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OSCILLATORY SOLUTIONS OF SOME AUTONOMOUS PARTIAL DIFFERENTIAL EQUATIONS WITH A PARAMETER

КОЛИВНІ РОЗВ'ЯЗКИ ДЕЯКИХ АВТОНОМНИХ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ ІЗ ЧАСТИННИМИ ПОХІДНИМИ З ПАРАМЕТРОМ

L. Herrmann

Inst. Techn. Math., Czech Techn. Univ.
Karlovo nám. 13, 121 35 Praha 2, Czech Republic
e-mail: Leopold.Herrmann@fs.cvut.cz

A class of partial differential equations of evolution (stemming from the groundwater flow problems) depending on a parameter τ is studied. The existence of an open interval $\mathcal{T}^0$ of parameter τ and of a function $\tau \mapsto \Theta(\tau)$, $\Theta\colon \mathcal{T}^0 \to (0,+\infty)$, is proved with the property that any nonzero global solution $u\colon \mathbb{R}^+ \times \Omega \to \mathbb{R}$ of the equation cannot remain nonnegative (nonpositive) throughout the set $J \times \Omega$, where $J \subset \mathbb{R}^+$ is any interval the length of which is greater than $\Theta(\tau)$. In other words, such solutions are globally oscillatory and $\Theta(\tau)$ is the uniform oscillatory time. The interval $\mathcal{T}^0$ as well as the function Θ are explicitly determined.

Вивчається клас еволюційних диференціальних рівнянь з частинними похідними із параметром τ, які розглядаються в задачах течії підземних вод. Доведено існування відкритого інтервалу $\mathcal{T}^0$ параметра τ та функції $\tau \mapsto \Theta(\tau)$, $\Theta\colon \mathcal{T}^0 \to (0,+\infty)$, які задовольняють таку властивість: будь-який ненульовий глобальний розв'язок $u\colon \mathbb{R}^+ \times \Omega \to \mathbb{R}$ рівняння не може залишатися невід'ємним (недодатним) на множині $J \times \Omega$, де $J \subset \mathbb{R}^+$ — будь-який інтервал, довжина якого перевищує $\Theta(\tau)$. Іншими словами, такі розв'язки є глобально коливними, а $\Theta(\tau)$ — рівномірним коливним часом. Інтервал $\mathcal{T}^0$ та функцію Θ знайдено в явному вигляді.

1. Setting up of the problem. We are interested in the hyperbolic reaction-diffusion equation

$$\tau \frac{\partial^2 u}{\partial t^2} + 2\delta(x,\tau)\frac{\partial u}{\partial t} + L\,u + f(x,\tau,u) = 0 \tag{1}$$

for the function

$$(t,x) \mapsto u(t,x), \quad u\colon \mathbb{R}^+ \times \Omega \to \mathbb{R}, \tag{2}$$

$\tau \in \mathcal{T} = (0,\, +\infty)$ plays a role of parameter.

Here $\mathbb{R}^+ = [0,+\infty)$, $\Omega \subset \mathbb{R}^n$ is a bounded domain (with sufficiently regular boundary), L is an elliptic operator, δ and f are functions satisfying reasonable smoothness, sign and growth conditions, $f(x,\tau,0) \equiv 0$.

Throughout this paper we use the following assumptions:

Let $H = L_2(\Omega)$, $V \subset W_2^1(\Omega)$ be closed, $\overset{\circ}{W}{}_2^1(\Omega) \subset V$. We identify H with its dual H' and H' with a dense subspace of the dual V' of V, thus $V \hookrightarrow H \hookrightarrow V'$, both embeddings are continuous and dense and we are entitled to denote the duality pairing on $V' \times V$ by the same symbol $\langle\cdot,\cdot\rangle$ as the scalar product in H.

Further, let $\ell(\cdot,\cdot)$ be a bilinear continuous V-elliptic form on $V\times V$. An isomorphism $L\colon V\to\to V'$ is defined by the formula $\ell(u,v)=\langle Lu,v\rangle$, $u,v\in V$. The operator L may be viewed also as an unbounded operator, $L\colon D(L)\subset H\to H$, $D(L)=\{u\in V\mid Lu\in H\}$.

Let $\ell^+(u,v)=\ell(v,u)$ be the adjoint form with the analogously defined adjoint operator L^+ for which $\langle Lu,v\rangle=\langle u,L^+v\rangle$, $u,v\in V$.

