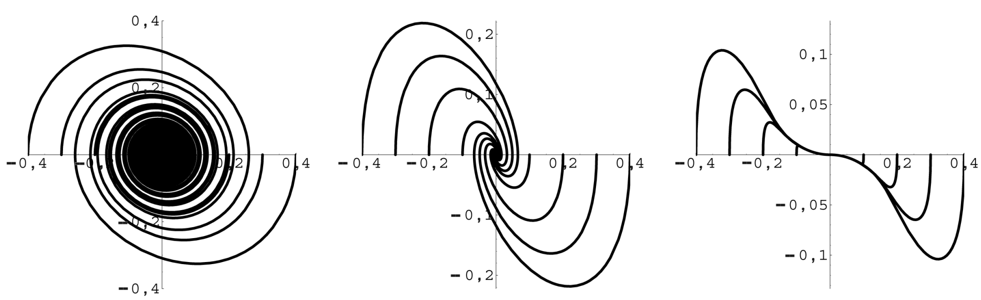
$\alpha = 2$ $\alpha = 1$ $\alpha = 0,5$

Fig. 1. Under damping: $x'' + \dfrac{1}{(1+t)^2}|x'|^{\alpha}\mathrm{sgn}\,(x') + x = 0.$



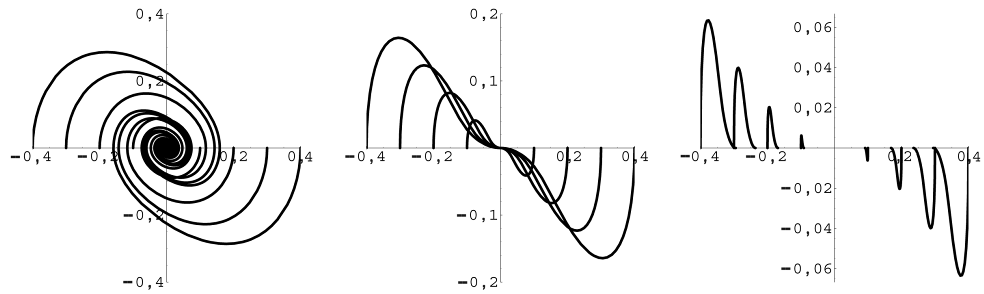
$\alpha = 2$ $\alpha = 1$ $\alpha = 0,5$

Fig. 2. Small damping: $x'' + \dfrac{1}{1+t}|x'|^{\alpha}\mathrm{sgn}\,(x') + x = 0.$



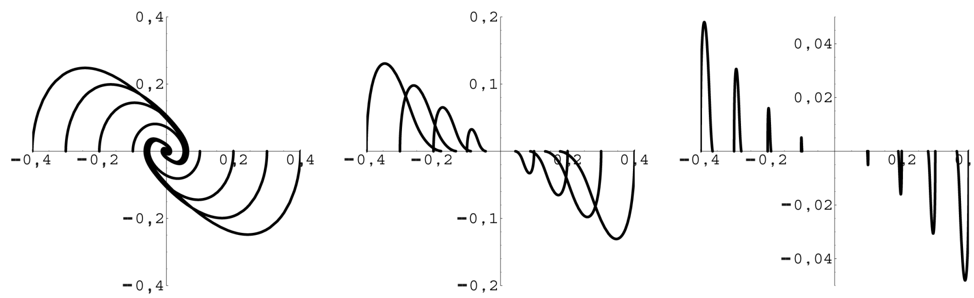
$\alpha = 2$ $\alpha = 1$ $\alpha = 0,5$

Fig. 3. Constant damping: $x'' + |x'|^{\alpha}\mathrm{sgn}\,(x') + x = 0.$



$\alpha = 2$ $\alpha = 1$ $\alpha = 0,5$

Fig. 4. Large damping: $x'' + (1+t)|x'|^{\alpha}\mathrm{sgn}\,(x') + x = 0$.



$\alpha = 2$ $\alpha = 1$ $\alpha = 0,5$

Fig. 5. Over damping: $x'' + (1+t)^2|x'|^{\alpha}\mathrm{sgn}\,(x') + x = 0$.

