UDC 517.9

OSCILLATION CRITERIA FOR DIFFERENCE EQUATIONS WITH SEVERAL RETARDED ARGUMENTS

КРИТЕРІЇ КОЛИВАНЬ ДЛЯ РІЗНИЦЕВИХ РІВНЯНЬ З ДЕЯКИМИ ЗАГАЮВАЛЬНИМИ АРГУМЕНТАМИ

R. Koplatadze, N. Khachidze

I. Javakhishvili Tbilisi State Univ.
University Str., 2, Tbilisi, 0186, Georgia
e-mail: r-koplatadze@yahoo.com
natiaa.xachidze@gmail.com

The difference equation

$$\Delta u(k) + \sum_{i=1}^{m} p_i(k)\, u(\tau_i(k)) = 0,$$

is considered, where $m \in N$, the functions, $p_i \colon N \to R_+,\ \tau_i \colon N \to N,\ \lim_{k\to+\infty} \tau_i(k) = +\infty,\ \tau_i(k) \le k-1,\ i = 1, \ldots, m,$ are defined on a set of natural numbers, and $\Delta u(k) = u(k+1) - u(k)$ is the difference operator. New oscillation criteria for all solutions of the equation are established.

Розглянуто різницеве рівняння

$$\Delta u(k) + \sum_{i=1}^{m} p_i(k)\, u(\tau_i(k)) = 0,$$

де $m \in N$, функції $p_i \colon N \to R_+,\ \tau_i \colon N \to N,\ \lim_{k\to+\infty} \tau_i(k) = +\infty,\ \tau_i(k) \le k-1,\ i = 1, \ldots, m,$ задані на множині натуральних чисел, а різницевий оператор має вигляд $\Delta u(k) = u(k+1) - u(k)$. Знайдено нові критерії коливань для всіх розв'язків цього рівняння.

1. Introduction. Consider the difference equation

$$\Delta u(k) + \sum_{i=1}^{m} p_i(k)\, u\big(\tau_i(k)\big) = 0, \tag{1.1}$$

where $m \ge 1$ is a natural number, $p_i \colon N \to R_+,\ \tau_i \colon N \to N$ are functions defined on the set $N = \{1, 2, \ldots\}$ and $\Delta u(k) = u(k+1) - u(k)$. Everywhere below it is assumed that

$$\lim_{k\to+\infty} \tau_i(k) = +\infty \qquad \text{and} \qquad \tau_i(k) \le k-1, \quad i = 1, \ldots, m. \tag{1.2}$$

For each $n \in N$ denote $N_n = \{n, n+1, \ldots\}$.

Definition 1.1. *Let $n \in N$. We will call a function $u \colon N \to R$ a proper solution of equation (1.1) on the set N_n, if it satisfies (1.1) on N_n and $\sup\{|u(i)| \colon i \ge k\} > 0$ for any $k \in N_n$.*

Definition 1.2. *We say that a proper solution $u \colon N_n \to R$ of equation (1.1) is oscillatory, if for any $k \in N_n$ there exist $n_1, n_2 \in N_k$ such that $u(n_1)u(n_2) \le 0$. Otherwise the solution is called nonoscillatory.*

Definition 1.3. *Equation (1.1) is said to be oscillatory, if any of its the proper solution is oscillatory.*

The problem of oscillation of solution of linear difference equation (1.1) for $m = 1$ has been studied by several authors, see [1–3] and reference therein. As to investigation of the analogous problem for equation (1.1) and for differential equations see [4–10].

Some results of this paper without proof is given in [11].

Remark 1.1. Below we assume that

$$\limsup_{k\to+\infty} p_i(k) < +\infty, \quad i = 1, \ldots, m,$$

otherwise equation (1.1) is oscillatory.

2. Main results. Throughout the paper we assume that

$$\tau_*(k) = \min\{\tau_i(k) \colon i = 1, \ldots, m\},$$

$$\eta^{\tau_*}(k) = \max\{s \colon s \in N, \tau_*(s) \le k\}, \tag{2.1}$$

$$\eta_1^{\tau_*}(k) = \eta^{\tau_*}(k), \qquad \eta_i^{\tau_*}(k) = \eta_1^{\tau_*} \circ \eta_{i-1}^{\tau_*} \qquad \text{for} \quad k \in N, \quad i = 2, 3, \ldots.$$

By (1.2) it is obvious that $\eta_i^{\tau_*}(k) < +\infty$ for any $k \in N$ and $\eta_i^{\tau_*}(k) \uparrow +\infty$ for $k \uparrow +\infty$.

