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MULTIPLE SOLUTIONS OF BOUNDARY-VALUE PROBLEMS FOR HAMILTONIAN SYSTEMS

КРАТНІ РОЗВ'ЯЗКИ ГРАНИЧНИХ ЗАДАЧ ДЛЯ ГАМІЛЬТОНОВИХ СИСТЕМ

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We consider two-point boundary-value problems for Hamiltonian system of the form $x' = f(k, y)$, $y' = g(x, \lambda)$, where k and λ are parameters. We estimate the number of solutions, both positive and oscillatory, for the boundary-value problems. Our main tool is the phase plane analysis combined with evaluations of time map functions. The bifurcation diagram and solution curves for Hamiltonian system are constructed. Examples are considered illustrating bifurcations with respect to the parameters k and λ.

Розглянуто двоточкову граничну задачу для гамільтонової системи вигляду $x' = f(k, y)$, $y' = g(x, \lambda)$, де k і λ — параметри. Наведено оцінку кількості додатних та осцилюючих розв'язків граничної задачі. Основним засобом є аналіз фазової площини та обчислення функцій часового відображення. Розглянуто біфуркаційні діаграми та криві розв'язків гамільтонової системи. Наведено приклади біфуркацій відносно параметрів k і λ.

1. Introduction. The theory of boundary-value problems for ordinary differential equations is a relatively well developed branch of mathematical analysis. There are books [1, 2, 4 – 6] devoted to the subject and chapters in the books concerning general theory of differential equations and dynamical systems. The classical problems of the theory are the existence of solutions, uniqueness of solutions, properties of solutions. It appears that often the problem of estimating the number of solutions arises in applications. In the vast literature on the subject this is one of less investigated items.

Often the two-point boundary-value problems are considered. The main tools in studying multiplicity of solutions are the theory of nonlinear eigenvalues and the bifurcation theory. The use of properties of solutions in phase spaces is rare. In the current article we consider certain generalization of the Dirichlet problem for the second order nonlinear differential equation. The problem of multiple solutions is studied. The important tool is the phase plane analysis. The two-dimensional system we deal with contains both nonlinearities and the two parameters. Under the change of parameters the structure of the phase plane changes substantially generating more solutions of the boundary-value problem (BVP in short). The system is of Hamiltonian type and the phase plane is a union of level sets of Hamiltonian. This allows to trace the changes in the structure of phase planes and results in estimations of the number of solutions of BVP.

We are looking for both positive and oscillatory solutions. In the study of positive solutions we consider auxiliary Cauchy problems and employ properties of the first zero function (the time map function). Our technique and partially our approach relate to those used in below listed articles.

We are interested in periodic solutions and our treatment follow the article [8] where the similar system was considered. We use much of the results and techniques concerning the first zero function as in [7, 9, 14]. Our results relate also to the earlier paper by the author devoted to Hamiltonian system generated by Trott curves [13] where the estimates of the number of oscillatory solutions to the Dirichlet BVP were given.

The paper has the following structure. In Section 2 we describe the system of differential equations and provide some useful information on the first zero functions for auxiliary initial value problems. Some definitions are given also.

In Section 3 we consider the Hamiltonian system of differential equations of the form $x' = -y(y^2 - k^2)$, $y' = -x(x^2 - \lambda^2)$, where k and λ are parameters. We analyze the problem with respect to the number of positive and oscillatory solutions.

In the final section we summarize the results and make conclusions.

2. Preliminary results and time-map formulae. We study Hamiltonian systems

$$\begin{aligned} x' &= -\frac{\partial H(x,y)}{\partial y}, \\ y' &= \frac{\partial H(x,y)}{\partial x} \end{aligned} \tag{1}$$

with the specific Hamiltonian functions given below. These systems are integrable and the level curves of Hamiltonians form the phase planes for related systems.

Namely, we consider the two-dimensional nonlinear system

$$\begin{aligned} x' &= f(k,y), \\ y' &= g(x,\lambda), \end{aligned} \tag{2}$$

where the right-hand sides are certain cubic polynomials, and $k > 0$, $\lambda > 0$ are parameters.

It will be shown below that the phase portrait of the system undergoes essential changes under the change of parameters. We are interested mainly in boundary-value problems on a fixed interval. Our goal is to trace changes that are made to the system and especially to the number of solutions of BVP if parameters k and λ vary.

Recall that (due to [3]) the appearance of a topologically nonequivalent phase portrait under variation of parameters is called a bifurcation.

In what follows we get the estimates of the number of solutions of two-point boundary-value problems. We are interested in oscillatory solutions and in positive solutions as well. Our technique is based on the phase plane analysis. To study positive solutions we will use the first zero function. The first zero function is denoted $t_1(\alpha)$ where t_1 is the first zero of a solution $x(t)$ depending on a parameter α and having zero at $t = 0$.

Since we are interested in solutions of the problem (2), (5) we mean by the first zero function (usually called also by time map function) the minimal time needed for $x(t)$, $x(0) = 0$, $y(0) = \alpha$ to vanish again.

For definitions and discussion and useful formulas related to the so called time map function, one can consult the paper [9]. When treating the oscillatory solutions we analyze corresponding regions of a phase plane that contain critical points and form the so-called period annuli. One of the standard stages in this analysis is the linearization around critical points.

Critical points of system (2) are to be determined from the relationships

$$\begin{aligned} f(k,y) &= 0, \\ g(x,\lambda) &= 0. \end{aligned} \tag{3}$$

Denote critical points of system (2) $(x_i; y_i)$.