Our interest in this paper is an examination of the impact of the nonlinear damping term on the resulting large and over damping effects that result in equation (1.1). To further illustrate the interaction between $a(t)$ and α, consider the following examples. Observe that $x_1(t) = 1+\dfrac{1}{1+t}$ is an over damped solution of each of the equations

$$x'' + \frac{\left(4+5\,t+4\,t^2+t^3\right)}{1+t}\,x' + x = 0,$$

$$x'' + (1+t)^3\left(4+5\,t+4\,t^2+t^3\right)\,x'^3 + x = 0,$$

$$x'' + \frac{\left(4+5\,t+4\,t^2+t^3\right)}{(1+t)^{\frac{5}{3}}}\,\mathrm{sgn}\,(x')\,|x'|^{\frac{2}{3}} + x = 0,$$

$$x'' + \frac{\left(4+5\,t+4\,t^2+t^3\right)}{(1+t)^2}\mathrm{sgn}\,(x')\,\sqrt{|x'|} + x = 0,$$

and

$$x'' + \frac{\left(4+5\,t+4\,t^2+t^3\right)}{(1+t)^{\frac{7}{3}}}\,\mathrm{sgn}\,(x')\,|x'|^{\frac{1}{3}} + x = 0.$$

To see the large damping effect, observe that $x_2(t) = \dfrac{1}{1+t}$ is a solution of each of the equations

$$x'' + \frac{\left(3 + 2\,t + t^2\right)}{1+t}\, x' + x = 0,$$

$$x'' + (1+t)^3 \left(3 + 2\,t + t^2\right)\, x'^3 + x = 0,$$

$$x'' + \frac{\left(3 + 2\,t + t^2\right)}{(1+t)^{\frac{5}{3}}}\, \mathrm{sgn}\,(x')\, |x'|^{\frac{2}{3}} + x = 0,$$

$$x'' + \frac{\left(3 + 2\,t + t^2\right)}{(1+t)^2}\, \mathrm{sgn}\,(x')\, \sqrt{|x'|} + x = 0,$$

and

$$x'' + \frac{\left(3 + 2\,t + t^2\right)}{(1+t)^{\frac{7}{3}}}\, \mathrm{sgn}\,(x')\, |x'|^{\frac{1}{3}} + x = 0.$$

Notice that the smaller the value of α in the damping term $a(t)|x'|^{\alpha}\mathrm{sgn}\,(x')$, the smaller the “degree"of $a(t)$ that is needed to result in an over damping effect. In addition, while in the case of linear damping, nonoscillatory solutions do not exist for $0 < a(t) \leq a_0$, where a_0 is small enough (e.g., $a(t) < 2$ for $f(x) = x$), the last example above shows that there can exist monotone solutions in the sublinear damping case even if $\lim_{t\to\infty} a(t) = 0$. On the other hand, Figures 4 and 5 show that solutions can oscillate in the case of superlinear damping with $\lim_{t\to\infty} a(t) = \infty$, while in the linear and sublinear cases, over damped behavior is observed.

The following type of lemma has proved to be fundamental in the study of these kinds of problems.

Lemma 1.1. *Let $x(t)$ be a solution of (1.1) with $x'(t_1) = x'(t_2) = 0$ and $x'(t) \neq 0$ for $t \in (t_1, t_2)$. Then, there exists $\tilde{t} \in (t_1, t_2)$ such that $x(\tilde{t}) = 0$. Consequently, every solution of (1.1) is either oscillatory or eventually monotonic.*

The proof is analogous to the proof of Lemma 1 in [5], and so in the interest of space, we omit the details. The next lemma is independent of the value of $\alpha > 0$ and guarantees that nonoscillatory solutions have to eventually decrease in absolute value. A proof of this lemma can be found in [6] where equation (1.1) with $\alpha > 1$ is considered.

Lemma 1.2. *Suppose that f is nondecreasing. Let $x(t)$ be a solution of (1.1) such that $x(t_1) \neq$ $\neq 0$ and $x(t)x'(t) \geq 0$ for all $t \in (t_1, t_2)$. Then $t_2 - t_1 < |x'(t_1)/f(x(t_1))|$.*

At times, we will make use of the energy function

$$V(t) = \frac{x'^2(t)}{2} + F(x(t)), \quad \text{where} \quad F(x) := \int h_0^x f(u)\, du. \tag{1.4}$$

It is easy to see that along solutions of (1.1), we have

$$V'(t) = -a(t)|x'(t)|^{\alpha+1} \leq 0 \tag{1.5}$$

so that the energy along solutions is nonincreasing. Our results in the remainder of this paper will require that the function f be nondecreasing. This ensures that $\lim_{x\to\infty} F(x) = \infty$ and guarantees that every solution is continuable and is bounded on $[0,\infty)$.