Let $k \in N$. Denote

$$\psi_1(k) = \prod_{i=1}^{m} \prod_{j=\tau_i(k)}^{k} \left(1 + mp^{1/m}(j)\right) \quad \text{for} \quad k \ge k_0, \tag{2.2}$$

$$\psi_s(k) = \left(\prod_{i=1}^{m} \prod_{j=\tau_i(k)}^{k} \left(1 + mp^{1/m}(s)\psi_{s-1}(j)\right)\right)^{1/m} \quad \text{for} \quad k \ge \eta_s^{\tau_*}(k_0), \quad s = 2, 3, \ldots, \tag{2.3}$$

$$p(j) = \prod_{i=1}^{m} p_i(j). \tag{2.4}$$

Theorem 2.1. *Let there exist $s_0 \in N$ and non-decreasing functions $\sigma_i \colon N \to N$, $i = 1, \ldots, m$, such that*

$$\tau_i(k) + 1 \le \sigma_i(k) \le k \quad \textit{for} \quad k \in N, \quad i = 1, \ldots, m, \tag{2.5}$$

and

$$\limsup_{k\to+\infty} \prod_{\ell=1}^{m} \left(\prod_{i=1}^{m} \sum_{s=\sigma_\ell(k)}^{k} p_i(s) \prod_{j=\tau_i(s)}^{\sigma_i(k)-1} \left(1 + m\, p^{1/m}(j)\, \psi_{s_0}^{1/m}(j)\right)\right)^{1/m} > \frac{1}{m^m}. \tag{2.6}$$

Then equation (1.1) is oscillatory, where ψ_{s_0} is given by (2.2), (2.3) and (2.4), when $s = s_0$.

Corollary 2.1. *Let there exist non-decreasing functions $\sigma_i \colon N \to N$, $i = 1, \ldots, m$, such that*

$$\limsup_{k\to+\infty} \prod_{\ell=1}^{m} \left(\prod_{i=1}^{m} \sum_{s=\sigma_\ell(k)}^{k} p_i(s) \prod_{j=\tau_i(s)}^{\sigma_i(k)-1} \left(1 + m\, p^{1/m}(j)\right)\right)^{1/m} > \frac{1}{m^m}. \tag{2.7}$$

Then equation (1.1) is oscillatory, where p is given by (2.4).

Theorem 2.2. *Let (2.5) be fulfilled,*

$$\liminf_{j\to+\infty} p(j) > 0 \tag{2.8}$$

and

$$\min\left\{\frac{\liminf\limits_{k\to+\infty}\prod_{i=1}^{m}\prod_{j=\tau_i(k)}^{k}\Big(1+m\alpha^{1/m}\,p^{1/m}(j)\Big)}{\alpha} : \alpha\ge 1\right\} > 1. \tag{2.9}$$

Then equation (1.1) is oscillatory, where p *is given by (2.4).*

Corollary 2.2. *Let for large* $j\in N$

$$p_i(j)\ge p_i>0,\quad i=1,\dots,m, \tag{2.10}$$

and

$$\overline{p}\,\liminf_{k\to+\infty}\frac{\Big(m+\sum_{i=1}^{m}(k-\tau_i(k))\Big)^{m+\sum_{i=1}^{m}(k-\tau_i(k))}}{\Big(\sum_{i=1}^{m}(k-\tau_i(k))\Big)^{\sum_{i=1}^{m}(k-\tau_i(k))}} > 1. \tag{2.11}$$

Then equation (1.1) is oscillatory, where $\overline{p}=\prod_{i=1}^{m}p_i$.