Let L^+ possess the (so-called principal) eigenvalue λ_1 and associated (principal) eigenfunction v_1 which are both *real*, the function v_1 is bounded and positive in Ω (cf. [1, 5, 6]).

A simple example: $H=L_2(0,\pi)$, $V=\overset{\circ}{W}{}_2^1(0,\pi)$, $\ell(u,v)=\displaystyle\int_0^\pi u'v'\,dx$, $\lambda_1=1$, $v_1=\sin x$.

More generally: $H=L_2(\Omega)$, $V=\overset{\circ}{W}{}_2^1(\Omega)$, let $A(x)=(a_{jk}(x))_{j,k=1}^n$ be a matrix of functions from $C^1(\bar\Omega)$, which is symmetric and positive definite uniformly with respect to $x\in\bar\Omega$, $B(x)==(b_j(x))_{j=1}^n$ be a vector of functions from $C^1(\bar\Omega)$, $c\in C(\bar\Omega)$,

$$\ell(u,\,v)=\int\limits_\Omega (A(x)\mathrm{grad}\,u\,\mathrm{grad}\,v+(B(x)\mathrm{grad}\,u)v+c(x)\,uv)\,dx,$$

hence

$$L\,u=-\mathrm{div}\,(A(x)\mathrm{grad}\,u)+B(x)\mathrm{grad}\,u+c(x)\,u,$$

$$L^+\,z=-\mathrm{div}\,(A(x)\mathrm{grad}\,z)-\mathrm{div}\,(B(x)\,z)+c(x)\,z.$$

2. Example of application. The problem may be considered an abstract extension of the problem studied in [12] — oscillations for an equation arising in groundwater flow with relaxation time.

A conventional form of groundwater flow equation (e. g., [11]) is obtained by means of the combination of the conservation law and the classical constitutive relation — Darcy's law. Using the Cattaneo approach [3] Bodvarsson [2] modified the Darcy law by adding a linear inertia term proportional to the time derivative of the fluid-phase flux density w:

$$\tau\,\frac{\partial w}{\partial t}+w=-\rho\,K\mathrm{grad}\,h,$$

where h is the hydraulic head, K the hydraulic conductivity, ρ the density of the fluid phase, τ the relaxation time accounting for inertial effects. The groundwater equation assumes then the form (see [12])

$$\tau\,\rho\,S_s\,\frac{\partial^2 u}{\partial t^2}+(\rho\,S_s+\tau\,D)\,\frac{\partial u}{\partial t}-\rho\,\mathrm{div}\;(K\mathrm{grad}\,u)+D\,u=0,$$

u is the hydraulic head (up to a constant), S_s the specific storativity, D a positive function of space coordinates. For another approach based on the Biot concepts see [13].

The so-called *relaxation time* τ is a measure of inertia effects incorporated into the final model — the flux does not start at time t when the gradient is imposed but at time $t+\tau$:

$$\tau\,\frac{\partial w}{\partial t}+w\approx\tau\,\frac{w(t+\tau,\cdot)-w(t,\cdot)}{\tau}+w(t,\cdot)=w(t+\tau,\cdot).$$

3. Global solutions. We assume that functions are smooth enough to ensure *the local result:* for any $(u_0, u_1) \in V \times H$ the initial-boundary value problem given by Eq. (1) and by initial conditions $u(0,\cdot) = u_0$ and $\dfrac{\partial u}{\partial t}(0,\cdot) = u_1$ has a unique solution $(t,x) \mapsto u(t,x)$ with finite energy defined on the maximal interval, that is, $u \in C([0,\, t_{\max}), V) \cap C^1([0, t_{\max}), H)$.

Sign, growth etc. conditions are assumed to make it possible *the extension of solutions* (e. g., by the à priori estimates method). Let $\mathcal{U}$ be the set of solutions u for which $t_{\max} = +\infty$ (the so-called *global* solutions) and $\mathcal{U} \neq \varnothing$. The next theory applies to whatever global solution $u \in \mathcal{U}$, provided it exists, this is why we are dispensing with specifying more detailed assumptions ensuring the global existence.

4. Two fundamental auxiliary functions. Main auxiliary tools in the proof are the so-called summit function and the universal comparison function introduced in [7]. These functions proved to be a useful device in developing a new method for the study of oscillatory properties of ordinary, partial and abstract differential equations (see, e. g., [8, 9, 10], for the idea see [4, 14]).

