UDC 517.983

WEAKLY NONLINEAR BOUNDARY-VALUE PROBLEMS FOR FREDHOLM INTEGRAL EQUATIONS WITH DEGENERATE KERNEL IN BANACH SPACES

СЛАБКОНЕЛІНІЙНІ КРАЙОВІ ЗАДАЧІ ДЛЯ ІНТЕГРАЛЬНИХ РІВНЯНЬ ФРЕДГОЛЬМА З ВИРОДЖЕНИМ ЯДРОМ У БАНАХОВИХ ПРОСТОРАХ

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Weakly nonlinear boundary-value problems for Fredholm integral equations with degenerate kernel in Banach spaces are considered. We obtain necessary and sufficient conditions for finding solutions of these problems and construct converging iterative procedures of determination of solutions of these boundary-value problems.

Розглянуто слабконелінійні крайові задачі для інтегральних рівнянь Фредгольма з виродженим ядром у банахових просторах. Одержано необхідні та достатні умови існування розв'язків таких задач, а також побудовано збіжні ітераційні процедури для знаходження розв'язків зазначених задач.

This paper is a continuation of the research on the conditions of solvability and the construction of solutions of weakly nonlinear integral Fredholm equations with a degenerate kernel in Banach spaces that were started in [1].

Constructive methods for the analysis of weakly nonlinear boundary-value problems for the systems of functional-differential and other equations traditionally occupy one of the important places in the qualitative theory of differential equations and continue the development of methods of perturbation theory, in particular, the methods of Lyapunov – Poincare small parameter [2, 3].

These methods were successfully developed in [4, 5] and applied in the study of weakly nonlinear boundary-value problems for systems of ordinary differential equations [6] and the construction of bounded solutions of weakly nonlinear differential equations [7] in Banach spaces.

In finite-dimensional Euclidean spaces, weakly nonlinear integral-differential equations and Fredholm integral equations with a nondegenerate kernel, which are not always solvable, were studied in [8, 9].

The specific nature of the study of boundary-value problems for systems of integral equations in Banach spaces lies in the fact that their linear part is an operator that does not have an inverse [10], which considerably complicates the study of boundary-value problems for such equations. Therefore, the problem of studying the conditions of existence and constructing the general solutions of weakly nonlinear boundary-value problems for not always solvable integral Fredholm equations with a degenerate kernel in Banach spaces is topical.

Statement of the problem. We consider a weakly nonlinear boundary-value problem

$$(Lz)(t) := z(t) - M(t)\int\limits_a^b N(s)z(s)ds = f(t) + \varepsilon\int\limits_a^b K(t,s)Z(z(s,\varepsilon),s,\varepsilon)ds, \tag{1}$$

$$\ell z(\cdot) = \alpha + \varepsilon J(z(\cdot,\varepsilon),\varepsilon), \tag{2}$$

where the operator-valued functions $M(t)$ and $N(t)$ are defined on the finite interval $\mathcal{I} = [a,b]$, act from the Banach space $\mathbf{B}$ into the same space, are strongly continuous with the norms $|||M||| = \sup\limits_{t\in\mathcal{I}}\|M(t)\|_{\mathbf{B}} = M_0 < \infty$ и $|||N||| = \sup\limits_{t\in\mathcal{I}}\|N(t)\|_{\mathbf{B}} = N_0 < \infty$; the operator-valued function $K(t,s)$ is defined in the square $\mathcal{I}\times\mathcal{I}$ and acts from the Banach space $\mathbf{B}$ into the same space with respect to each variable, is strongly continuous with respect to each variable with the norm $|||K||| = \sup\limits_{t\in\mathcal{I}}\|K(t,s)\|_{\mathbf{B}} = K_0 < \infty$; $Z(z(t,\varepsilon),t,\varepsilon)$ is nonlinear z bounded operator function, $J(z(\cdot,\varepsilon),\varepsilon)$ is nonlinear z vector-functional that in the neighborhood of the generating solution $\|z - z_0\| \leq q$ have a strongly continuous Frechet derivative with respect to z and are continuous for the set of variables $z,\ t,\ \varepsilon,\ q$ and ε_0, that are rather small constants; $Z(0,t,0) = 0$, $Z'_z(0,t,0) = 0,\ J(0,0) = 0,\ J'_z(0,0) = 0;\ f(t)$ is a vector-valued function in the Banach space $\mathbf{C}(\mathcal{I},\mathbf{B})$ that are continuous vector functions on the interval $\mathcal{I};\ \alpha$ is an element of the Banach space $\mathbf{B}_1\colon \alpha \in \mathbf{B}_1$.