We look for solutions of BVP (2), (5) in some neighborhoods of critical points. For critical points located on the vertical axis $x = 0$ we consider segments of the form

$$x(0) = 0, \quad y(0) = \gamma. \tag{4}$$

The estimates of the number of solutions for the Dirichlet BVP

$$x(0) = 0, \quad x(1) = 0 \tag{5}$$

are obtained then by using phase-plane analysis and linearization at critical points. This approach was used in [10, 12, 13].

The condition (5) can be replaced by the symmetrical one $y(0) = 0$, $y(1) = 0$. A similar proposition about the number of solutions can be formulated for this case also.

Suppose values of k and λ are given and let $x(t,\gamma)$ stand for the first component of a solution $(x(t), y(t))$ of the problem (2), (4). Denote by $t_i(\gamma)$ the ith zero of $x(t,\gamma)$. Then $t_1(\gamma)$ is the first zero function.

Suppose that k and λ are flexible. We denote $x(t,\gamma,k,\lambda)$ a solution of (2) with the initial conditions (4) and $U(k,\lambda,\gamma)$ the first zero function (time-map) for this solution (the first point t where $x(t)$ vanishes).

For the Dirichlet boundary conditions (5) the relation

$$U(k,\lambda,\gamma) = 1 \tag{6}$$

defines *the solution surface* for BPV (2), (4).

By a *solution surface* we mean a set of parameters k, λ, γ such that the relation (6) is satisfied. For these particular values of parameters a solution to the problem (2), (5) exists.

The solution surface for this problem seems to be somewhat complicated object. We try to consider it drawing attention to projections on (k,λ), (k,γ) and (λ,γ) coordinate planes. Consequently, we consider three types of level curves. First, we treat the relation (6) with fixed parameter γ

$$U(k,\lambda,\gamma_i) = 1, \tag{7}$$

then the parameter k is fixed in (6)

$$U(k_j,\lambda,\gamma) = 1, \tag{8}$$

and, finally, we look for (k,γ) level curves defined by (6) where the parameter λ is fixed

$$U(k,\lambda_s,\gamma) = 1. \tag{9}$$

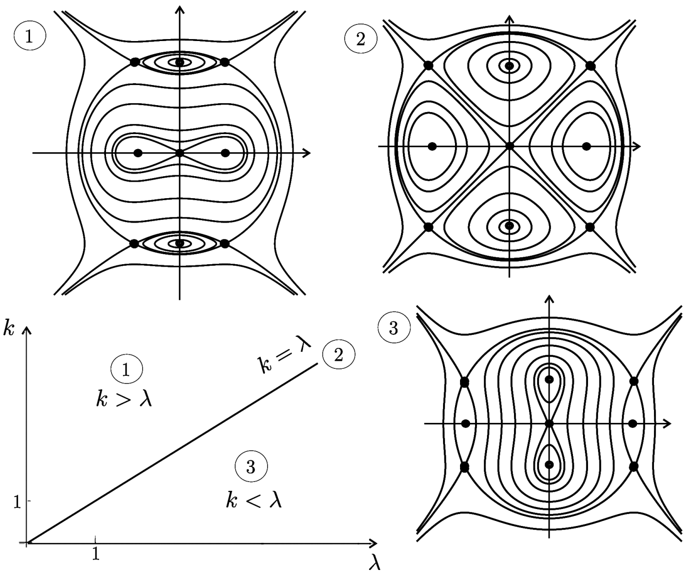

Fig. 1. The bifurcation diagram of system (10), $k > 0$, $\lambda > 0$.

Looking at the relation (7) where γ is fixed we can found (or, at least, estimate) values of k and λ, for which the problem (2), (5) has a solution. Similarly, looking at the level curves (8) or (9) of the first zero function we are facing the following task: for given k or λ to find those values of γ that define solutions to the Dirichlet problem (2), (5). The relations (8) and (9) can be then interpreted as *solution curves* bearing important information on the number of solutions to the problem.

3. Hamiltonian system. The system we study below is the Hamiltonian system

$$\begin{aligned} x' &= -y(y^2 - k^2), \\ y' &= -x(x^2 - \lambda^2), \end{aligned} \tag{10}$$

where λ and k are parameters (λ, $k > 0$).

The respective Hamiltonian function is

$$H_1(x, y) = \frac{1}{4}\, y^4 - \frac{1}{2}\, k^2 y^2 - \frac{1}{4}\, x^4 + \frac{1}{2}\, x^2 \lambda^2 + C. \tag{11}$$

System (10) has 9 critical points, 4 of them are the points of type "center" and 5 are the points of type "saddle": $(-\lambda, -k)$, $(\lambda, -k)$, $(-\lambda, k)$, (λ, k), $(0, 0)$.

The phase portraits for different regions of parameters k and λ are depicted in Fig. 1.

We are interested in the number of solutions of system (10) that satisfy given boundary conditions, for instance the Dirichlet boundary conditions (5).

Linearization of system (10) at a critical point (x^*, y^*) yields

$$\begin{aligned} u' &= \left(-3y^{*2} + k^2\right) v, \\ v' &= \left(-3x^{*2} + \lambda^2\right) u. \end{aligned} \tag{12}$$

Consider the linearized system at the critical point $(0, k)$

$$u' = -2k^2 v, \qquad v' = \lambda^2 u. \tag{13}$$

The eigenvalues of linearized system (13) are $\mu_{1,2} = \pm\sqrt{2}k\lambda\, i$, where i is an imaginary unity and the critical point $(0, k)$ is of type "center".