Let us denote

$$\mathcal{O} = \left\{ (q,p) \in \mathbb{R}^2 \mid q > 0, p > -\sqrt{q} \right\}.$$

The set $\mathcal{O}$ is the set of couples $(q,p) \in \mathbb{R}^2$ for which solutions of the equation $\ddot{u} + 2p\dot{u}^+ + qu = 0$, $t \in \mathbb{R}$, admit positive local maxima. Here $\dot{u}^+(t) = \max\{\dot{u}(t),\, 0\}$. The *summit function* is a function $(q,p) \mapsto \vartheta^q_p$, $\vartheta^q_p \colon \mathcal{O} \to \mathbb{R}$, that to any (q,p) assigns the first positive t where the maximum of the solution satisfying $u(0) = 0$, $\dot{u}(0) = c (> 0)$ is attained (the value of ϑ is independent of c). The summit function is continuous on $\mathcal{O}$ and monotonically decreasing in each variable while the other is fixed. Its explicit form is:

$$\vartheta^q_p = \begin{cases} \dfrac{\pi}{\sqrt{q-p^2}} + \dfrac{1}{\sqrt{q-p^2}} \arctan \dfrac{\sqrt{q-p^2}}{p}, & -\sqrt{q} < p < 0, \\[2ex] \dfrac{\pi}{2\sqrt{q}}, & p = 0, \\[2ex] \dfrac{1}{\sqrt{q-p^2}} \arctan \dfrac{\sqrt{q-p^2}}{p}, & 0 < p < \sqrt{q}, \\[2ex] \dfrac{1}{\sqrt{q}}, & p = \sqrt{q}, \\[2ex] \dfrac{1}{\sqrt{p^2-q}} \operatorname{argtanh} \dfrac{\sqrt{p^2-q}}{p}, & p > \sqrt{q}. \end{cases}$$

Further, the *universal comparison function* C is a real function $(t,q,p,n) \mapsto C(t,q,p,n)$ defined for $t \in \mathbb{R}$, $(q,p) \in \mathcal{O}$ and $(q,n) \in \mathcal{O}$ and possessing as a function $t \mapsto c(t) = C(t,q,p,n)$ with q, p, n fixed, the following properties (remember that $\dot{c}^{\pm} = \max\{\pm\dot{c}(t), 0\}$) (see Fig. 1):

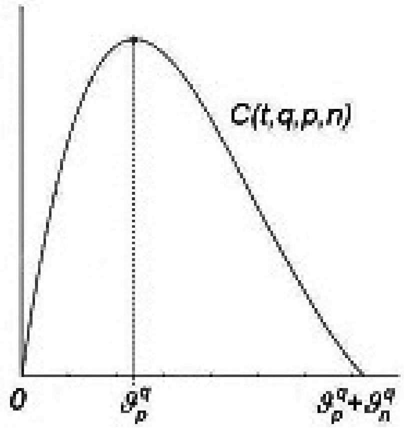

Fig. 1. Universal comparison function.

$$c \in C^2\left(\left[0, \vartheta_p^q + \vartheta_p^q\right]\right), \quad \ddot{c} + 2\left(p\dot{c}^+ + n\dot{c}^-\right) + qc = 0, \quad t \in \left[0, \vartheta_p^q + \vartheta_p^q\right],$$

$$c(0) = c\left(\vartheta_p^q + \vartheta_p^q\right) = 0, \quad c(t) > 0, \quad t \in \left(0, \vartheta_p^q + \vartheta_p^q\right),$$

$$\dot{c}(0) = 1, \quad \dot{c}(t) > 0, \quad t \in \left[0, \vartheta_p^q\right), \quad \dot{c}\left(\vartheta_p^q\right) = 0,$$

$$\dot{c}(t) < 0, \quad t \in \left(\vartheta_p^q, \vartheta_p^q + \vartheta_p^q\right], \quad \dot{c}\left(\vartheta_p^q + \vartheta_p^q\right) = -\exp\left(-p\vartheta_p^q + n\vartheta_p^q\right).$$

(See [7] for the explicit formula of c.)