Along with the problem (1), (2) we consider the linear generating boundary-value problem

$$z_0(t) - M(t)\int\limits_a^b N(s)z_0(s)ds = f(t), \tag{3}$$

$$\ell z_0(\cdot) = \alpha, \tag{4}$$

which is obtained from (1), (2) for $\varepsilon = 0$.

The problem is to find the necessary and sufficient conditions for the existence of the solutions of the weakly nonlinear boundary-value problem (1), (2). We seek solutions in the class of vector-valued functions $z(t,\varepsilon)$, that are continuous with respect to the variable t and with respect to the parameter ε, and turning at $\varepsilon = 0$ into the generating solution of the linear boundary-value problem (3), (4).

Auxiliary information. Suppose that a bounded linear operator $D = I_{\mathbf{B}} - \int_a^b N(s)M(s)\,ds$, $D\colon \mathbf{B}\to\mathbf{B}$ is generalized invertible. Then [11, 12] there is a bounded projector $\mathcal{P}_{N(D)}\colon \mathbf{B}\to \to N(D)$, that projects a Banach space $\mathbf{B}$ onto the null space $N(D)$ of D operator, a bounded projector $\mathcal{P}_{Y_D}\colon \mathbf{B}\to Y_D$, that projects a Banach space $\mathbf{B}$ on the subspace $Y_D = \mathbf{B}\ominus R(D)$ and D^- is a bounded generalized inverse operator to the operator D [4, 5, 13].

The class of bounded linear generalized invertible operators that act from the Banach space $\mathbf{B}$ to the Banach space $\mathbf{B}$ will be denoted as $\mathbf{GI}(\mathbf{B},\mathbf{B})$. It is obvious that the operator belonging to $\mathbf{GI}(\mathbf{B},\mathbf{B})$ is normally solvable [14].

It is shown in [15] that if the operator $D \in \mathbf{GI}(\mathbf{B},\mathbf{B})$, then, under the condition

$$M(t)\mathcal{P}_{Y_D}\int\limits_a^b N(s)f(s)ds = 0$$

and only under it, the operator gather (3) is solvable and has a family of solutions

$$z_0(t) = M(t)\mathcal{P}_{N(D)}c + \big(L^- f\big)(t), \tag{5}$$

where c is an arbitrary element of the Banach space $\mathbf{B}$,

$$\big(L^- f\big)(t) = f(t) + M(t)D^- \int\limits_a^b N(s)f(s)ds$$

is a bounded generalized inverse operator to the integral operator L [10].

Substituting the solution (5) of the inhomogeneous operator gather (3) into the boundary condition (4), we obtain the operator gather

$$Qc + \ell f(\cdot) + \ell M(\cdot)D^- \int\limits_a^b N(s)f(s)ds = \alpha,$$

where $Q = \ell M(\cdot)\mathcal{P}_{N(D)} \colon \mathbf{B} \to \mathbf{B}_1$ is a bounded linear operator.

Let the operator $Q \in \mathbf{GI}(\mathbf{B}, \mathbf{B}_1)$. Denote $\mathcal{P}_{N(Q)} \colon \mathbf{B} \to N(Q)$ is the bounded projector of the Banach space $\mathbf{B}$ to the null space $N(Q)$ of the operator Q, $\mathcal{P}_{Y_Q} \colon \mathbf{B}_1 \to Y_Q$ is the bounded projector of the Banach space $\mathbf{B}_1$ onto the subspace $Y_Q = \mathbf{B}_1 \ominus R(Q)$, Q^- is a bounded generalized inverse operator to the operator Q.

Theorem 1 [15]**.** *Let the operators* $D \in \mathbf{GI}(\mathbf{B}, \mathbf{B})$ *and* $Q \in \mathbf{GI}(\mathbf{B}, \mathbf{B}_1)$.

Then the corresponding (3), (4) homogeneous $(f(t) = 0,\ \alpha = 0)$ *boundary-value problem has the family of solutions*

$$z(t) = \widetilde{M}(t)c,$$

where $\widetilde{M}(t) = M(t)\mathcal{P}_{N(D)}\mathcal{P}_{N(Q)}$, c *is an arbitrary element of the Banach space* $\mathbf{B}$.

The nonhomogeneous boundary-value problem (3), (4) is solvable for those and only those $f(t) \in \mathbf{C}(\mathcal{I}, \mathbf{B})$ *and* $\alpha \in \mathbf{B}_1$, *that satisfy the system of conditions*

$$\begin{cases} M(t)\mathcal{P}_{Y_D} \displaystyle\int_a^b N(s)f(s)ds = 0, \\ \mathcal{P}_{Y_Q}\left[\alpha - \ell f(\cdot) - \ell M(\cdot)D^- \displaystyle\int_a^b N(s)f(s)ds\right] = 0 \end{cases} \tag{6}$$

and it has the family of solutions

$$z_0(t) = \widetilde{M}(t)c + (Gf)(t) + M(t)\mathcal{P}_{N(D)}Q^-\alpha, \tag{7}$$

where

$$(Gf)(t) := \big[f(t) - M(t)\mathcal{P}_{N(D)}Q^-\ell f(\cdot)\big] +$$

$$+ M(t)\left[I_{\mathbf{B}} - \mathcal{P}_{N(D)}Q^-\ell M(\cdot)\right]D^- \int\limits_a^b N(s)f(s)ds \tag{8}$$

is a generalized Green operator of the corresponding (3), (4) semi-homogeneous $(\alpha = 0)$ *boundary-value problem.*

