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THE APPLICATION OF VARIATIONAL METHODS TO SOME STATIC CONTACT PROBLEMS FOR PLIANT SHELLS OF ROTATION*

ЗАСТОСУВАННЯ ВАРІАЦІЙНИХ МЕТОДІВ В ДЕЯКИХ КОНТАКТНИХ ЗАДАЧАХ СТАТИКИ М'ЯКИХ ОБОЛОНОК ОБЕРТАННЯ

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We propose a method for finding the finite deformations of hyperelastic domal shells of rotation due to hydrostatic load. The problem is considered under the condition that the deformed surface of the shell enters the domain of perfect contact with coaxial rigid surface. The efficiency of the suggested approach is assured by taking into account the singular properties of solutions on the line of transition of shell surface from free domain to the contact one and their asymptotic behavior in vicinity of the shell pole.

For a set of examples, which illustate the advantages of the algorithm, we present the basic dependences describing stressly-deformed state of a shell.

На основі варіаційного методу пропонується розв'язок задачі про скінченні деформації гіперпружних оболонок обертання під дією гідростатичного навантаження за умови, що деформована поверхня оболонки вступає в зону ідеального контакту із співвісно розміщеною жорсткою поверхнею обертання. Ефективність запропонованого підходу зумовлена врахуванням властивостей шуканих розв'язків на лінії переходу поверхні оболонки від вільної зони до контактної та асимптотичної поведінки їх в околі полюса оболонки.

На конкретних прикладах проілюстровано можливості запропонованого алгоритму.

Consider the pliant domal shell of rotation Σ_0 in undeformed state rigidly fixed on a parallel of radius R_0 on walls of an absolutely rigid surface of rotation S so that the axes of symmetry of Σ_0 and S coincide. We suppose that the shell is made of an isotropic incompressible and highly elastic material, and its thickness h_0 is constant and sufficiently small with respect to R_0. The distance of points of the shell generatrix to the axis of symmetry in the initial state is posed by

$$\xi = \xi(s),$$

where s is the length of a meridian arc measured from the shell pole. In order to describe the shape of deformed membrane, we introduce the cylindrical coordinate system (z, η, r) whose axis Oz coincides with the symmetry axis of the considered mechanical system.

We assume that the shell is under hydrostatic pressure Q. Hence, for some values of parameters of load, the deformed shell Σ can contact with the rigid surface S, as shown in

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Fig. 1. We define the contact domain as the part of the surface of the deformed shell Σ which is in contact with the surface S. The remaining part of the shell surface will be referred as free domain.

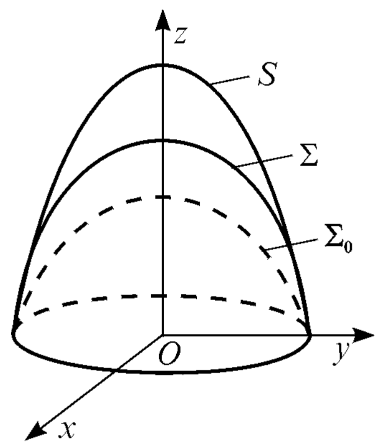

Fig. 1

Let the generatrix of the surface S have positive (negative) curvature and be described by the equation

$$r = \varphi(z). \tag{1}$$

In what follows we assume that, as the load Q increases, the contact domain grows from the side of rigid fixing of the shell on the surface S.

By virtue of the assumptions made, the geometrical and physical quantities describing the deformed state of the shell are the functions of the arc length s only and, hence, we may seek for the equations describing the meridian of the deformed shell in the following form:

$$z = z(s), \qquad r = r(s). \tag{2}$$

The equilibrium conditions for the infinitesimal element of the free domain of the deformed shell enable us to obtain the governing equations [1]

$$\begin{aligned} &\frac{dT_1}{ds} + \frac{1}{r}\frac{dr}{ds}(T_1 - T_2) = 0, \\ &\frac{T_1}{R_1} + \frac{T_2}{R_2} = Q, \qquad Q = C - Dz. \end{aligned} \tag{3}$$