System (13) can be rewritten in the form

$$u'' = -2k^2\lambda^2 u \tag{14}$$

and there is the solution

$$u(t) = \sin\sqrt{2}\,k\lambda\, t, \tag{15}$$

that satisfies the initial condition $u(0) = 0$, $u'(0) \neq 0$.

3.1. Case $k > \lambda$. Let us introduce the notions of the trivial and nontrivial period annuli that we found in the literature.

Definition 1 [11]. *We will call a period annulus associated with a central region*[1] *a trivial period annulus. Periodic trajectories of a trivial period annulus encircle exactly one critical point of the type "center".*

Definition 2 [11]. *Respectively a period annulus enclosing several (more than one) critical points will be called a nontrivial period annulus.*

Recall that a trivial period annulus contains a set of periodic trajectories that encircle exactly one critical point of the type "center". By a nontrivial period annulus we call a period annulus enclosing several (more than one) critical points. The phase portrait for hamiltonian system (10) contains four trivial period annuli around critical points of the type "center": $(0, -k)$, $(-\lambda, 0)$, $(\lambda, 0)$, $(0, k)$ and one nontrivial periodic annulus — around three critical points $(-\lambda, 0)$, $(0, 0)$, $(\lambda, 0)$; so totally there are five period annuli (Fig. 2).

Let us consider critical points of the type "center". We would like to evaluate the number of solutions of the BVP in a neighborhood of a "center".

Appropriate solutions $(u(t), v(t))$ of the linear system (13) provide approximations to solutions of the Cauchy problems (10), $x(0) = 0$, $y(0) = k \pm \epsilon$, where $\epsilon > 0$ is a small value. We suppose that ϵ is so small that $k + \epsilon < \gamma_2$, where $(0, \gamma_2)$ is a point of intersection of the "upper" heteroclinic solution with y axis. Also $k - \epsilon > \gamma_1$, where $(0, \gamma_1)$ is a point of intersection of the "lower" heteroclinic solution with y axis. Two heteroclinic trajectories are connecting two saddle points $(-\lambda, k)$, (λ, k).

Therefore, if $t(k \pm \epsilon)$ is the time needed for a point to move along phase trajectory of the system (10) from the point $(0, k \pm \epsilon)$ to $(0, k \mp \epsilon)$, then $t(k \pm \epsilon) \approx \dfrac{\tau}{2} = \dfrac{\pi}{\sqrt{2}k\lambda}$, where $\tau = \dfrac{2\pi}{\sqrt{2}\,k\lambda}$ is the period of solution (15). This is true for ϵ close to zero.

[1] *Sabatini M.* Liénard limit cycles enclosing period annuli, or enclosed by period annuli // Rocky Mount. J. Math. — 2005. — **35**, № 1. — P. 253–266.

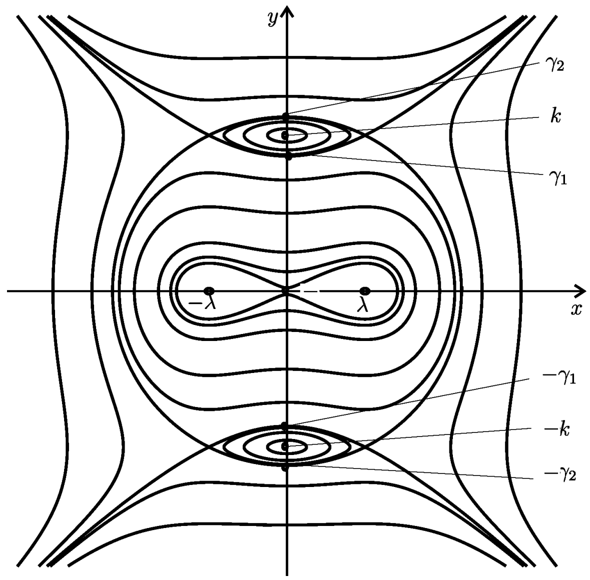

Fig. 2. The phase portrait of system (10), $k > \lambda$.

Consider solutions of system (10), $x(0) = 0$, $y(0) = k + \epsilon$, where ϵ is close to zero. Then $x(t)$ has exactly n zeros in the interval $(0, 1)$ (and $x(1) \neq 0$) if the inequalities

$$\frac{\pi n}{\sqrt{2}\,k\lambda} < 1 < \frac{\pi(n+1)}{\sqrt{2}\,k\lambda} \tag{16}$$

hold. On the other hand, if $k + \epsilon \to \gamma_2$ then the respective $x(t)$ has not zeros in the interval $(0, 1]$. Therefore by continuity arguments there are at least n solutions of the problem (10), (5). By considering solutions of (10) with the initial values $x(0) = 0$, $y(0) = k - \epsilon$, where ϵ changes from zero to $k - \gamma_1$, we get additional at least n solutions. Hence at least $2n$ solutions with the trajectories in the upper trivial annulus.

Remark 1. Similarly estimation of the number of solutions for the neighborhood of the critical point $(0, -k)$ can be obtained.

The linearization at the critical point $(0, -k)$ is the same as for the point $(0, k)$, namely, the system (13) and equation (14) for $x(t)$. Therefore we have the time $t(-k \pm \epsilon)$ needed for a point to move along phase trajectory from the point $(0, -k \pm \epsilon)$ to $(0, -k \mp \epsilon)$ and $t(-k \pm \epsilon) = \dfrac{\pi}{\sqrt{2}\,k\lambda}$. Note that $-k+\epsilon < -\gamma_1$ and $-k-\epsilon > -\gamma_2$ where $(0, -\gamma_1)$ and $(0, -\gamma_2)$ are points of intersection of the heteroclinic solutions with y axis.