The remainder of the paper is devoted primarily to the study of equation (1.1) with a sublinear damping term, i.e., with $0 < \alpha < 1$. The next section contains our results on the asymptotic behavior of nonoscillatory solutions of equation (1.1). We give sufficient conditions for all nonoscillatory solutions to converge to zero and sufficient conditions for the existence of a nonoscillatory solution that does not converge to zero. In the last section, we give sufficient conditions for all solutions to be nonoscillatory.

2. Properties of nonoscillatory solutions. Necessary and sufficient conditions to guarantee that every nonoscillatory solution of (1.1) with a linear damping term ($\alpha = 1$) tends to zero were given by Smith [4].

Theorem 2.1 ([4], Theorem 1). *Assume that $a(t) \geq a_0 > 0$. Then every nonoscillatory solution of the equation*

$$x'' + a(t)x' + x = 0 \tag{2.1}$$

tends to zero as $t \to \infty$ if and only if

$$\int_0^\infty e^{-H(s)} \int_0^s e^{H(u)} du ds = \infty, \tag{2.2}$$

where $H(t) = \int_0^t a(s)ds$.

The proof of this theorem is based on the fact that by integrating equation (2.1), the derivative of a solution can be written in the form

$$x'(t) = \exp\left[-\int_T^t a(s)ds\right]\left(x'(T) - \int_0^t x(s)\,\exp\left[\int_0^s a(u)du\right]\,ds\right).$$

For each nonoscillatory solution of (1.1), we can also obtain a similar expression. Let the solution $x(t)$ be monotonic and satisfy $x(t)x'(t) \leq 0$ on the interval $[T,\infty)$. Then, equation (1.1) can be written in the form

$$x''(t) + a(t)|x'(t)|^{\alpha-1}x'(t) + f(x(t)) = 0.$$

Observe that the function $|u|^{\alpha-1}$ $(u > 0)$ is increasing for $\alpha > 1$, and is decreasing and has a singularity at $u = 0$ if $0 < \alpha < 1$. Integrating this equation, we obtain

$$x'(t) = \exp\left[-\int_T^t a(s)|x'(s)|^{\alpha-1}ds\right] \times$$

$$\times \left(x'(T) - \int_T^t f(x(s))\,\exp\left[\int_T^s a(u)|x'(u)|^{\alpha-1}du\right]\,ds\right). \tag{2.3}$$

This expression is the basis for generalizations of Smith's result to more general equations, such as

$$x'' + g(t, x, x')x' + f(x) = 0;$$

see, for example [3, 5, 7–11].

By deriving appropriate estimates for $x'(t)$, we will be able to obtain results for the case of sublinear damping.

Theorem 2.2. *Assume that* $0 < \alpha \le 1$, f *is nondecreasing, and*

$$\int_0^\infty \exp\left[-\int_0^\tau L\frac{a}{B^{\frac{1-\alpha}{\alpha}}}\right]\int_0^\tau \exp\left[\int_0^s L\frac{a}{B^{\frac{1-\alpha}{\alpha}}}\right] ds\, d\tau = \infty \tag{2.4}$$

for every constant $L > 0$, *where*

$$B(t) := \exp\left[-\int_0^t a(s)ds\right]\int_0^t \exp\left[\int_0^s a(u)du\right] ds.$$

Then $\lim_{t\to\infty} x(t) = 0$ *for every nonoscillatory solution* $x(t)$ *of (1.1).*

Proof. To the contrary, assume that $x(t)$ is solution of (1.1) that is nonoscillatory on some interval $[T_1, \infty)$, $T_1 \ge 0$, and that $x(t)$ does not tend to zero as $t \to \infty$. In view of Lemmas 1.1 and 1.2, we can assume that $x(t) > 0$ and $x'(t) < 0$ on $[T_1, \infty)$. Moreover, since $x(t)$ is bounded, it must be the case that $x'(t) \to 0$ as $t \to \infty$, so choose $T \ge T_1$ such that $|x'(t)| < 1$ for $t > T$. Letting $\delta = \inf_{t\ge T} f(x(t))$ and replacing T in (2.3) by $2T$, we have

$$x'(t) \le -\delta \exp\left[-\int_{2T}^t a(s)|x'(s)|^{\alpha-1}ds\right]\int_{2T}^t \exp\left[\int_{2T}^s a(u)|x'(u)|^{\alpha-1}du\right] ds =$$

$$= -\delta\int_{2T}^t \exp\left[-\int_s^t a(u)|x'(u)|^{\alpha-1}du\right] ds.$$

An integration from $2T$ to t yields

$$0 < x(t) \le x(2T) - \delta\int_{2T}^t\int_{2T}^\tau \exp\left[-\int_s^\tau a(u)|x'(u)|^{\alpha-1}du\right] ds\, d\tau. \tag{2.5}$$

We wish to obtain a lower estimate for the integral on the right-hand side of the above inequality so that condition (2.4) will imply that the right-hand side of (2.5) tends to $-\infty$ as $t \to \infty$. This will contradict the fact that $x(t)$ is positive and complete the proof of the theorem.