It is necessary to note that the first of the conditions (6) will always be satisfied if the condition $\mathcal{P}_{Y_D}\int_a^b N(s)f(s)ds = 0$ is satisfied.

To solve the problem, we need the information about the solvability conditions and the representation of the solutions of the operator gathers with linear operator B_0 that is an operator matrix

$$B_0 = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$

where $B_1\colon \mathbf{B} \to \mathbf{B}$ and $B_2\colon \mathbf{B} \to \mathbf{B}_1$ are linear bounded generalized invertible operators [11].

In this case [11, 12] there are bounded projectors $\mathcal{P}_{N(B_1)}\colon \mathbf{B} \to N(B_1)$ and $\mathcal{P}_{N(B_2)}\colon \mathbf{B} \to$ $\to N(B_2)$ to the null spaces of the operators B_1 and B_2, and also the bounded projectors $\mathcal{P}_{Y_{B_1}}\colon \mathbf{B} \to Y_{B_1}$ and $\mathcal{P}_{Y_{B_2}}\colon \mathbf{B}_1 \to Y_{B_2}$ to the subspaces $Y_{B_1} = \mathbf{B}\ominus R(B_1)$ and $Y_{B_2} = \mathbf{B}_1\ominus R(B_2)$, respectively, and also bounded generalized inverse operators B_1^- and B_2^-.

Then, using [16] for the system of operator equations

$$B_0 c = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} c = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \qquad b_1 \in \mathbf{B}, \quad b_2 \in \mathbf{B}_1 \tag{9}$$

the following theorem is true.

Theorem 2 [17]. *Let $B_1 \in \mathbf{GI}(\mathbf{B},\mathbf{B})$ and $B_2 \in \mathbf{GI}(\mathbf{B},\mathbf{B}_1)$. Then the system of operator gathers (9) is solvable for those and only those $\operatorname{col}[b_1, b_2]$ that satisfy the condition*

$$\mathcal{P}_{Y_{B_0}}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = 0,$$

under which it has a family of solutions

$$c = \mathcal{P}_{N(B_0)}\hat{c} + B_0^- \begin{bmatrix} b_1 \\ b_2 \end{bmatrix},$$

where $\mathcal{P}_{Y_{B_0}} = \begin{bmatrix} I_{\mathbf{B}} - B_1\mathcal{P}_{N(B_2)}B_1^- & -B_1B_2^- \\ 0 & \mathcal{P}_{Y_{B_2}} \end{bmatrix}$ is a bounded projector onto the subspace $Y_{B_0} =$ $= I_{\mathbf{B}\times\mathbf{B}_1} \ominus R(B_0),$ $\mathcal{P}_{N(B_0)} = \mathcal{P}_{N(B_2)}\mathcal{P}_{N(B_1)}$ is a bounded projector onto the null space $N(B_0)$ of the operator B_0, $\hat{c}$ is an arbitrary element of the Banach space $\mathbf{B}$,

$$B_0^- = \begin{bmatrix} \mathcal{P}_{N(B_2)}B_1^- & B_2^- \end{bmatrix}$$

is a bounded generalized inverse operator to the operator B_0.

Main result. Using the generalized Green operator (8) of the linear semi-homogeneous boundary-value problem, we seek the existence conditions for the solutions $z = z(t,\varepsilon)$ of the boundary-value problem (1), (2) that are defined in the class of vector functions: $z(\cdot,\varepsilon) \in \mathbf{C}(\mathcal{I},\mathbf{B})$, $z(t,\cdot) \in \mathbf{C}(0,\varepsilon_0]$ and turn for $\varepsilon = 0$ to one of the generating solutions $z_0(t,c)$

Performing in (1), (2) the change of the variable

$$z(t,\varepsilon) = z_0(t,c) + x(t,\varepsilon)$$

for the deviation $x(t,\varepsilon)$ from the generating solution, we obtain the boundary-value problem

$$x(t) - M(t)\int_a^b N(s)x(s)ds = \varepsilon\int_a^b K(t,s)Z(z_0(s,c) + x(s,\varepsilon), s, \varepsilon)\,ds, \tag{10}$$

$$\ell x(\cdot) = \varepsilon J\big(z_0(\cdot, c) + x(\cdot, \varepsilon), \varepsilon\big). \tag{11}$$

We find the necessary condition of the existence of solutions $z(t,\varepsilon)$ of the boundary-value problem (1), (2), which for $\varepsilon = 0$ become one of the generating solutions $z_0(t,c) \in \mathbf{C}(\mathcal{I}, \mathbf{B})$ of the generating boundary-value problem (3), (4).