Thus we arrive at the following assertion.

Proposition 1. *Let n be a positive integer such that the inequalities (16) hold. Then the Dirichlet problem (10), (5) has at least $4n$ nontrivial oscillatory solutions. The trajectories of these solutions are lying entirely in two trivial period annuli containing critical points $(0, -k)$ and $(0, k)$.*

Proof by considering both trivial period annuli separately. There are at least $2n$ solutions for the upper period annulus and at least $2n$ solutions for the lower one.

Remark 2. Looking for positive solutions of the BVP we can make use of the time map function $U(k, \lambda, \gamma)$. The graph of the time map function has U-shaped components and the number of positive solutions can be one, two or more solutions depending on whether the

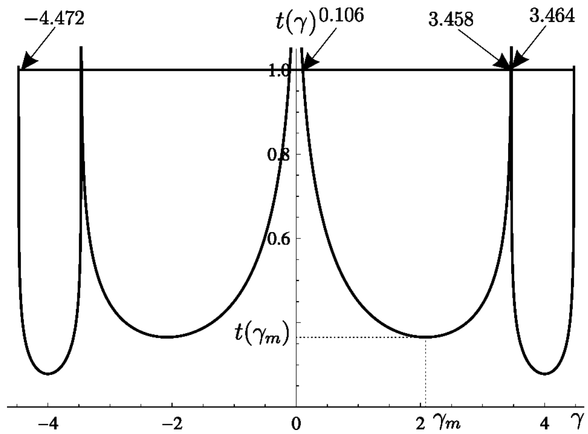

Fig. 3. The graph of first zero function of system (17) and $t(\gamma) = 1$.

graph of $U(k, \lambda, \gamma)$ has one, two or more intersections with the unity level (the unity refers to the length of the interval).

This linearization technique is not applicable for estimating the number of solution of the Dirichlet problem where the initial conditions are $\gamma \in (0, \gamma_1)$ or $\gamma \in (-\gamma_1, 0)$.

Instead we are looking at the first zero function and detecting the number of solutions of problem (10), (5).

Consider solutions $x(t, \gamma)$ for $0 < \gamma < \gamma_1$, where $t(\gamma)$ is the time needed for a point to move along phase trajectory from the point $(0, \gamma)$ to $(0, -\gamma)$. Then $t(\gamma)$ tends to $+\infty$ as γ goes to γ_1 and $t(\gamma)$ tends to $+\infty$ as γ goes to zero. Therefore there is a minimum of $t(\gamma)$ in $(0, \gamma_1)$ at some γ_m and in this case also, the time map function has a "U" shaped graph.

We can formulate the following assertion.

Proposition 2. *If* $1 < t(\gamma_m)$, *then the Dirichlet problem (10), (5) with initial condition* $x(0) = 0$, $y(0) = \gamma > 0$, *where* $\gamma \in (0, \gamma_1)$ *has no positive nontrivial solutions; if* $1 = t(\gamma_m)$, *then the Dirichlet problem (10), (5) has at least one positive solution; if* $1 > t(\gamma_m)$, *then the Dirichlet problem (10), (5) has at least two positive solutions.*

Similar analysis can be made for solutions with the initial condition $x(0) = 0, y(0) = \gamma < 0$. For $-\gamma_1 < \gamma < 0$ a point moves clockwise along a phase trajectory from the point $(0, -\gamma)$ to the point $(0, \gamma)$ and therefore the Dirichlet problem has not positive solutions with these initial data.

Example 1. Consider the system

$$\begin{aligned} x' &= -y(y^2 - 4^2), \\ y' &= -x(x^2 - 2^2). \end{aligned} \tag{17}$$

Using Proposition 1 and noting that $\dfrac{3\pi}{8\sqrt{2}} < 1 < \dfrac{4\pi}{8\sqrt{2}}$ we conclude that the problem (17), (5) has at least 12 oscillatory solutions.

Now we look for positive solutions. For $0 < \gamma < \gamma_1$ the first zero function is equal to 1 at least two times because the minimum of time map function is $t(\gamma_m) \approx 0.365$.

In Fig. 3 we see the graph of first zero function of system (17) which shows both positive and negative solutions. The arrowheads point to initial conditions corresponding to positive solutions.

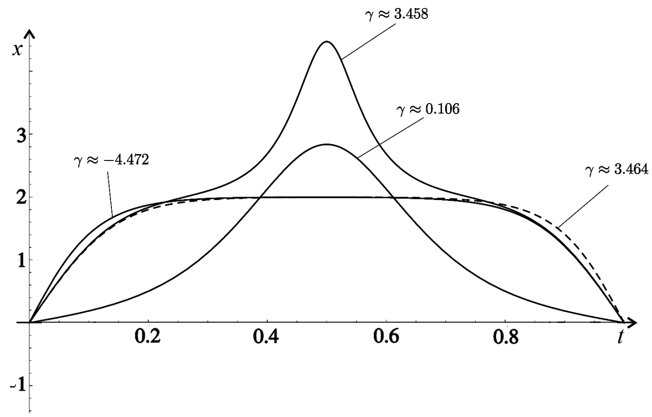

Fig. 4. Positive solutions of problem (17), (5).