Fundamental lemma. *Let q, M and m be constants such that*

$$q > 0, \quad -\sqrt{q} < m \leq M < \sqrt{q}.$$

In other terms: let

$$(q, -M) \in \mathcal{O}, \quad (q, m) \in \mathcal{O}.$$

Let $J \subset \mathbb{R}^+$ be an interval of the length

$$|J| > \vartheta_{-M}^q + \vartheta_m^q.$$

Then there is a constant $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and any interval $J_1 = [t_1, t_2] \subseteq J$, $|J| \geq |J_1| = \vartheta_{-M}^{q-\varepsilon} + \vartheta_m^{q-\varepsilon}$ there exists a function $t \mapsto \gamma(t)$ with the properties

$$\gamma \in C^2([t_1, t_2]), \tag{2a}$$

$$\gamma > 0 \quad \textit{in} \quad (t_1, t_2), \quad \gamma(t_1) = \gamma(t_2) = 0, \tag{2b}$$

$$\dot{\gamma}(t_1) > 0, \quad \dot{\gamma}(t_2) < 0, \tag{2c}$$

and satisfying the equation

$$\ddot{\gamma} + 2\left(-M\dot{\gamma}^+ + m\dot{\gamma}^-\right) + (q - \varepsilon)\gamma = 0 \quad \textit{in} \quad (t_1, t_2).$$

For the proof it is sufficient to choose ε_0 such that

$$(q-\varepsilon_0, -M) \in \mathcal{O}, \quad (q-\varepsilon_0, m) \in \mathcal{O},$$

that is,

$$q-\varepsilon_0 > 0, \quad -\sqrt{q-\varepsilon_0} < m \le M < \sqrt{q-\varepsilon_0},$$

to use the fact that the function $(q,p) \mapsto \vartheta_p^q$ is monotonically decreasing in q for p fixed, hence, for any $\varepsilon \in (0, \varepsilon_0)$,

$$|J| \ge |J_1| = \vartheta_{-M}^{q-\varepsilon} + \vartheta_m^{q-\varepsilon} > \vartheta_{-M}^q + \vartheta_m^q,$$

and finally to define

$$\gamma(t) = C(t-t_1, q-\varepsilon, -M, m),$$

where C is the universal comparison function.

Corollary. *Let $\tau > 0$ be fixed. Let q, M and m be constants such that for*

$$q(\tau) = \frac{q}{\tau}, \quad M(\tau) = \frac{M}{\tau}, \quad m(\tau) = \frac{m}{\tau} \tag{3}$$

the following holds:

$$(q(\tau), -M(\tau)) \in \mathcal{O}, \quad (q(\tau), m(\tau)) \in \mathcal{O}. \tag{4}$$

Let $J \subset \mathbb{R}^+$ be any interval of the length

$$|J| > \vartheta_{-M(\tau)}^{q(\tau)} + \vartheta_{m(\tau)}^{q(\tau)}.$$

Then there exists a constant $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and any interval $J_1 = [t_1, t_2] \subseteq J$ of the length $\vartheta_{-M(\tau)}^{q(\tau)-\varepsilon/\tau} + \vartheta_{m(\tau)}^{q(\tau)-\varepsilon/\tau}$ there exists a function $t \mapsto \gamma(t)$ with the properties (2) and satisfying the equation

$$\tau\ddot{\gamma} + 2\left(-M\dot{\gamma}^+ + m\dot{\gamma}^-\right) + (q-\varepsilon)\gamma = 0 \quad \textit{in} \quad (t_1, t_2). \tag{5}$$

5. Main results.

Theorem. *Let the following assumptions be satisfied:*

Let there exist nonnegative constants α_0 and β_1 and a positive constant α_1 such that

$$0 \le \alpha_0 \le \delta(x,\tau) \le \alpha_1 + \beta_1\tau, \quad x \in \bar{\Omega}, \quad \tau \in \mathcal{T}. \tag{6}$$

Let there exist constants f_+ and f_- such that

$$\lambda_1 + f_\pm > 4\alpha_1\beta_1, \quad \begin{cases} f(x,\tau,u) \ge f_+u, & u \ge 0, \\ f(x,\tau,u) \le f_-u, & u \le 0, \end{cases} \quad x \in \bar{\Omega}, \quad \tau \in \mathcal{T}. \tag{7}$$