Corollary 2.3. *Let* $n_i\in N,\ i=1,\dots,m,$ *for large* $k\in N$

$$\tau_i(k)\le k-n_i,\quad n_i\in N,\qquad p_i(k)\ge p_i>0,\quad i=1,2,\dots,m, \tag{2.12}$$

and

$$\frac{\overline{p}\Big(m+\sum_{i=1}^{m}n_i\Big)^{m+\sum_{i=1}^{m}n_i}}{\Big(\sum_{i=1}^{m}n_i\Big)^{\sum_{i=1}^{m}n_i}} > 1. \tag{2.13}$$

Then equation (1.1) *is oscillatory, where* $\overline{p}=\prod_{i=1}^{m}p_i$.

Corollary 2.4. *Let* $n_i\in N,\ i=1,\dots,m,$ *for large* $k\in N$*, (2.12) holds and*

$$\overline{p}\sum_{i=1}^{m}n_i > \left(\frac{\sum_{i=1}^{m}n_i}{m+\sum_{i=1}^{m}n_i}\right)^{1+\frac{1}{m}\sum_{i=1}^{m}n_i}.$$

Then equation (1.1) is oscillatory, where $\overline{p}=\prod_{i=1}^{m}p_i$.

Remark 2.1. It is obvious that

$$\left(\frac{\sum_{i=1}^{m}n_i}{m+\sum_{i=1}^{m}n_i}\right)^{1+\frac{1}{m}\sum_{i=1}^{m}n_i} \to \frac{1}{e}$$

when $\sum_{i=1}^{m}n_i\to+\infty$.

3. Some auxiliary statements. In this section we establish the estimate of the quotient

$$\frac{u\left(\tau_i(k)\right)}{u(k+1)}, \quad i=1,\ldots,m, \qquad \text{and} \qquad \left(\prod_{i=1}^{m} \frac{u\left(\tau_i(k)\right)}{u(k+1)}\right)^{1/m},$$

where $u\colon N_{k_0} \to (0,+\infty)$ is nonoscillatory solution of equation (1.1).

Lemma 3.1. *Let $k_0 \in N$ and $u\colon N_{k_0} \to (0,+\infty)$ is a positive solution of equation (1.1). Then for any $s \in \{1,2,\ldots\}$ we have*

$$\frac{\left(\prod_{i=1}^{m} u\left(\tau_i(k)\right)\right)^{1/m}}{u(k+1)} \geq \left[\prod_{i=1}^{m} \prod_{j=\tau_i(k)}^{k} \left(1+m\left(\prod_{\ell=1}^{m} p_\ell(j)\right)^{1/m} \psi_s(j)\right)\right]^{1/m} \tag{3.1}$$

$$\text{for} \quad k \geq \eta_s^{\tau_*}(k_0), \quad s=1,2,\ldots,$$

where functions ψ_s and $\eta_s^{\tau_*}$ are defined by (2.1) – (2.4).

Proof. From (1.1) we have

$$\frac{u(k)}{u(k+1)} = 1+\sum_{i=1}^{m} p_i(k)\,\frac{u\left(\tau_i(k)\right)}{u(k+1)} \quad \text{for} \quad k \geq \eta_1(k_0) \tag{3.2}$$

and

$$\frac{u\left(\tau_i(k)\right)}{u(k+1)} = \prod_{j=\tau_i(k)}^{k}\left(1+\sum_{\ell=1}^{m} p_\ell(j)\,\frac{u\left(\tau_\ell(j)\right)}{u(j+1)}\right) \quad \text{for} \quad k \geq \eta_2(k_0), \quad i=1,\ldots,m.$$

Using the arithmetic mean-geometric mean inequality, from the last equality we get

$$\frac{u\left(\tau_i(k)\right)}{u(k+1)} \geq \prod_{j=\tau_i(k)}^{k}\left(1+m\left(\prod_{\ell=1}^{m} p_\ell(j)\right)^{1/m} \frac{\left(\prod_{\ell=1}^{m} u\left(\tau_\ell(j)\right)\right)^{1/m}}{u(j+1)}\right)$$

$$\text{for} \quad k \geq \eta_2(k_0), \quad i=1,\ldots,m.$$

Therefore,

$$\frac{\left(\prod\limits_{\ell=1}^{m} u\left(\tau_\ell(k)\right)\right)^{1/m}}{u(k+1)} \geq \left(\prod_{i=1}^{m}\prod_{j=\tau_i(k)}^{k}\left(1+m\left(\prod_{\ell=1}^{m} p_\ell(j)\right)^{1/m} \frac{\left(\prod_{\ell=1}^{m} u(\tau_\ell(j))\right)^{1/m}}{u(j+1)}\right)\right)^{1/m} \tag{3.3}$$