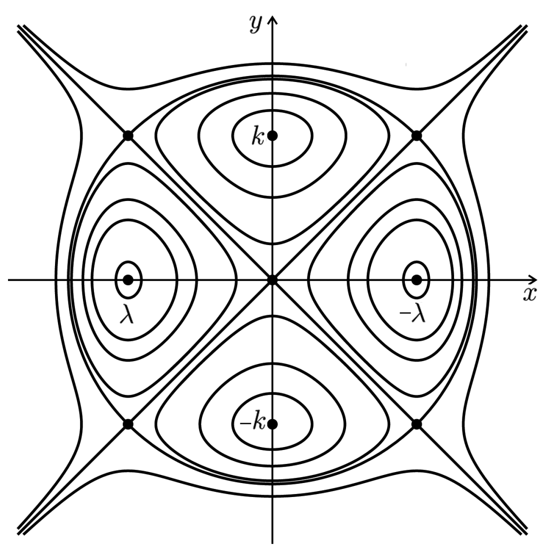

Fig. 5. The phase portrait of system (10), $k = \lambda$.

Fig. 4 shows that the Dirichlet problem (17), (5) has three positive solutions with $y(0) = \gamma \approx 0.106; 3.458; 3.464$ and since the rotation in neighborhood of the critical point $(0, -k)$ is in counter-clockwise direction, then one positive solution is for $y(0) = \gamma \approx -4.472$.

If we look for oscillatory solutions for γ in the interval $(0, \gamma_1)$ then the number of solutions is n_1, because $1 \geq n_1 t(\gamma_m) \approx 2 \cdot 0.365$, and the same number of such solutions is for γ in $(-\gamma_1, 0)$. In this example $n_1 = 2$, therefore in the interval $(-\gamma_1, \gamma_1)$ the Dirichlet problem (17), (5) has four oscillatory solutions. Totally at least 16 oscillatory solutions.

3.2. Case $k = \lambda$. The Hamiltonian system (10) contains four trivial period annuli around points of type “center”: $(0, -k)$, $(-\lambda, 0)$, $(\lambda, 0)$, $(0, k)$ (Fig. 5).

The solutions of Dirichlet boundary-value problem can be found only in the regions that surround the “center” type points $(0, k)$ and $(0, -k)$.

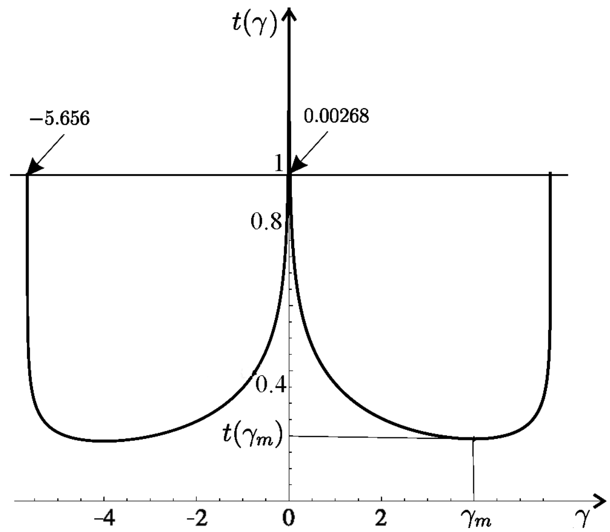

Fig. 6. The graph of first zero function of system (18) and $t(\gamma) = 1$.

Proposition 3. *Let n be a positive integer such that $\dfrac{\pi n}{\sqrt{2}\,k^2} < 1 < \dfrac{\pi(n+1)}{\sqrt{2}\,k^2}$. The Dirichlet problem (10), (5) with the initial conditions $x(0) = 0$, $y(0) = \pm k \pm \epsilon$ has at least $4n$ nontrivial oscillatory solutions.*

The proof is similar to that of Proposition 1.

Appropriate solutions $(u(t), v(t))$ of the linear system (13) provide approximations to solutions of the Cauchy problems (10), $x(0) = 0$, $y(0) = \pm k \pm \epsilon$, where $\epsilon > 0$ is a small value. For the case $k = \lambda$ one has that $u(t) = \sin \sqrt{2}\,k^2\, t$, $u(0) = 0$, $u'(0) \neq 0$. Therefore $t(\pm k \pm \epsilon) = \dfrac{\pi}{\sqrt{2}\,k^2}$. Then for a positive integer n such that $\dfrac{\pi n}{\sqrt{2}\ k^2} < 1 < \dfrac{\pi(n+1)}{\sqrt{2}\,k^2}$ the Dirichlet problems (10), (5) with the initial conditions $x(0) = 0$, $y(0) = k \pm \epsilon$ or $x(0) = 0$, $y(0) = -k \pm \epsilon$ have at least $2(n + n)$ nontrivial oscillatory solutions.

Remark 3. The number of positive solutions for the case $k = \lambda$ is two because the rotation in neighborhoods of the critical points $(0, k)$ and $(0, -k)$ is in the counter-clockwise direction.

Example 2. Consider the system

$$
\begin{aligned}
x' &= -y(y^2 - 4^2), \\
y' &= -x(x^2 - 4^2).
\end{aligned}
\tag{18}
$$

Using Proposition 3 and noting that $\dfrac{7\pi}{16\sqrt{2}} < 1 < \dfrac{8\pi}{16\sqrt{2}}$ we conclude that the problem (18), (5) has 28 nontrivial oscillatory solutions.

The graph of the first zero function for system (18) is depicted in Fig. 6.

There are one positive solution for $y(0) = \gamma \approx 0.00268$ and one solution for $y(0) = \gamma \approx \approx -5.656$ to the Dirichlet problem (18), (5) as shown in Fig. 7.