The internal strains T_1 and T_2 of the deformed shell in the meridian and parallel directions are determined from the relations

$$T_i = 2h_0\lambda_3\left(\lambda_i^2-\lambda_3^2\right)\left(\frac{\partial W}{\partial I_1}+\lambda_{3-i}^2\frac{\partial W}{\partial I_2}\right), \qquad i=1,2,$$

$$\lambda_1 = \left[\left(\frac{dz}{ds}\right)^2+\left(\frac{dr}{ds}\right)^2\right]^{1/2}, \qquad \lambda_2=\frac{r(s)}{\xi(s)}, \qquad \lambda_3=\frac{1}{\lambda_1\lambda_2}=\frac{h(s)}{h_0}, \tag{4}$$

$$I_1=\lambda_1^2+\lambda_2^2+\lambda_3^2, \qquad I_2=\lambda_1^{-2}+\lambda_2^{-2}+\lambda_3^{-2}, \qquad W=W\left(I_1,I_2\right).$$

Here λ_1, λ_2 and λ_3 are the main degrees of lengthening in the directions of meridian, parallel and normal to deformed surface, respectively, W is the energy function of deformation for the shell material, I_1, I_2 are the invariants of deformations, and $h(s)$ is the thickness of the shell in the deformed state. The main radii of the curvature of the deformed shell median surface are given by the formulae

$$R_1=\lambda_1^3\Big/\left(\frac{d^2r}{ds^2}\frac{dz}{ds}-\frac{dr}{ds}\frac{d^2z}{ds^2}\right), \qquad R_2=-(r\lambda_1)\Big/\frac{dz}{ds}.$$

Let us choose the energy function of deformation in Mooney's form [2]

$$W=C_1\left(I_1-3\right)+C_2\left(I_2-3\right), \tag{5}$$

where C_1, C_2 are physical constants, which may be found experimentally.

When assuming that no friction between the shell Σ and the surface S occur, we arrive at the equilibrium equations (3) for the contact domain

$$\begin{aligned} &\frac{dT_1}{ds}+\frac{1}{r}\frac{dr}{ds}(T_1-T_2)=0,\\ &\frac{T_1}{R_1}+\frac{T_2}{R_2}=Q-Q_R, \end{aligned} \tag{6}$$

where Q_R is the pressure exerted by the surface S on contacting shell.

The account of the relation (2) between the solutions $r(s)$ and $z(s)$ in the contact domain makes the second equation from (6) auxiliary and enables us to calculate the distribution of normal pressure Q_R between the surface S and the shell Σ, using the solution $z(s)$.

Let us introduce the following dimensionless variables

$$\{s^*,r^*,z^*\}=\frac{\{s,r,z\}}{R_0}, \qquad T_i^*=T_i/\left(2h_0C_1\right), \qquad W^*=\frac{W}{C_1}, \qquad Q^*=\frac{QR_0}{(2h_0C_1)}.$$

In what follows, we use these dimensionless variables but omit the asterisk for the sake of simplicity.

Let us place the origin of coordinate system $Oz\eta r$ in the plane of the fixed contour and write the boundary values of $z(s)$ and $r(s)$ at the ends of integration interval of equilibrium equations

$$\left.\frac{dz}{ds}\right|_{s=0}=r(0)=z(s_0)=0. \tag{7}$$

Thus, we have obtained the nonlinear boundary value problem possessing an interesting feature with respect to the case of free deformation of the shell. Namely, the shell contact domain is a priori unknown and should be determined while solving the problem.

Let us apply the variational formulation of the considered problem to the construction of the approximate solution. For the free deformation of the domal shells under the hydrostatic load, the solution of relevant nonlinear boundary value problem is equivalent to finding the stationary points of the functional [3]

$$I = \int_0^{s_0} \Phi(s, r, z, r', z') ds, \tag{8}$$

$$\Phi = \left[(I_1 - 3) + \Gamma\,(I_1 - 3)\right] \xi(s) + Q r^2 \frac{dz}{ds}, \qquad \Gamma = \frac{C_2}{C_1}.$$

In the case where the deformed shell contacts with the rigid surface S with generatrix (1), we must impose on the class of admissible functions both the conditions (7) and an additional constraint of geometrical nature, which has the form of an inequality:

$$\psi(z, r) = r(s) - \varphi\,(z(s)) \leq 0,$$

where the equality holds for the contact domain of the shell.