$$\text{for} \quad k \geq \eta_2(k_0).$$

Since

$$\frac{\left(\prod_{\ell=1}^{m} u(\tau_\ell(k))\right)^{1/m}}{u(k+1)} \geq 1,$$

the inequality (3.1) is obviously fulfilled for $s = 1$. Assume its validity for some $s \in \{1, 2, \ldots\}$. Then by (3.3) we obtain

$$\frac{\left(\prod_{i=1}^{m} u(\tau_i(k))\right)^{1/m}}{u(k+1)} \geq \left(\prod_{i=1}^{m} \prod_{j=\tau_i(k)}^{k} \left(1 + m\left(\prod_{\ell=1}^{m} p_\ell(j)\right)^{1/m} \psi_s(j)\right)\right)^{1/m} \quad \text{for} \quad k \geq \eta_{s+1}^{\tau_*}(k_0).$$

where function $\psi_s,\ s = 1, 2, \ldots$ are given by (2.2) and (2.3). The proof of the lemma is complete.

Lemma 3.2. *Let*

$$\liminf_{k\to+\infty} p(k) > 0 \tag{3.4}$$

and

$$\min\left\{\frac{\liminf_{k\to+\infty} \prod_{i=1}^{m} \prod_{j=\tau_i(k)}^{k} \left(1 + m\, p^{1/m}(j)\, \alpha\right)}{\alpha} : \alpha \geq 1\right\} > 1. \tag{3.5}$$

Then

$$\lim_{s\to+\infty}\left(\liminf_{k\to+\infty} \psi_s(k)\right) = +\infty, \tag{3.6}$$

where functions p and ψ_s are given by (2.2)–(2.4).

Proof. Suppose contrary. Let

$$\lim_{s\to+\infty}\left(\liminf_{k\to+\infty} \psi_s(k)\right) = \beta < +\infty. \tag{3.7}$$

Then by (3.4) for any $\varepsilon \in (0, \beta - 1)$ there exist $k_0,\ s_0 \in N$ such that

$$\psi_s(k) \geq \beta - \varepsilon \geq 1 \qquad \text{for} \quad k \in N_{k_0} \quad \text{and} \quad s \in N_{s_0}. \tag{3.8}$$

According to (3.8) from (2.3) we get

$$\psi_s(k) \geq \left(\prod_{i=1}^{m} \prod_{j=\tau_i(k)}^{k} \left(1 + m\, p^{1/m}(j)(\beta - \varepsilon)\right)\right)^{1/m} \quad \text{for} \quad s \geq \eta_{s_0}^{\tau_*},$$

where p is given by (2.4).

That is

$$\liminf_{k\to+\infty} \psi_s(k) \geq \liminf_{k\to+\infty} \left(\prod_{i=1}^{m} \prod_{j=\tau_i(k)}^{k} \left(1 + m\, p^{1/m}(j)(\beta - \varepsilon)\right)\right)^{1/m}.$$

By (3.8) of the last inequality, we have

$$\beta \geq \liminf_{k\to+\infty} \left(\prod_{i=1}^{m} \prod_{j=\tau_i(k)}^{k} \left(1 + m\, p^{1/m}(j)(\beta - \varepsilon)\right)\right)^{1/m}.$$

Therefore,

$$\frac{\liminf_{k\to+\infty}\left(\prod_{i=1}^{m}\prod_{j=\tau_i(k)}^{k}\left(1+m(\beta-\varepsilon)\,p^{1/m}(j)\right)\right)^{1/m}}{\beta}\leq 1. \tag{3.9}$$