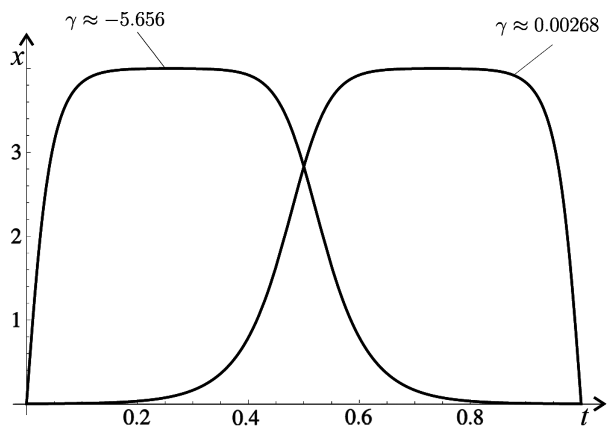

Fig. 7. Positive solutions of problem (18), (5).

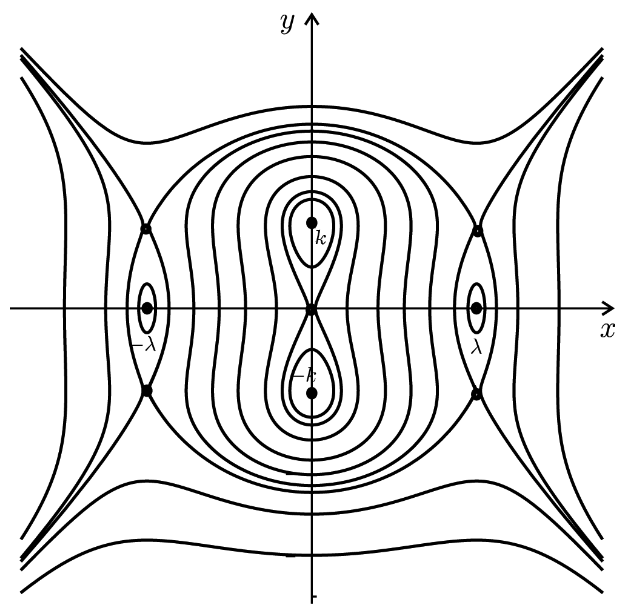

Fig. 8. The phase portrait of system (10), $k < \lambda$.

3.3. Case $k < \lambda$. The hamiltonian system (10) contains four trivial period annuli around points of the type "center": $(0,-k)$, $(-\lambda,0)$, $(\lambda,0)$, $(0,k)$ and one nontrivial period annulus around the "figure eight" region including the three critical points $(-k,0)$, $(0,0)$, $(k,0)$, and there are totally five period annuli (Fig. 8).

This case is similar to the first case where $k > \lambda$ and can be obtained from the system (10) by swapping the right-hand sides of equations. It is possible to formulate similar propositions about number of positive or oscillatory solutions.

Proposition 4. *Let there exist a positive integer n such that $\dfrac{\pi n}{\sqrt{2}\,k\lambda} < 1 < \dfrac{\pi(n+1)}{\sqrt{2}\;k\lambda}$. Then the Dirichlet problem (10), (5) has at least $4n$ nontrivial oscillatory solutions. The trajectories of these solutions are lying entirely in a "figure eight" region bounded by two homoclinic solutions surrounding the "center" points $(0,k)$ and $(0,-k)$.*

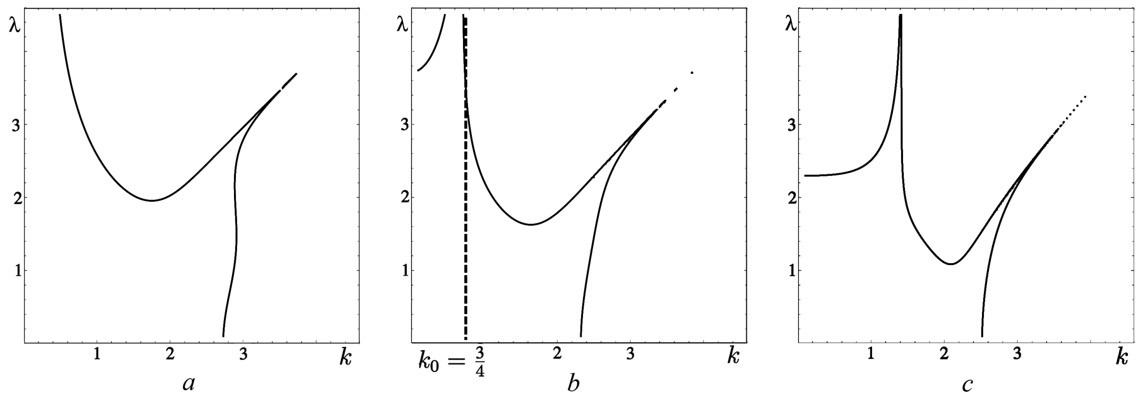

Fig. 9. The graph of time map $U(k, \lambda, \gamma_i) = 1$ for system (10), $\gamma = 0.5$ (*a*), $\gamma = 1$ (*b*) and $\gamma = 2$ (*c*).