To obtain this lower bound on the above integral, we must derive an upper estimate for $a(t)|x'|^{\alpha-1}(t)$ on $[T, t)$. Now since $0 < \alpha \le 1$, this in turn means we need a lower estimate for

$|x'(t)|$. Let $y(t) = |x'(t)| = -x'(t)$; then from (1.1), we obtain

$$y(t) = y(T) - \int_T^t a(s)y^{\alpha}(s)\,ds + \int_T^t f(x(s))\,ds,$$

and since $|y(t)| < 1$ and $0 < \alpha \leq 1$, this implies

$$z(t) = y^{\alpha}(t) \geq y(T) - \int_T^t a(s)z(s)\,ds + \delta(t-T).$$

By a standard differential inequalities argument, we have

$$z(t) \geq \exp\left[-\int_T^t a(s)ds\right]\left(y(T) + \delta\int_T^t \exp\left[\int_T^s a(u)du\right]\,ds\right) \geq$$

$$\geq \delta\exp\left[-\int_T^t a(s)ds\right]\int_T^t\left[\exp\int_T^s a(u)du\right]\,ds = \delta B(t). \qquad (2.6)$$

Thus,

$$a(t)|x'(t)|^{\alpha-1} \leq a(t)\left(\frac{1}{\delta B(t)}\right)^{\frac{1-\alpha}{\alpha}} = k\frac{a(t)}{B(t)^{\frac{1-\alpha}{\alpha}}}$$

for $t > T$, where $k = \delta^{(\alpha-1)/\alpha}$. It follows that

$$\int_{2T}^t\int_{2T}^{\tau}\exp\left[-\int_s^{\tau} a(u)|x'(u)|^{\alpha-1}du\right]\,ds\,d\tau \geq$$

$$\geq \int_{2T}^t\int_{2T}^{\tau}\exp\left[-k\int_s^{\tau} k\frac{a(u)}{B(u)^{\frac{1-\alpha}{\alpha}}}du\right]\,ds\,d\tau.$$

Note that taking the integral on $[2T,\infty)$, we avoided the singularity of $1/B(t)$ at $t = T$. This completes the proof of the theorem.

The following result is the counterpart to Theorem 2.2; it gives conditions for the existence of a nonoscillatory solution that does not tend to zero.

Theorem 2.3. *Assume that* $0 < \alpha \leq 1$, $0 < a_0 \leq a(t)$, *and* f *is nondecreasing. If*

$$\int_0^{\infty}\exp\left[-\int_0^{\tau} L\frac{a}{B^{1-\alpha}}\right]\int_0^{\tau}\exp\left[\int_0^s L\frac{a}{B^{1-\alpha}}\right]\,ds\,d\tau < \infty \qquad (2.7)$$

for every constant $L > 0$, where $B(t)$ is defined in Theorem 2.2, then there exists a nonoscillatory solution of (1.1) such that $\lim_{t\to\infty} x(t) \neq 0$.

Proof. We are going to construct a solution of (1.1) that does not tend to zero. Consider the solution with the initial conditions $x(T) > 0,\ F(x(T)) = 1/2$ and $x'(T) = 0$, where T will be chosen later. Assuming that the statement is not true, this solution is either oscillatory, or it is monotone decreasing on $[T, \infty)$ and $\lim_{t\to\infty} x(t) = 0$. Let $K := \sup_{t\geq T} |f(x(t))| < \infty$; note that the value of K depends only on the initial conditions since the energy function defined in (1.4) is nonincreasing. Let $[T, \bar{T})$ be the maximal interval to the right of T on which $x(t) > 0$. Note that we may have $\bar{T} = \infty$. Now, $x'(t)$ is negative on some interval to the right of T, so choose $T_1 > T$ such that $x(t) > x(T)/2$ for $t \in [T, T_1]$. Taking T_1 in (2.3) in place of T, for $t \in [T_1, \bar{T})$, we then have

$$x'(t) = \exp\left[-\int_{T_1}^{t} a(s)|x'(s)|^{\alpha-1} ds\right] \times$$

$$\times \left(x'(T_1) - \int_{T_1}^{t} f(x(s)) \exp\left[\int_{T_1}^{s} a(u)|x'(u)|^{\alpha-1} du\right] ds\right). \tag{2.8}$$