Suppose that the boundary-value problem (1), (2) has a solution $z(t,\varepsilon),$ then by Theorem 1, the system of solvability conditions must be valid

$$\begin{cases} \mathcal{P}_{Y_D} \displaystyle\int_a^b N(s) \left[f(t) + \varepsilon \int_a^b K(s,\tau) Z(z(\tau,\varepsilon),\tau,\varepsilon) d\tau \right] ds = 0, \\ \mathcal{P}_{Y_Q} \left[\alpha + \varepsilon J(z(\cdot,\varepsilon),\varepsilon) - \ell f(\cdot) - \right. \\ \qquad \left. - \ell M(\cdot) D^- \displaystyle\int_a^b N(s) \left[f(s) + \varepsilon \int_a^b K(s,\tau) Z(z(\tau,\varepsilon),\tau,\varepsilon) d\tau \right] ds \right] = 0, \end{cases}$$

which, taking into account (6) and $\varepsilon \neq 0,$ take the form

$$\begin{cases} \mathcal{P}_{Y_D} \left[\displaystyle\int_a^b N(s) \int_a^b K(s,\tau) Z(z(\tau,\varepsilon),\tau,\varepsilon) d\tau \right] ds = 0, \\ \mathcal{P}_{Y_Q} \left[J(z(\cdot,\varepsilon),\varepsilon) - \ell M(\cdot) D^- \displaystyle\int_a^b N(s) \int_a^b K(s,\tau) Z(z(\tau,\varepsilon),\tau,\varepsilon) d\tau \, ds \right] = 0. \end{cases} \tag{12}$$

Taking into account the continuity of the operator-valued functions $Z(z,t,\varepsilon)$ and $J(z(\cdot,\varepsilon),\varepsilon)$ with respect to the totality of the variables $z,\ t,\ \varepsilon,$ passing to the limit at $\varepsilon \to 0$ in the system (12), we obtain the necessary condition of the existence of solutions of the boundary-value problem (1), (2)

$$F(c) = \begin{cases} \mathcal{P}_{Y_D} \left[\displaystyle\int_a^b N(s) \int_a^b K(s,\tau) Z(z_0(\tau,c),\tau,0) d\tau \right] ds = 0, \\ \mathcal{P}_{Y_Q} \left[J(z_0(\cdot,c),0) - \ell M(\cdot) D^- \displaystyle\int_a^b N(s) \int_a^b K(s,\tau) Z(z_0(\tau,c),\tau,0) d\tau \, ds \right] = 0. \end{cases}$$

Thus, the theorem is valid for the boundary-value problem (1), (2).

Theorem 3. *Suppose that with respect to the above conditions, the boundary-value problem (1), (2) has the solution* $z(t,\varepsilon),$ *continuous on* $\varepsilon \in [0,\varepsilon_0],$ *which converts at* $\varepsilon = 0$ *to some generating solution* $z_0(t,c)$ *of the form (7) obtained at* $c = c_0$. *Then the element* $c_0 \in \mathbf{B_1}$ *satisfies the system of gathers*

$$F(c_0) = \begin{cases} \mathcal{P}_{Y_D} \displaystyle\int_a^b N(s) \int_a^b K(s,\tau) Z((z_0(\tau,c_0)),\tau,\varepsilon) d\tau ds = 0, \\ \mathcal{P}_{Y_Q} \left[J(z_0(\cdot,c_0)) - \ell M(\cdot) D^- \displaystyle\int_a^b N(s) \int_a^b K(s,\tau) Z(z_0(s,c_0)),\tau,\varepsilon) d\tau \, ds \right] = 0. \end{cases} \tag{13}$$

By analogy with weakly nonlinear problems for ordinary differential gathers [2, 4, 5], the system of gathers (13) will be called a system of equations for generating constants.

Therefore, if the system of equations (13) has a solution $c = c_0 \in \mathbf{B},$ then the element c_0 determines the generating solution $z_0(t, c_0)$ to which the solution $z(t, \varepsilon)$ of the original nonlinear boundary-value problem (1), (2) can correspond. If the system of equations (13) does not have solutions, then the boundary-value problem (1), (2) does not have the desired solution. Thus, the necessary condition of the existence of a solution of the boundary-value problem (1), (2) is satisfied by choosing the constant c in the generating solution (7), as the real root of the system of equations (13).