Then there exist open intervals $\mathcal{T}_+^0 \subset \mathcal{T}$ and $\mathcal{T}_-^0 \subset \mathcal{T}$, respectively, of parameter τ such that for any $\tau \in \mathcal{T}_\pm^0$ any non identically zero global solution $u \in \mathcal{U}$ cannot remain nonnegative (non-positive, respectively) throughout the set $J \times \Omega$, where $J \subset \mathbb{R}^+$ is any interval the length $|J|$ of which is greater than

$$\vartheta_{-M(\tau)}^{q_+(\tau)} + \vartheta_{m(\tau)}^{q_+(\tau)} \quad \left(\vartheta_{-M(\tau)}^{q_-(\tau)} + \vartheta_{m(\tau)}^{q_-(\tau)},\ \textit{respectively}\right),$$

where

$$q_\pm(\tau) = \frac{\lambda_1 + f_\pm}{\tau}, \quad M(\tau) = \frac{\alpha_1}{\tau} + \beta_1, \quad m(\tau) = \frac{\alpha_0}{\tau}.$$

In other words, let $u \in \mathcal{U}$ be any global solution, then

$$u \equiv 0 \quad \textit{on} \quad \mathbb{R}^+ \times \Omega$$

if either

$$u \geq 0, \quad t \in J, \quad J \subset \mathbb{R}^+, \quad x \in \Omega, \quad |J| > \vartheta_{-M(\tau)}^{q_+(\tau)} + \vartheta_{m(\tau)}^{q_+(\tau)} \quad (\textit{if } \tau \in \mathcal{T}_+^0),$$

or

$$u \leq 0, \quad t \in J, \quad J \subset \mathbb{R}^+, \quad x \in \Omega, \quad |J| > \vartheta_{-M(\tau)}^{q_-(\tau)} + \vartheta_{m(\tau)}^{q_-(\tau)} \quad (\textit{if } \tau \in \mathcal{T}_-^0).$$

Moreover, there exists an interval $\mathcal{T}^0 \subset \mathcal{T}$ of parameter τ and a function $\tau \mapsto \Theta(\tau)$, $\Theta\colon \mathcal{T}^0 \to \to (0, +\infty)$, such that the implication is valid:

$$\tau \in \mathcal{T}^0, \quad |J| > \Theta(\tau) \implies \begin{cases} \operatorname{meas}\{(t,x) \in J \times \Omega \mid u(t,x) > 0\} > 0, \quad \textit{and simultaneously} \\ \operatorname{meas}\{(t,x) \in J \times \Omega \mid u(t,x) < 0\} > 0. \end{cases}$$

In terms of [9]: for $\tau \in \mathcal{T}^0$ Eq. (1) is uniformly globally oscillatory with the oscillatory time $\Theta(\tau)$.

Proof. 1. For a $\tau \in \mathcal{T}$ assume that $u \in \mathcal{U}$, u is nonnegative on $J \times \Omega$, $J \subset \mathbb{R}^+$ is any interval with the length $|J| > \Theta_+(\tau)$, where

$$\Theta_+(\tau) = \vartheta_{-M(\tau)}^{q_+(\tau)} + \vartheta_{m(\tau)}^{q_+(\tau)},$$

$q_+(\tau)$, $M(\tau)$ and $m(\tau)$ are defined as in (3) with

$$q = q_+ = \lambda_1 + f_+, \quad M = \alpha_1 + \beta_1\tau, \quad m = \alpha_0.$$

We prove that then necessarily $u \equiv 0$ on $J \times \Omega$ and hence on $\mathbb{R}^+ \times \Omega$.

2. We verify that assumptions (4) of Corollary from Sect. 4 are fulfilled. First, $q_+ > 0$ since

$$\lambda_1 + f_+ > 4\alpha_1\beta_1 \geq 0 \tag{8}$$

by (7). In the case $\beta_1 = 0$ the assumption (4) is ensured for

$$\tau \in \mathcal{T}_+^0 = \left(\frac{\alpha_1^2}{\lambda_1 + f_+}, +\infty\right).$$

If $\beta_1 > 0$ it is easy to check (we repeat partly the idea of the proof from [12] for the sake of completeness) that the validity of the quadratic inequality in τ

$$\beta_1^2\tau^2 + [2\alpha_1\beta_1 - (\lambda_1 + f_+)]\,\tau + \alpha_1^2 < 0 \tag{9}$$

ensures that (4) is true. In view of (8) the discriminant is positive,

$$D_+ = (\lambda_1 + f_+)\,(\lambda_1 + f_+ - 4\alpha_1\beta_1) > 0,$$

and the inequality (9) holds for

$$\tau \in \mathcal{T}_+^0 = \left(\tau_+^1, \tau_+^2\right), \quad \tau_+^{1,2} = \frac{-2\alpha_1\beta_1 + \lambda_1 + f_+ \pm \sqrt{D_+}}{2\beta_1^2},$$