According to (3.9) for any $\varepsilon\in(0,\beta)$ we get

$$\min\left\{\frac{\liminf\limits_{k\to+\infty}\left(\prod_{i=1}^{m}\prod_{j=\tau_i(k)}^{k}\left(1+m\,p^{1/m}(j)\,\alpha\right)\right)^{1/m}}{\alpha}:\alpha\geq 1\right\}\leq$$

$$\leq\frac{\liminf\limits_{k\to+\infty}\left(\prod_{i=1}^{m}\prod_{j=\tau_i(k)}^{k}\left(1+m\,p^{1/m}(j)(\beta-\varepsilon)\right)\right)^{1/m}}{\beta-\varepsilon}=$$

$$=\frac{\beta}{\beta-\varepsilon}\,\frac{\liminf\limits_{k\to+\infty}\left(\prod_{i=1}^{m}\prod_{j=\tau_i(k)}^{k}\left(1+m\,p^{1/m}(j)(\beta-\varepsilon)\right)\right)^{1/m}}{\beta}\leq\frac{\beta}{\beta-\varepsilon},$$

with implies

$$\min\left\{\frac{\liminf_{k\to+\infty}\left(\prod_{i=1}^{m}\prod_{j=\tau_i(k)}^{k}\left(1+m\,p^{1/m}(j)\,\alpha\right)\right)^{1/m}}{\alpha}:\alpha\geq 1\right\}\leq 1,$$

where p is given by (2.4). This however contradicts (3.5). The obtained contradiction proves that (3.6) holds.

4. Proofs of the theorems. ***Proof of Theorem 2.1.*** Suppose to the contrary, that equation (1.1) has a non oscillatory proper solution $u\colon N_{k_0}\to R.$ Since $-u(k)$ is also a solution (1.1), we confine ourselves only to the case that u is eventually positive solution of equation (1.1). Then there exists $k_1\geq k_0$ such that $u(\tau_i(k))>0$ for $k\in N_i(k),\ i=1,\ldots,m.$ As seen while proving Lemma 3.1

$$\left(\prod_{i=1}^{m}\frac{u(\tau_i(k))}{u(k+1)}\right)^{1/m}\geq\psi_s(k)\quad\text{for}\quad k\geq\eta_s^{\tau_*}(k_1),\quad s=1,2,\ldots, \tag{4.1}$$

where the functions $\eta_s^{\tau_*}$ and ψ_s are defined by (2.1) – (2.4).

From (1.1) by (2.1), (2.5) and (3.2) we have

$$u(\sigma_i(k))\geq\sum_{s=\sigma_i(k)}^{k}\sum_{j=1}^{m}p_j(s)\,u(\tau_j(s))\quad\text{for}\quad k\geq\eta_s^{\tau_*}(k_1),\quad s=1,2,\ldots, \tag{4.2}$$

and

$$u(\tau_j(s))\geq u(\sigma_j(k))\prod_{t=\tau_j(s)}^{\sigma_j(k)-1}\left(1+\sum_{\ell=1}^{m}p_\ell(t)\,\frac{u(\sigma_\ell(t))}{u(t+1)}\right) \tag{4.3}$$

$$\text{for}\quad \eta_1^{\tau_*}(k_1) \le s \le k, \quad j = 1, \dots, m.$$

Therefore, by (4.2) and (4.3) we get

$$u(\sigma_i(k)) \ge \sum_{j=1}^{m} \left(\sum_{s=\sigma_i(k)}^{k} p_j(s)\, u(\sigma_j(k)) \prod_{t=\tau_j(s)}^{\sigma_j(k)-1} \left(1 + \sum_{\ell=1}^{m} p_\ell(t) \, \frac{u(\sigma_\ell(t))}{u(t+1)} \right) \right) =$$

$$= \sum_{j=1}^{m} u(\sigma_j(k)) \left(\sum_{s=\sigma_i(k)}^{k} p_j(s) \prod_{t=\tau_j(s)}^{\sigma_j(k)-1} \left(1 + \sum_{\ell=1}^{m} p_\ell(t) \, \frac{u(\sigma_\ell(t))}{u(t+1)} \right) \right), \quad i = 1, \dots, m.$$