To prove the sufficiency, using the conditions on the nonlinear operator-valued functions $Z(z, t, \varepsilon)$ and $J(z, \varepsilon),$ we single out the linear parts with respect to x and terms of order zero with respect to $\varepsilon.$ As a result, we obtain expansions

$$Z(z_0(t, c_0) + x(t, \varepsilon), t, \varepsilon) = Z_0(t, c_0) + T(t)x(t, \varepsilon) + R(x(t, \varepsilon), t, \varepsilon),$$

$$J(z_0(\cdot, c_0) + x(\cdot, \varepsilon), \varepsilon) = J_0(\cdot, c_0) + \ell_1 x(\cdot, \varepsilon) + R_1(x(\cdot, \varepsilon), \varepsilon),$$

where

$$Z_0(t, c_0) = Z(z_0(t, c_0), t, 0) \in \mathbf{C}(\mathcal{I}, \mathbf{B}),$$

$$J_0(\cdot, c_0) = J_0(z_0(\cdot, c_0), 0) \in \mathbf{B}_1;$$

$$T(t) = T(t, c_0) = \left.\frac{\partial Z(z, t, 0)}{\partial z}\right|_{z=z(t,c_0)} \in \mathbf{C}(\mathcal{I}, \mathbf{B}),$$

$$\ell_1 = \left.\frac{\partial J(z, 0)}{\partial z}\right|_{z=z(\cdot,c_0)}, \qquad \ell_1 \colon \mathbf{C}(\mathcal{I}, \mathbf{B}) \to \mathbf{B}_1;$$

$R(x(t, \varepsilon), t, \varepsilon)$ is nonlinear vector-valued function, $R_1(x(\cdot, \varepsilon), \varepsilon)$ is nonlinear vector-valued functional.

Considering the nonlinearities in the boundary-value problem (10), (11) as inhomogeneities and applying theorem 1 to it, we obtain the following expression for the representation of its solution $x(t, \varepsilon)$:

$$x(t, \varepsilon) = \widetilde{M}(t)c + \bar{x}(t, \varepsilon).$$

In this case, the unknown vector $c = c(\varepsilon) \in \mathbf{B}_1$ is determined from the solvability conditions of the type (12)

$$\begin{cases} \mathcal{P}_{Y_D} \displaystyle\int_a^b N(s) \int_a^b K(s, \tau) \Big\{ Z_0(\tau, c_0) + T(\tau)x(\tau, \varepsilon) + \\ \quad + R(x(\tau, \varepsilon), \tau, \varepsilon) \Big\} d\tau\, ds = 0, \\ \mathcal{P}_{Y_Q} \Bigg[J_0(\cdot, c_0) + \ell_1 x(\cdot, \varepsilon) + R_1(x(\cdot, \varepsilon), \varepsilon) - \\ \quad - \ell M(\cdot) D^- \displaystyle\int_a^b N(s) \int_a^b K(s, \tau) \Bigg\{ Z_0(\tau, c_0) + \\ \quad + T(\tau)x(\tau, \varepsilon) + R(x(\tau, \varepsilon), \tau, \varepsilon) \Bigg\} d\tau ds \Bigg] = 0. \end{cases} \tag{14}$$

The unknown vector function $\bar{x}(t,\varepsilon)$ is defined by the formula

$$\bar{x}(t,\varepsilon) = \varepsilon \left(G \int\limits_a^b K(\cdot, s) \left\{ Z_0(s,c_0) + T(s)x(s,\varepsilon) + R(x(s,\varepsilon),s,\varepsilon) \right\} ds \right)(t) +$$

$$+ M(t)Q^- \big[J_0(\cdot, c_0) + \ell_1 x(\cdot,\varepsilon) + R_1(x(\cdot,\varepsilon),\varepsilon) \big],$$

where the operator G acts on the vector function

$$\varphi(t,\varepsilon) = \int\limits_a^b K(t,s) \big\{ Z_0(s,c_0) + T(s)x(s,\varepsilon) + R(x(s,\varepsilon),s,\varepsilon) \big\} ds$$

by the rule (8).