Next observe that $y^2(t) = [x'(t)]^2 \leq 2V(t) \leq 2V(T) = 2F(x(T)) = 1$, so $y(t) \leq 1$ for $t \geq T$. Let $z(t) = |x'(t)| = -x'(t)$; then from (1.1), we have

$$z' = -a(t)z^{\alpha}(t) + f(x(t)) \leq -a(t)z(t) + K \leq -a(t)z(t) + K,$$

from which it follows that

$$z(t) \leq K \exp\left[-\int_{T}^{t} a(s)ds\right] \int_{T}^{t} \exp\left[\int_{T}^{s} a(u)du\right] ds \leq KB(t).$$

We then have

$$a(t)|x'|^{\alpha-1}(t) \geq \frac{a(t)}{z^{1-\alpha}(t)} \geq K_1 \frac{a(t)}{B^{1-\alpha}(t)}$$

for $t \geq T$, where $K_1 = K^{\alpha-1}$. We can then bound the first term in (2.8) as follows:

$$x'(T_1) \exp\left[-\int_{T_1}^{t} a(s)|x'(s)|^{\alpha-1} ds\right] \geq -|x'(T_1)| \exp\left[-\int_{T_1}^{t} K_1 \frac{a(s)}{B^{1-\alpha}(s)} ds\right].$$

For the second term in (2.8), we have

$$-\exp\left[-\int_{T_1}^{t} a(s)|x'(s)|^{\alpha-1}ds\right]\int_{T_1}^{t} f(x(s))\exp\left[\int_{T_1}^{s} a(u)|x'(u)|^{\alpha-1}du\right]ds \geq$$

$$\geq -\int_{T_1}^{t} f(x(s))\exp\left[-\int_{s}^{t} a(u)|x'(u)|^{\alpha-1}du\right]ds \geq$$

$$\geq -\int_{T_1}^{t} f(x(s))\exp\left[-\int_{s}^{t} K_1\frac{a(u)}{B^{1-\alpha}(u)}du\right]ds \geq$$

$$\geq -K\int_{T_1}^{t} \exp\left[-\int_{s}^{t} K_1\frac{a(u)}{B^{1-\alpha}(u)}du\right]ds.$$

Applying the above estimates to (2.8) and then integrating from T_1 to t, we obtain

$$x(t) \geq x(T_1) - |x'(T_1)|\int_{T_1}^{t} \exp\left[-\int_{T_1}^{s} K_1\frac{a(u)}{B^{1-\alpha}(u)}du\right]ds-$$

$$-K\int_{T_1}^{t}\int_{T_1}^{s} \exp\left[-\int_{u}^{s} K_1\frac{a(w)}{B^{1-\alpha}(w)}dw\right]du\,ds. \tag{2.9}$$

Now l'Hôpital's rule shows that $1/B(t)$ is bounded below away from zero, and since $a(t)$ is as well, we see that the first integral in (2.9) converges. Hence, T can be chosen such that

$$|x'(T_1)|\int_{T_1}^{t} \exp\left[-\int_{T_1}^{s} K_1\frac{a(u)}{B^{1-\alpha}(u)}du\right]ds < F^{-1}(\tfrac{1}{2})/8 = x(T)/8.$$

The last term in (2.9) can be estimated from above by

$$\int_{T_1}^{t}\int_{T_1}^{s} \exp\left[-\int_{u}^{s} K_1\frac{a(w)}{B^{1-\alpha}(w)}dw\right]du\,ds \leq$$

$$\leq K\int_{T_1}^{t}\int_{T_1}^{\tau} \exp\left[-\int_{s}^{\tau} K_1\frac{a(u)}{B^{1-\alpha}(u)}du\right]ds\,d\tau.$$

The integral on the right is equivalent to (2.7) and so it converges; for sufficiently large T, it can be bounded above by $x(T)/8 = F^{-1}\Big(\frac{1}{2}\Big)/8$, where F^{-1} is the inverse function of F on $[0, \infty)$. Thus, we are able to choose $T > 0$ so that (2.9) yields

$$x(t) \geq x(T_1) - x(T)/8 - x(T)/8 = x(T)/4 = F^{-1}\left(\frac{1}{2}\right)/4,$$

and we see that the solution $x(t)$ does not tend to zero.