Using the arithmetic mean-geometric mean inequality, from the last inequality we get

$$u(\sigma_i(k)) \ge m \left(\prod_{j=1}^{m} u(\sigma_j(k)) \right)^{1/m} \left(\prod_{j=1}^{m} \sum_{s=\sigma_i(k)}^{k} p_j(s) \times \right.$$

$$\left. \times \prod_{t=\tau_j(s)}^{\sigma_j(k)-1} \left(1 + m \left(\prod_{\ell=1}^{m} p_\ell(t) \right)^{1/m} \left(\prod_{j=1}^{m} \frac{u(\sigma_\ell(t))}{u(t+1)} \right)^{1/m} \right) \right)^{1/m}$$

$$\text{for}\quad k \ge \eta_1^{\tau_*}(k_1), \quad i = 1, \dots, m.$$

On the other hand by Lemma 3.1 we have

$$\prod_{i=1}^{m} u(\sigma_i(k)) \ge m^m \prod_{i=1}^{m} u(\sigma_i(k)) \prod_{i=1}^{m} \left(\sum_{s=\sigma_i(k)}^{k} p_i(s) \prod_{t=\tau_j(s)}^{\sigma_j(k)-1} \left(1 + m \left(\prod_{\ell=1}^{m} p_\ell(t) \right)^{1/m} \psi_s(t) \right) \right)^{1/m}$$

$$\text{for}\quad k \ge \eta_s^{\tau_*}(k_1), \quad s = 1, 2, \dots,$$

where functions $\eta_s^{\tau_*}$ and ψ_s are given by (2.1) – (2.3). Therefore, for $s = k_0$

$$\limsup_{k \to +\infty} \prod_{i=1}^{m} \left(\prod_{j=1}^{m} \sum_{s=\sigma_i(k)}^{k} p_j(s) \prod_{t=\tau_j(s)}^{\sigma_j(k)-1} \left(1 + m \left(\prod_{\ell=1}^{m} p_\ell(t) \right)^{1/m} \psi_{k_*}(t) \right) \right)^{1/m} \le \frac{1}{m^m},$$

which contradicts (2.6). The proof of the theorem is complete.

Proof of Corollary 2.1. Since $\psi_s(k) \ge 1$ for large $s, k \in N$, this corollary immediately follows from Theorem 2.1.

Proof of Theorem 2.2. Suppose to the contrary, that equation (1.1) has a nonoscillatory solution $u \colon N_{k_0} \to (0, +\infty)$. Then according to (2.8), (2.9) and Lemma 3.2 condition (3.6) holds. By (2.5) and (2.8), choose $M > 0$ such that for large k we have

$$\prod_{\ell=1}^{m} \left(\prod_{i=1}^{m} \sum_{s=\sigma_i(k)}^{k} p_i(s) \prod_{j=\tau_i(s)}^{\sigma_i(k)-1} \left(1 + m p^{1/m}(j) M^{1/m} \right) \right)^{1/m} > \frac{2}{m^m}. \tag{4.4}$$

On the other hand according to (3.6) there exist $s_0 \in N$ and $k_0 \in N$ such that $\psi_{s_0}(k) \geq M$ for $k \geq k_0$. Therefore, by (4.4) the condition (2.6) be fulfilled for $s = s_0$, which proved the validity of the theorem.

Proof of Corollary 2.2. It is suffices to show that condition (2.9) holds. Indeed by (2.10) for large k we have

$$\frac{\liminf_{k\to+\infty} \prod_{i=1}^{m} \prod_{j=\tau_i(k)}^{k} \left(1 + m\alpha^{1/m}\, p^{1/m}(j)\right)}{\alpha} \geq$$

$$\geq \frac{\liminf_{k\to+\infty} \prod_{i=1}^{m} \left(1 + m\,\alpha^{1/m}\, p^{1/m}\right)^{(k-\tau_i(k))+1}}{\alpha} =$$

$$= \liminf_{k\to+\infty} \frac{\left(1 + m\,\alpha^{1/m}\, p^{1/m}\right)^{\sum_{i=1}^{m}(k-\tau_i(k))+m}}{\alpha} \geq$$

$$\geq \liminf_{k\to+\infty} \left\{ \min \frac{\left(1 + m\,\alpha^{1/m}\, p^{1/m}\right)^{m+\sum_{i=1}^{m}(k-\tau_i(k))}}{\alpha} : \alpha \geq 1 \right\}, \tag{4.5}$$

where $p = \prod_{i=1}^{m} p_i$. Since

$$\min_{\alpha\geq 1} \frac{\left(1 + m\,\alpha^{1/m}\, p^{1/m}\right)^{m+\sum_{i=1}^{m}(k-\tau_i(k))}}{\alpha}$$

$$= \frac{p\left(m + \sum_{i=1}^{m}(k-\tau_i(k))\right)^{m+\sum_{i=1}^{m}(k-\tau_i(k))}}{\left(\sum_{i=1}^{m}(k-\tau_i(k))\right)^{\sum_{i=1}^{m}(k-\tau_i(k))}}, \tag{4.6}$$

according to (4.5) and (4.6) condition (2.9) holds, which proved the validity of the corollary.