Substituting the expression $x(t,\varepsilon)$ in (14) for $\widetilde{M}(t)c + \bar{x}(t,\varepsilon)$ and isolating the terms containing the constant $c,$ taking (13) into account, we obtain the operator equation

$$B_0 c = - \left[\begin{array}{l} \mathcal{P}_{Y_D} \displaystyle\int_a^b N(s) \int_a^b K(s,\tau) \overline{R}(x(\tau,\varepsilon), \bar{x}(\tau,\varepsilon), \tau, \varepsilon) d\tau \, ds, \\ \mathcal{P}_{Y_Q} \left[\overline{R}_1(x(\cdot,\varepsilon), \bar{x}(\cdot,\varepsilon), \varepsilon) - \right. \\ \left. \qquad - \ell M(\cdot) D^- \displaystyle\int_a^b N(s) \int_a^b K(s,\tau) \overline{R}(x(\tau,\varepsilon), \bar{x}(\tau,\varepsilon), \tau, \varepsilon) d\tau \, ds \right] \end{array} \right],$$

where

$$B_0 = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$

$$B_1 = \mathcal{P}_{Y_D} \int\limits_a^b N(s) \int\limits_a^b K(s,\tau) T(\tau) \widetilde{M}(\tau) d\tau ds,$$

$$B_2 = \mathcal{P}_{Y_Q} \left[\ell_1 \widetilde{M}(\cdot) - \ell M(\cdot) D^- \int\limits_a^b N(s) \int\limits_a^b K(s,\tau) T(\tau) \widetilde{M}(\tau) d\tau ds \right],$$

$$\overline{R}(x(s,\varepsilon), \bar{x}(s,\varepsilon), s, \varepsilon) := T(s)\bar{x}(s,\varepsilon) + R(x(s,\varepsilon),s,\varepsilon),$$

$$\overline{R}_1(x(\cdot,\varepsilon), \bar{x}(\cdot,\varepsilon), \varepsilon) := \ell_1 \bar{x}(\cdot,\varepsilon) + R_1(x(\cdot,\varepsilon),\varepsilon).$$

The operator B_0 acts from the Banach space $\mathbf{B}$ to the direct product of the Banach spaces $\mathbf{B}$ and $\mathbf{B}_1,$ $B_0 \colon \mathbf{B} \to \mathbf{B} \times \mathbf{B}_1.$

Using the fact that the vector constant $c_0 \in \mathbf{B_1}$ satisfies the system of equations for the generating constants (13), to find a continuous in ε solution of $x(\cdot,\varepsilon) \in \mathbf{C}(\mathcal{I},\mathbf{B}),$ $x(t,0) = 0$ of the weakly nonlinear boundary-value problem (1), (2), go to the equivalent operator system

$$x(t,\varepsilon) = \widetilde{M}(t)c(\varepsilon) + \bar{x}(t,\varepsilon),$$

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} c = -\begin{bmatrix} \mathcal{P}_{Y_D} \int_a^b N(s) \int_a^b K(s,\tau)\overline{R}(x(\tau,\varepsilon), \bar{x}(\tau,\varepsilon), \tau, \varepsilon) d\tau ds, \\ \mathcal{P}_{Y_Q} \Big[\overline{R}_1(x(\cdot,\varepsilon), \bar{x}(\cdot,\varepsilon), \varepsilon) - \\ - \ell M(\cdot) D^- \int_a^b N(s) \int_a^b K(s,\tau)\overline{R}(x(\tau,\varepsilon), \bar{x}(\tau,\varepsilon), \tau, \varepsilon) d\tau\, ds \Big] \end{bmatrix}, \qquad (15)$$

$$\bar{x}(t,\varepsilon) = \varepsilon \left(G \int\limits_a^b K(\cdot,s) \left[Z_0(s,c_0) + T(s)x(s,\varepsilon) + R(x(s,\varepsilon),s,\varepsilon) \right] ds \right)(t) +$$

$$+ M(t)Q^- \left[J_0(\cdot,c_0) + \ell_1 x(\cdot,\varepsilon) + R_1(x(\cdot,\varepsilon),\varepsilon) \right],$$

Let $B_1 \in \mathbf{GI}(\mathbf{B},\mathbf{B})$ and $B_2 \in \mathbf{GI}(\mathbf{B},\mathbf{B}_1)$. Then, by Theorem 2, because of normal solvability, the second equation of the operator system (15) is solvable if and only if its right-hand side satisfies the condition

$$\begin{bmatrix} \widetilde{\mathcal{P}}_{Y_{B_1}} & B_{12} \\ 0 & \mathcal{P}_{Y_{B_2}} \end{bmatrix} \begin{bmatrix} \mathcal{P}_{Y_D} \int_a^b N(s) \int_a^b K(s,\tau)\overline{R}(x(\tau,\varepsilon), \bar{x}(\tau,\varepsilon), \tau, \varepsilon) d\tau\, ds, \\ \mathcal{P}_{Y_Q} \Big[\overline{R}_1(x(\cdot,\varepsilon), \bar{x}(\cdot,\varepsilon), \varepsilon) - \\ -\ell M(\cdot) D^- \int_a^b N(s) \int_a^b K(s,\tau) \times \\ \times \overline{R}(x(\tau,\varepsilon), \bar{x}(\tau,\varepsilon), \tau, \varepsilon) d\tau\, ds \Big] \end{bmatrix} = 0_{2\times 1}, \qquad (16)$$

where $0_{2\times 1}$ is a dimensional zero matrix, $\widetilde{\mathcal{P}}_{Y_{B_1}} = I_{\mathbf{B}} - B_1 \mathcal{P}_{N(B_2)} B_1^-$, $B_{12} = -B_1 B_2^-$.