Remark 2.1. Notice that if $\alpha = 1$, then Theorems 2.2 and 2.3 taken together become equivalent to Theorem 2.1 of Smith above.

Applying our results above can be difficult due to complicated structure of the expression

$$\exp\left[-\int_0^t h(\tau)d\tau\right]\int_0^t \exp\left[\int_0^\tau h(s)ds\right]d\tau$$

which appears nested within both conditions (2.4) and (2.7). The following two lemmas will aid in formulating some criteria that are somewhat easier to apply than Theorems 2.2 and 2.3.

Lemma 2.1. *Assume that* $h(t) > 0$ *is nondecreasing. Then*

$$\lim_{t\to\infty} h(t)\,\exp\left[-\int_0^t h(\tau)\,d\tau\right]\int_0^t \exp\left[\int_0^\tau h(s)\,ds\right]d\tau \geq 1.$$

The proof of the above lemma as well as the following lemma can be found in [6].

Lemma 2.2. *Assume that* $\liminf_{t\to\infty} h(t) > 0$ *and* $\lim_{t\to\infty} \dfrac{h'(t)}{h(t)} = 0$*. Then*

$$\lim_{t\to\infty} h(t)\,\exp\left[-\int_0^t h(\tau)\,d\tau\right]\int_0^t \exp\left[\int_0^\tau h(s)\,ds\right]d\tau = 1.$$

Now assume that $a(t) \leq \bar{a}(t)$, where $\bar{a}(t)$ is nondecreasing. From Lemma 2.1, we have

$$B(t) = \exp\left[-\int\limits_0^t a(s)ds\right]\int\limits_0^t \exp\left[\int\limits_0^s a(u)du\right]\,ds \geq$$

$$\geq \exp\left[-\int\limits_0^t \bar{a}(s)ds\right]\int\limits_0^t \exp\left[\int\limits_0^s \bar{a}(u)du\right]\,ds \geq$$

$$\geq \exp\left[-\int\limits_0^t \bar{a}(s)ds\right]\int\limits_0^t \frac{\bar{a}(s)}{\bar{a}(t)} \exp\left[\int\limits_0^s \bar{a}(u)du\right]\,ds =$$

$$= \left(1 - \exp\left[-\int\limits_0^t \bar{a}(s)ds\right]\right)\Big/\,\bar{a}(t) \geq k/\bar{a}(t),$$

where the last inequality holds for $t \geq T$ for some $k > 0$ and some sufficiently large $T > 0$. Thus,

$$a(t)/B^{\frac{1-\alpha}{\alpha}}(t) \leq k_1 \bar{a}(t)^{1/\alpha}$$

for $t \geq T$, where $k_1 = k^{\frac{\alpha-1}{\alpha}}$. Repeating the above argument, we obtain the following corollary to Theorem 2.2.

Corollary 2.1. *Assume that* $0 < \alpha \leq 1$, *f is nondecreasing,* $0 \leq a(t) < \bar{a}(t)$, *$\bar{a}(t)$ is nondecreasing, and*

$$\int\limits_0^\infty \frac{1}{\bar{a}^{1/\alpha}(t)}\,dt = \infty. \tag{2.10}$$

Then $\lim_{t\to\infty} x(t) = 0$ *for every nonoscillatory solution $x(t)$ of (1.1).*

Similar arguments, together with an application of Lemma 2.2 using a lower estimate on $a(t)$, yield an upper estimate for the integral in (2.7). As a consequence, we have the following corollary to Theorem 2.3.

Corollary 2.2. *Assume that* $0 < \alpha \leq 1$, *f is nondecreasing,* $0 < a_0 \leq \underline{a}(t) \leq a(t)$, *$\underline{a}(t)$ is differentiable,* $\lim_{t\to\infty} \underline{a}'(t)/\underline{a}(t) = 0$, *and*

$$\int\limits_0^\infty \frac{1}{\underline{a}^{2-\alpha}(t)}\,dt < \infty. \tag{2.11}$$

Then there exists a nonoscillatory solution $x(t)$ of (1.1) such that $\lim_{t\to\infty} x(t) = 0$.