where the roots $\tau_+^{1,2}$ of the corresponding quadratic equation fulfil

$$\tau_+^1 + \tau_+^2 = \frac{-2\alpha_1\beta_1 + \lambda_1 + f_+}{\beta_1^2}, \quad \tau_+^1\tau_+^2 = \frac{\alpha_1^2}{\beta_1^2},$$

hence they are positive. We set

$$\mathcal{T}_+^0 = \left(\tau_+^1, \tau_+^2\right).$$

By Corollary from Sect. 4 there exists a constant $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and any interval $J_1 = [t_1, t_2] \subseteq J$ of the length $\vartheta_{-M(\tau)}^{q(\tau)-\varepsilon/\tau} + \vartheta_{m(\tau)}^{q(\tau)-\varepsilon/\tau}$ we have a function $t \mapsto \gamma(t)$ with the properties (2) and (5).

3. We use $\gamma(t)\,v_1(x)$ as a *test function* in Eq. (1). In virtue of (2a) and (2b)

$$0 = \tau \int\limits_{\Omega} [\dot\gamma(t_1)u(t_1,x) - \dot\gamma(t_2)\,u(t_2,x)]\,v_1(x)\,dx +$$

$$+ \int\limits_{t_1}^{t_2}\int\limits_{\Omega} \left[\left(\tau\ddot\gamma - 2\delta(x,\tau)(\dot\gamma^+ - \dot\gamma^-) + \lambda_1\gamma\right)uv_1 + f(x,\tau,u)\gamma v_1\right] dx\,dt.$$

Estimating, for u nonnegative in $J_1 \times \Omega$, we get in view of (6), (7) and (2c) that

$$0 \ge \int\limits_{t_1}^{t_2}\int\limits_{\Omega} \left[\tau\ddot\gamma + 2(-\alpha_1 - \beta_1\tau)\dot\gamma^+ + 2\alpha_0\dot\gamma^- + (\lambda_1 + f_+)\,\gamma\right]\,u\,v_1\,dx\,dt =$$

$$= \varepsilon\int\limits_{t_1}^{t_2}\int\limits_{\Omega} \gamma u\,v_1\,dx\,dt. \tag{10}$$

4. In virtue of $\varepsilon > 0$, $v_1 > 0$ on Ω, $\gamma > 0$ on $J_1 = (t_1, t_2)$ and inequality (10) we get $u \equiv 0$ on $J_1 \times \Omega$, hence (due to the unique solvability of the initial-boundary value problem) on $\mathbb{R}^+ \times \Omega$.

5. For u nonpositive the proof goes along the same lines by replacing f_+, D_+, $\tau_+^{1,2}$, $\mathcal{T}_+^0$ and $\Theta_+(\tau)$, respectively, by f_-, D_-, $\tau_-^{1,2}$, $\mathcal{T}_-^0$ and $\Theta_-(\tau)$, respectively. We set

$$\mathcal{T}_-^0 = \left(\tau_-^1, +\infty\right) \quad \text{if} \quad \beta_1 = 0, \quad \mathcal{T}_-^0 = \left(\tau_-^1, \tau_-^2\right) \quad \text{if} \quad \beta_1 > 0.$$

If $\beta_1 = 0$ we define

$$\mathcal{T}^0 = \left(\max\left\{\tau_-^1, \tau_+^1\right\}, +\infty\right).$$

If $\beta_1 > 0$ it is easy to verify that $\tau_-^1 < \tau_+^2$ provided that $f_- \leq f_+$ and $\tau_+^1 < \tau_-^2$ provided that $f_+ \leq f_-$. Hence, the intersection of intervals $\mathcal{T}_+^0$ and $\mathcal{T}_-^0$ is nonempty and we put

$$\mathcal{T}^0 = \mathcal{T}_+^0 \cap \mathcal{T}_-^0 = \left(\tau_-^1, \tau_+^2\right) \quad \text{or} \quad \left(\tau_+^1, \tau_-^2\right),$$

according to whether $f_- \leq f_+$ or $f_+ \leq f_-$. Finally, we define

$$\Theta(\tau) = \max\left\{\Theta_+(\tau), \Theta_-(\tau)\right\}.$$

The theorem is proved.

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