At

$$\begin{bmatrix} \widetilde{\mathcal{P}}_{Y_{B_1}} & B_{12} \\ 0 & \mathcal{P}_{Y_{B_2}} \end{bmatrix} \begin{bmatrix} \mathcal{P}_{Y_D} \\ \mathcal{P}_{Y_Q} \end{bmatrix} = 0_{2\times 1} \qquad (17)$$

the condition (16) will always be satisfied and, by Theorem 2, the second equation of the operator system (15) will have a family of solutions

$$c(\varepsilon) = \mathcal{P}_{N(B_0)}\hat{c} - \left[\mathcal{P}_{N(B_2)} B_1^- \;\; B_2^- \right] \times$$

$$\times \begin{bmatrix} \mathcal{P}_{Y_D} \int_a^b N(s) \int_a^b K(s,\tau)\overline{R}(x(\tau,\varepsilon), \bar{x}(\tau,\varepsilon), \tau, \varepsilon) d\tau\, ds, \\ \mathcal{P}_{Y_Q} \Big[\overline{R}_1(x(\cdot,\varepsilon), \bar{x}(\cdot,\varepsilon), \varepsilon) - \\ -\ell M(\cdot) D^- \int_a^b N(s) \int_a^b K(s,\tau)\overline{R}(x(\tau,\varepsilon), \bar{x}(\tau,\varepsilon), \tau, \varepsilon) d\tau\, ds \Big] \end{bmatrix},$$

where $\mathcal{P}_{N(B_0)} = \mathcal{P}_{N(B_2)}\mathcal{P}_{N(B_1)}$ is the projector onto the zero-space $N(B_0)$ of the operator $B_0,\ \hat{c}$ is an arbitrary element of the Banach space $\mathbf{B},\ \left[\mathcal{P}_{N(B_2)}B_1^- \ \ B_2^-\right]$ is a generalized inverse operator to the operator $B_0 = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$.

Setting $\hat{c} \equiv 0,$ when conditions (17) are satisfied, one of the solutions of the second equation of the operator system (15) will have the form

$$c(\varepsilon) = \widetilde{B}_1^- \int\limits_a^b N(s) \int\limits_a^b K(s,\tau)\overline{R}(x(\tau,\varepsilon),\bar{x}(\tau,\varepsilon),\tau,\varepsilon)d\tau\, ds +$$

$$+ \widetilde{B}_2^- \Bigg[\overline{R}_1(x(\cdot,\varepsilon),\bar{x}(\cdot,\varepsilon),\varepsilon) -$$

$$- \ell M(\cdot)D^- \int\limits_a^b N(s) \int\limits_a^b K(s,\tau)\overline{R}(x(\tau,\varepsilon),\bar{x}(\tau,\varepsilon),\tau,\varepsilon)d\tau\, ds\Bigg],$$

where $\widetilde{B}_1^- = -\mathcal{P}_{N(B_2)}B_1^-\mathcal{P}_{Y_D},\ \widetilde{B}_2^- = -B_2^-\mathcal{P}_{Y_Q}.$

Thus, if conditions (17) are satisfied, the operator system (15) will have the form

$$x(t,\varepsilon) = \widetilde{M}(t)c(\varepsilon) + \bar{x}(t,\varepsilon),$$

$$c(\varepsilon) = \widetilde{B}_1^- \int\limits_a^b N(s) \int\limits_a^b K(s,\tau)\overline{R}(x(\tau,\varepsilon),\bar{x}(\tau,\varepsilon),\tau,\varepsilon)d\tau\, ds +$$

$$+ \widetilde{B}_2^- \Bigg[\overline{R}_1(x(\cdot,\varepsilon),\bar{x}(\cdot,\varepsilon),\varepsilon) - \tag{18}$$

$$- \ell M(\cdot)D^- \int\limits_a^b N(s) \int\limits_a^b K(s,\tau)\overline{R}(x(\tau,\varepsilon),\bar{x}(\tau,\varepsilon),\tau,\varepsilon)d\tau\, ds\Bigg],$$

$$\bar{x}(t,\varepsilon) = \varepsilon\left(G\int\limits_a^b K(\cdot,s)\left\{Z_0(s,c_0) + T(s)x(s,\varepsilon) + R(x(s,\varepsilon),s,\varepsilon)\right\}ds\right)(t) +$$

$$+ M(t)Q^-\left[J_0(\cdot,c_0) + \ell_1 x(\cdot,\varepsilon) + R_1(x(\cdot,\varepsilon),\varepsilon)\right],$$

By analogy with [1, 4, 5, 9], it can be shown that the operator system (18) belongs to the class of the systems for which the convergent method of simple iterations is applicable.