Proof of Corollary 2.3. Since function $\dfrac{(m+x)^{m+x}}{x^x}$ $(1 \leq x < +\infty)$ is non-decreasing, according to (2.12) and (2.13) condition (2.9) be fulfilled, which proved the validity of the corollary.

Remark 4.1. It is obvious that if (2.12) be fulfilled and $m = 1$, then Corollary 2.4 is Theorem 7.5.1, [2].

Remark 4.2. The following example illustrate the significance of our results.

Consider difference equation

$$\Delta u(k) + p(k)\, u(k-1) = 0$$

then condition (7.5.2) [2] is reduced to the condition

$$\liminf_{k\to+\infty} p(k) > \frac{1}{4} \tag{4.7}$$

and condition (2.9) is reduced by the condition

$$\liminf_{k\to+\infty} \left(\sqrt{p(k-1)} + \sqrt{p(k)}\right)^2 > 1. \tag{4.8}$$

It is obvious that condition (4.7) implies condition (4.8).

On the other hand if $\varepsilon \in \left(0, \dfrac{1}{4}\right]$, $p(2k) = \varepsilon$ and $p(2k+1) = 1 - \varepsilon$, $k = 1, 2, \ldots$ then it is obvious that condition (4.8) be fulfilled but condition (4.7) is not fulfilled.

References

1. *Ladas, G., Philos, Ch. G., Sficas, Y. G.* Sharp conditions for the oscillation of delay difference equations // J. Appl. Math. Simulat. – 1989. – **2**, № 2. – P. 101 – 111.
2. *Györi I., Ladas G.* Oscillation theory of delay differential equations with applications // Oxford Math. Monogr. – Clarendon Press, 1991.
3. *Chatzarakis G. E., Koplatadze R., Stavroulakis I. P.* Optimal oscillation criteria for first order difference equations with delay argument // Pacific J. Math. – 2008. – **235**. – № 1. – P. 15 – 33.
4. *Koplatadze R., Pinelas S.* Oscillation criteria for first-order linear difference equation with several delay arguments // Nelin. Kolyvannya. – 2014. – **17**, № 2. – P. 248 – 267.
5. *Koplatadze R. G., Chanturiya T. A.* Oscillating and monotone solutions of first-order differential equations with deviating argument // Differents. Uravneniya. – 1982. – **18**, № 8. – P. 1463 – 1465, 1472.
6. *Koplatadze R., Kvinikadze G.* On the oscillation of solutions of first-order delay differential inequalities and equations // Georgian Math. J. – 1994. – **1**, № 6. – P. 675 – 685.
7. *Grammatikopoulos M. K., Koplatadze R., Stavroulakis I. P.* On the oscillation of solutions of first order differential equations with retarded arguments // Georgian Math. J. – 2003. – **10**, № 1. – P. 63 – 76.
8. *Berikelashvili G. K., Dzhokhadze O. M., Koplatadze R. G.* On an approach to the investigation of the asymptotic properties of solutions of first-order ordinary differential equations with delay // Differents. Uravneniya. – 2008. – **44**, № 1. – P. 19 – 38.
9. *Infante G., Koplatadze R., Stavroulakis I. P.* Oscillation criteria for differential equations with several retarded arguments // Funkcial. Ekvac. – 2015. – **58**, № 3. – P. 347 – 364.
10. *Koplatadze R.* Specific properties of solutions of first order differential equations with several delay arguments // Izv. Nat. Akad. Nauk Armenii. Ser. Mat. – 2015. – **50**, № 5. – P. 24 – 33.
11. *Koplatadze R.* On asymptotic behavior of first order difference equations with several delay // Bull. TICMI. – 2017. – **21**, № 2. – P. 1 – 5.

Received 17.02.2018