Proof. First note that an upper estimate for the integral $B(t)$ can be obtained by replacing $a(t)$ by $\underline{a}(t)$. Hence, from Lemma 2.2 with $h(t) \equiv \underline{a}(t)$, we have

$$B(t) = \exp\left[-\int_0^t a(s)ds\right]\int_0^t \exp\left[\int_0^s a(u)du\right] ds \leq$$

$$\leq \exp\left[-\int_0^t \underline{a}(s)ds\right]\int_0^t \exp\left[\int_0^s \underline{a}(u)du\right] ds \leq \frac{K}{\underline{a}(t)}$$

for some $K > 0$ and $t \geq t_1$ with t_1 sufficiently large. This implies that the function $a(t)/B^{1-\alpha}(t)$ in condition (2.7) can be estimated from below by

$$\frac{a(t)}{B^{1-\alpha}(t)} \geq \frac{\underline{a}^{2-\alpha}(t)}{K^{1-\alpha}}.$$

If we again apply Lemma 2.2 this time with $h(t) \equiv \underline{a}^{2-\alpha}(t)$, we see that there exists a number K_1 such that

$$\int_0^\infty \exp\left[-\int_0^\tau L\frac{a}{B^{1-\alpha}}\right]\int_0^\tau \exp\left[\int_0^s L\frac{a}{B^{1-\alpha}}\right] ds\, d\tau \leq K_1\int_0^\infty \frac{1}{\underline{a}^{2-\alpha}(\tau)}d\tau,$$

and this completes the proof of the corollary.

Applying Corollaries 2.1 and 2.2 to the case $c_1t^\gamma \leq a(t) \leq c_2t^\sigma$ with $0 < \gamma \leq \sigma$ and $c_1 < c_2$, we obtain that if $\sigma \leq \alpha$, then every nonoscillatory solution of (1.1) tends to zero as $t \to \infty$, and if $\gamma(2-\alpha) > 1$, then there exists a nonoscillatory solution that does not tend to zero. These results verify our preliminary observations that over damping appears for much smaller functions $a(t)$ in the case of sublinear damping than it does for the case of linear damping.

Remark 2.2. Lemmas 2.1 and 2.2 show that conditions (2.4) and (2.7) are essentially equivalent to (2.10) and (2.11) above, respectively, in case $a(t)$ is nondecreasing. It remains an open question as to whether these conditions can be improved.

Remark 2.3. In Lemma 1.2, and consequently in Theorems 2.2 and 2.3 as well as Corollaries 2.1 and 2.2, the condition that f be nondecreasing can be replaced by

$f(u)$ is bounded away from zero if u is bounded away from zero.

3. Sufficient conditions for nonoscillation. Although Lemma 1.1 allows for the existence of both oscillatory and nonoscillatory solutions, our preliminary examples with $f(x) = x$ and our results in [1, 2] suggest that, under some mild additional conditions, all solutions of (1.1) may in fact be nonoscillatory. The following theorem holds.

Theorem 3.1. *Assume that $a(t) \geq a_0 > 0$ and $\lim_{(x,y)\to(0,0)} f'(x)|y|^{2(1-\alpha)} = 0$ on the set $\{(x,y) : Cf(x) = |y|^\alpha \operatorname{sgn}(y)\}$ for some $C > 1/a_0$. Then every solution $x(t)$ of (1.1) is nonoscillatory and*

$$\limsup_{t\to\infty}\left|\frac{[x'(t)]^\alpha}{f(x(t))}\right| \leq \frac{1}{a_0}.$$

For the proof of Theorem 3.1, we need the following lemma that is somewhat interesting in its own right; its conclusion is independent of the form of the nonlinearity $f(x)$ as well as the value of $\alpha > 0$.

Lemma 3.1. *If* $a(t) \geq a_0 > 0$*, then* $\lim_{t\to\infty} V(t) = 0$ *for every oscillatory solution of (1.1).*

Proof. Let $x(t)$ be an oscillatory solution of (1.1) such that $\lim_{t\to\infty} V(t) = \varepsilon > 0$. Then, there exists an infinite sequence of intervals $[t_n, s_n]$ such that $t_n < s_n < t_{n+1}$, $x(t_n) = 0$, $F(x(s_n)) = \varepsilon/2$, and $x(t) > 0$ and $x'(t) > 0$ on $[t_n, s_n]$ for $n = 1, 2, \ldots$. From the definition of $V(t)$, we obtain that $\varepsilon/2 \leq [x'(t)]^2 \leq 2K$ on the intervals $[t_n, s_n]$, where $K = V(t_1)$. Hence,