Theorem 4. *Suppose that the generating boundary-value problem (3), (4) under the conditions (6) has a family of generating solutions (7). Then for each element* $c_0 \in \mathbf{B_1}$, *which satisfies the system of equations for the generating constants (13), under the conditions*

$$\mathcal{P}_{N(B_0)} \neq 0, \qquad \begin{bmatrix} \widetilde{\mathcal{P}}_{Y_{B_1}} & B_{12} \\ 0 & \mathcal{P}_{Y_{B_2}} \end{bmatrix} \begin{bmatrix} \mathcal{P}_{Y_D} \\ \mathcal{P}_{Y_Q} \end{bmatrix} = 0_{2\times 1}$$

the boundary-value problem (1), (2) has at least one solution $z(t,\varepsilon) = z_0(t,c_0) + x(t,\varepsilon)$ *continuous with respect to* ε, *which turns into a generating solution* $z_0(t,c_0)$ *at* $\varepsilon = 0$. *This solution is found by converging to* $[0,\varepsilon_*] \subset [0,\varepsilon_0]$ *of the iterative process*

$$z_{k+1}(t,\varepsilon) = z_0(t,c_0) + x_{k+1}(t,\varepsilon),$$

$$x_{k+1}(t,\varepsilon) = \widetilde{M}(t)c_k(\varepsilon) + \bar{x}_{k+1}(t,\varepsilon), \quad k = 0,1,2,\ldots,$$

$$\begin{aligned} c_k(\varepsilon) = \widetilde{B}_1^- \int\limits_a^b N(s) \int\limits_a^b K(s,\tau)\overline{R}(x_k(\tau,\varepsilon),\bar{x}_k(\tau,\varepsilon),\tau,\varepsilon)d\tau\, ds + \\ + \widetilde{B}_2^- \Bigg[\overline{R}_1(x_k(\cdot,\varepsilon),\bar{x}_k(\cdot,\varepsilon),\varepsilon) - \\ - \ell M(\cdot)D^- \int\limits_a^b N(s) \int\limits_a^b K(s,\tau)\overline{R}(x_k(\tau,\varepsilon),\bar{x}_k(\tau,\varepsilon),\tau,\varepsilon)d\tau\, ds \Bigg], \end{aligned} \tag{19}$$

$$\begin{aligned} \bar{x}_{k+1}(t,\varepsilon) = \varepsilon \Bigg[G\Bigg(\int\limits_a^b K(\cdot,s) \Bigg\{ Z_0(s,c_0) + \\ + T(s)\left[\widetilde{M}(s)c_k(\varepsilon) + \bar{x}_k(s,\varepsilon)\right] + R(x_k(s,\varepsilon),s,\varepsilon)ds) \Bigg\} \Bigg)(t) + \\ + M(t)Q^- \left\{ J_0(\cdot,c_0) + \ell_1\left[\widetilde{M}(\cdot)c_k(\varepsilon) + \bar{x}_k(\cdot,\varepsilon)\right] + R_1(x_k(\cdot,\varepsilon),\varepsilon) \right\} \Bigg]. \end{aligned}$$

Remark 1. If $\mathcal{P}_{N(B_0)} \neq 0$ and $\begin{bmatrix} \widetilde{\mathcal{P}}_{Y_{B_1}} & B_{12} \\ 0 & \mathcal{P}_{Y_{B_2}} \end{bmatrix} = 0_{2\times 1}$, then the operator B_0 is d-normal. In this case, the condition (17) will always be satisfied and the second equation of the operator system (15) will be always solvable, and the generalized inverse operator B_0^- will be a right inverse operator $(B_0)_r^{-1}$ [13]. Then the boundary-value problem (1), (2) will have at least one solution that is found by means of the convergent iterative process (19), in which $B_0^- = (B_0)_r^{-1}$.

Remark 2. If $\mathcal{P}_{N(B_0)} = 0$ and $\mathcal{P}_{Y_{B_0}} = \begin{bmatrix} \widetilde{\mathcal{P}}_{Y_{B_1}} & B_{12} \\ 0 & \mathcal{P}_{Y_{B_2}} \end{bmatrix} \neq 0_{2\times 1}$, then the operator B_0 is n-normal. In this case, the generalized inverse operator B_0^- is the left inverse operator $(B_0)_l^{-1}$ and under the condition (17) the second equation of the operator system (15) is definitely solvable [13]. Then for each c_0 of the system for the generating constants (13) the boundary-value problem (1), (2) has the only solution that is found by means of a convergent iterative process (19), in which $B_0^- = (B_0)_l^{-1}$.

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Received 21.05.18