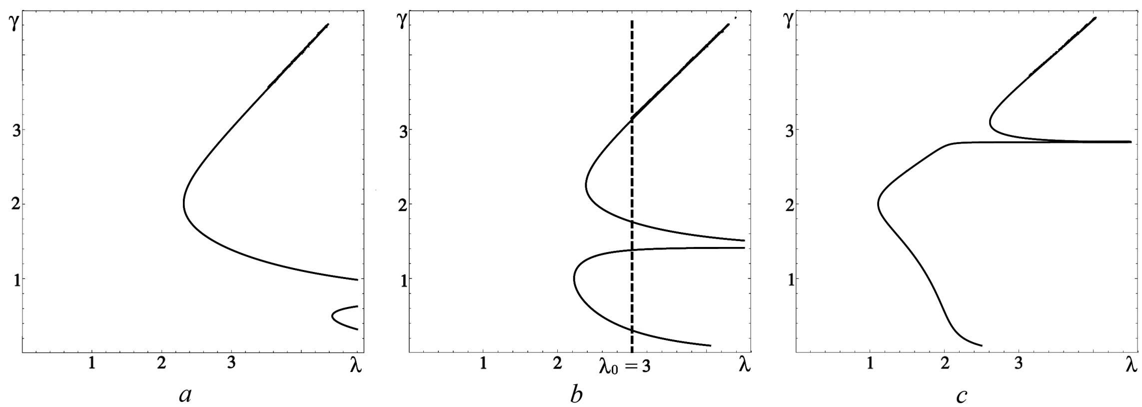

Fig. 10. The graph of time map $U(k_j, \lambda, \gamma) = 1$ for system (10), $k = 0.5$ (*a*), $k = 1$ (*b*) and $k = 2$ (*c*).

For trajectories in the nontrivial period annulus one has the following statement.

Proposition 5. *The graph of the time-map function for solutions in the nontrivial period annulus has two U-shaped components and the number of positive solutions may be one, two or no solutions in each U-shaped case. The number of solutions is equal to the number of solutions of the equation $U(k, \lambda, \gamma) = 1$, where k and λ are given and γ is from the two segments of the axis $x = 0$ lying in the nontrivial period annulus.*

3.4. Solution curves for the Hamiltonian system. The time map for Hamiltonian system (10) is a solution surface (6).

Consider level curves of the first type, where γ is a fixed parameter (Fig. 9). Fix some value k_0. Then we can determine (from Fig. 9, *b*) values of λ such that there exists a solution of BVP (10), (5) with given $\gamma = y(0)$.

Let us interpret the case of Fig. 9, *b*. The value of $\gamma = y(0)$ is fixed $(\gamma = 1)$. We look for values of (k, λ) such that there exists a solution of BVP (10), (5). Namely for $k_0 = \dfrac{3}{4}$ we find exactly one value of λ, approximately $\lambda \approx 4.499$, such that the problem (10), (5) with these λ and k_0 has a solution of BVP (10), (5) with $y(0) = \gamma = 1$.

Consider level curves of the second type, where in equation (6) the parameter k is fixed (Fig. 10).

Fix now some value of λ_0. Then we can determine the values of γ such that the problem (10), (5) is solvable for a given k.

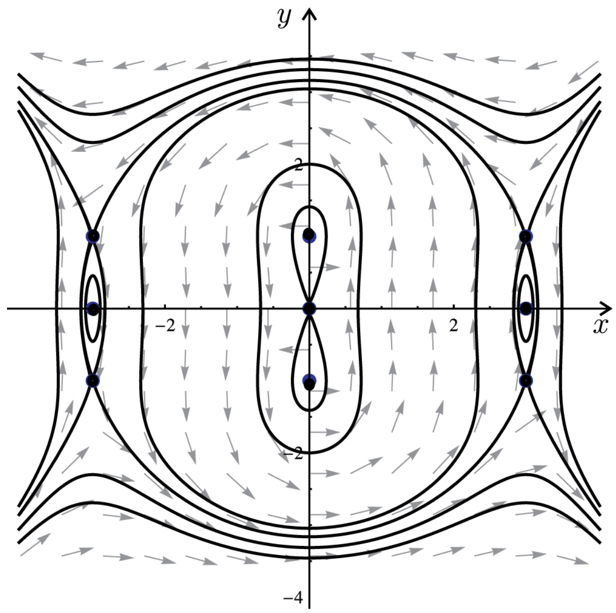

Fig. 11. The phase portrait of system (19).

Let us consider Fig. 10, *b*, where $k = 1$. Changing the value of λ we can evaluate the number of solutions of the problem (10), (5). For example, if we fix a value of λ also $\lambda_0 = 3$, we have four intersections with the straight line $\lambda_0 = 3$. These points of intersections are $\gamma_1 \approx 0.303423$, $\gamma_2 \approx 1.38128$, $\gamma_3 \approx 1.75823$, $\gamma_4 \approx 3.13218$. We can assume that the problem (10), (5) have four solutions with the initial conditions $x(0) = 0$, $y(0) = \gamma_i$, $i = 1, 2, 3, 4$.

To detect the number of solutions we use the phase-plane method. Consider the phase portrait for Hamiltonian system (10), where $k = 1$, $\lambda = 3$

$$\begin{aligned} x' &= -y\left(y^2 - 1^2\right), \\ y' &= -x\left(x^2 - 3^2\right) \end{aligned} \tag{19}$$

in Fig. 11 and solutions of the problem (19), (5) with the initial conditions $x(0) = 0$, $y(0) = \gamma_i$, $i = 1, 2, 3, 4$, in Fig. 12, *a*.

We conclude that there is one positive solution for $\gamma_1 \approx 0.303423$, and the solutions of the Dirichlet problem for γ_i, $i = 2, 3, 4$, are negative.

Consider solutions of the Dirichlet problem (19), (5) with the initial conditions $-\gamma_i$, $i = 1, 2, 3, 4$: $-\gamma_1 \approx -0.303423$, $-\gamma_2 \approx -1.38128$, $-\gamma_3 \approx -1.75823$, $-\gamma_4 \approx -3.13218$ (Fig. 12, *b*).