$$x(s_n) = F^{-1}(\varepsilon/2) < \sqrt{2K}\,(s_n - t_n).$$

Now, $[x'(t)]^2 \geq \varepsilon/2$ on $[t_n, s_n]$ implies $|x'(t)|^{\alpha+1} \geq (\varepsilon/2)^{(\alpha+1)/2}$, so an integration of $V'(t)$ yields

$$V(t) = V(t_1) - \int_{t_1}^{t} a(u)|x'(u)|^{\alpha+1} du \leq V(t_1) - \sum_{s_n < t} \int_{t_n}^{s_n} a(u)|x'(u)|^{\alpha+1} du \leq$$

$$\leq V(t_1) - \sum_{s_n < t} a_0 (\varepsilon/2)^{(\alpha+1)/2}(s_n - t_n) \to -\infty$$

as $t \to \infty$. This contradicts the nonnegativity of $V(t)$ and completes the proof of the lemma.

Proof of Theorem 3.1. Assume that there exists a nontrivial oscillatory solution $x(t)$ of (1.1). By Lemma 3.1, $\lim_{t\to\infty} V(t) = 0$ along this solution. Let $\varepsilon > 0$ be given and choose $T \geq 0$ such that $V(t) \leq \varepsilon$ for $t \geq T$. The trajectory $(x(t), y(t)) = (x(t), x'(t))$ passes infinitely many times through the section defined by

$$\Gamma_C = \{(x, y) : x \geq 0,\ y \leq 0,\ y^2/2 + F(x) \leq \varepsilon,\ Cf(x) \geq |y|^\alpha\}.$$

We will show that if ε is small enough, the trajectory cannot leave this region. This means that the solution is nonoscillatory, which contradicts our assumption.

Clearly, Γ_C is closed and is bounded by the curves $G_1 = \{(x, y) : y = 0\}$, $G_2 = \{(x, y) : y^2/2 + F(x) = \varepsilon\}$, and $G_3 = \{(x, y) : Cf(x) = |y|^\alpha\}$. Since the tangent vector at $y = 0$ is $(0, -f(x))$, the trajectory enters Γ_C through G_1. The energy is nonincreasing along $x(t)$, so the trajectory cannot leave Γ_C through G_2. Finally, consider the curve $Cf(x) = |y|^\alpha$. Let the numbers $s_n,\ n = 1, 2, \ldots$, be given by $T < s_1 < \ldots < s_n < s_{n+1}$, $Cf(x(s_n)) = |y(s_n)|^\alpha$, $x(s_n) > 0$, and $y(s_n) < 0$. The tangent vector to the trajectory is $(y(s_n),\, a(s_n)|y(s_n)|^\alpha - -f(x(s_n))$, and the normal vector to the curve G_3 is $(-Cf'(x(s_n))/\alpha|y(s_n)|^{\alpha-1}, -1)$. Using the notation $x(s_n) = x_n$, $y(s_n) = y_n$, and $a(s_n) = a_n$, their scalar product is

$$-C\frac{f'(x_n)}{\alpha|y_n|^{\alpha-1}} y_n - a_n|y_n|^\alpha + f(x_n) = Cf(x_n)\left(\frac{C}{\alpha} f'(x_n)|y_n|^{2(1-\alpha)} - a_n + \frac{1}{C}\right). \tag{3.1}$$

Since, $C > 1/a_0$ and $\lim_{n\to\infty} f'(x_n)|y_n|^{2(1-\alpha)} = 0$, it follows that T can be chosen large enough so that (3.1) is negative, i.e., the tangent vector to the trajectory at every intersection point

$(x(s_n), y(s_n))$ is directed towards the interior of Γ_C. This shows the monotonicity of the solutions. Since the trajectories are trapped in the region Γ_C, the second part of the theorem is also proved.

Observe that the hypotheses of the above theorem do not explicitely use the assumption $0 < \alpha < 1$; the nonlinearity of $f(x)$ is also involved (see the autonomous case in [1, 2]. We easily can see that if $f(x) = |x|^\beta \mathrm{sgn}\,(x)$, the continuity condition takes the simple form $\beta > \alpha/(2-\alpha)$, which clearly holds if, for example, $0 < \alpha < 1$ and $\beta \geq 1$.

Finally, note that Fig. 2 and the examples in the introduction show that for $0 < \alpha < 1$, there can exist nonoscillatory solutions of equation (1.1) with $a(t)$ not bounded away from zero, for example, if $a(t) = 1/t^\omega$ with $\omega > 0$. The extension of our nonoscillation result to this case remains an open problem.

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