For the initial conditions $-\gamma_i$, $i = 2, 3, 4$, the solutions of the Dirichlet problem (19), (5) are positive.

Consider now level curves of the third type. These are curves corresponding to fixed parameter λ in equation (6) (Fig. 13).

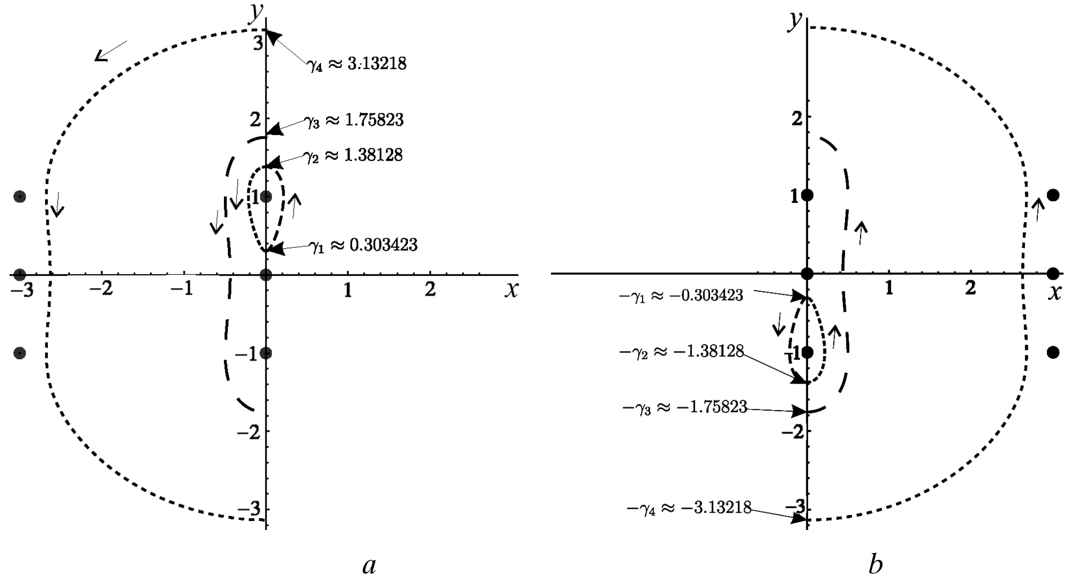

Fig. 12. The solutions of Dirichlet problem (19), (5), $\gamma > 0$ (a) and $\gamma < 0$ (b).

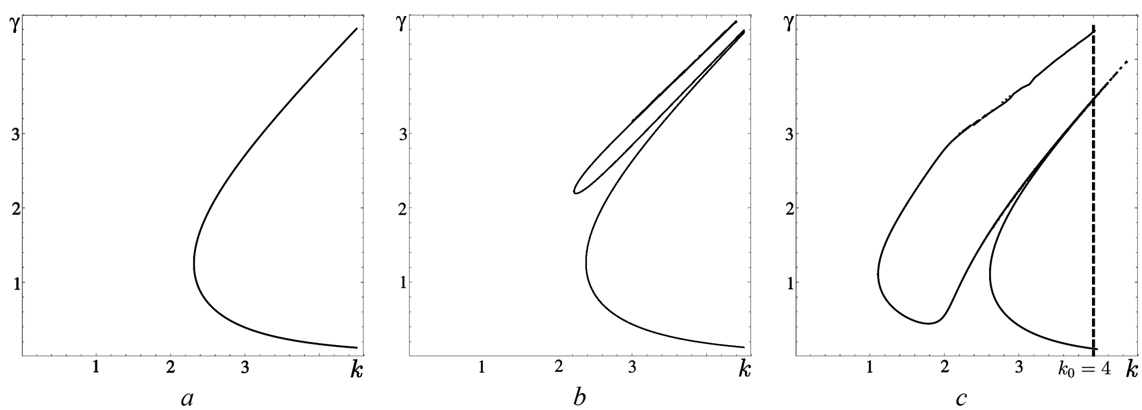

Fig. 13. The graph of time map $U(k, \lambda_s, \gamma) = 1$ of system (10), $\lambda = 0.5$ (a), $\lambda = 1$ (b) and $\lambda = 2$ (c).

Let us consider Fig. 13, c, where $\lambda = 2$ in the system (10). By changing the parameter k we get estimates of the number of solutions of the problem (10), (5) provided that calculations were carried out with sufficient precision. For instance, for $\gamma > 0$ and $k_0 = 4$ we get four intersections of the straight line $k_0 = 4$ with the level curves $U(4, 2, \gamma)$. These points of intersections are in one-to-one correspondence with the points $\gamma_1 \approx 0.106$, $\gamma_2 \approx 3.458$, $\gamma_3 \approx 3.464$, $\gamma_4 \approx 4.472$ in Fig. 3. This means that a solution $(x(t), y(t))$ with the initial values $x(0) = 0$, $y(0) = \gamma_i$, $i = 1, 2, 3, 4$, satisfies also the condition $x(1) = 0$. Therefore, there are exactly four solutions of problem (10), (5) positive or negative, where $k = 4$, $\lambda = 2$.

In a similar manner other possible variants can be considered.

The graphs of time map in Fig. 10 and Fig. 13 can be interpreted as solution curves.

4. Conclusions. In this article we have investigated the two-dimensional differential system with parameters. We show how evaluate the number of solutions of a two-point BVP by considering the time map function and studying the projections on three related coordinate planes. As a by-product, we have got descriptions of solution curves containing information about the number of solutions.

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