The solution of this problem was first constructed by methods for solving similar problems in the theory of nonlinear programming [4]. This approach introduces a penalty function so that the constraint in the form of inequality is reduced to that in the form of equality, and the relevant problem of finding the conditional extremum of the functional is solved by well-known methods of calculus of variations. However, this approach turned out to be inefficient because it does not take into account the differential properties of solutions at the contact line.

Assume that the transition of shell generatrix from contact domain to free one takes place for $s = a$. Let us study the properties of solutions $r(s)$ and $z(s)$ in the vicinity of this point. To do this, let us represent the functional (8) in the form

$$I = \int_0^a \Phi(s, z, r, z', r') ds + \int_a^{s_0} \Phi(s, z, r, z', r')_{r=\varphi(z)} ds = I_1 + I_2. \tag{9}$$

In order to evaluate the variation of the functional I_1, we can use the formula for the variation of a functional with two unknown functions, where the left boundary point is fixed, but the right one can move along the line $r = \varphi(z)$. Thus, we obtain [5]

$$\delta I_1 = \left[\Phi - z'\Phi_{z'} - r'\Phi_{r'}\right]_{s=a} \delta a + \left[\left(\Phi_{z'} + \frac{\partial \varphi}{\partial z}\Phi_{r'}\right)\delta z\right]_{s=a} + \\ + \int_0^a \left[\left(\Phi_z - \frac{d}{ds}\Phi_{z'}\right)\delta z + \left(\Phi_r - \frac{d}{ds}\Phi_{r'}\right)\delta r\right] ds. \tag{10}$$

In turn, the functional I_2 can be represented in the form

$$I_2 = \int\limits_a^{s_0} \Phi\left(s, \varphi(z), z', \frac{\partial\varphi}{\partial z} z'\right) ds = \int\limits_a^{s_0} F(s, z, z') ds,$$

and its variation will be evaluated as that of a functional with one function with movable left boundary point. This yields the formula

$$\delta I_2 = -[F - z' F_{z'}]_{s=a} \delta a - [F_{z'} \delta z]_{s=a} + \int\limits_0^{s_0} \left(F_z - \frac{d}{ds} F_{z'}\right) \delta z ds.$$

Taking into account the relations

$$F_z = \Phi_z + \frac{\partial\varphi}{\partial z}\Phi_r + \frac{d}{dz}\left(\frac{\partial\varphi}{\partial z} z'\right)\Phi_{z'}, \qquad F_{z'} = \Phi_{z'} + \frac{\partial\varphi}{\partial z}\Phi_{r'}$$

and introducing the notation

$$\widetilde{\Phi} = \Phi(s, z, r, \widetilde{z}', \widetilde{r}'),$$

where $\widetilde{z}', \widetilde{r}'$ are the values of the derivatives z' and r' at the point $s = a$ when tending to it from the right, we can represent the variation of the functional I_2 in the form

$$\delta I_2 = -\left[\widetilde{\Phi} - \widetilde{z}'\widetilde{\Phi}_{z'} - \widetilde{r}'\widetilde{\Phi}_{r'}\right]_{s=a} \delta a - \left[\left(\widetilde{\Phi}_{z'} + \frac{\partial\varphi}{\partial z}\widetilde{\Phi}_{r'}\right)\delta z\right]_{s=a} + \\ + \int\limits_a^{s_0} \left[\Phi_z - \frac{d}{ds}\Phi_{z'} + \frac{\partial\varphi}{\partial z}\left(\Phi_r - \frac{d}{ds}\Phi_{r'}\right)\right]_{r=\varphi(z)} \delta z ds. \tag{11}$$

Since $z(s)$ and $r(s)$ ensure the extremum of the functional I, we have

$$\delta I = \delta I_1 + \delta I_2 = 0.$$

If the functions $z(s)$ and $r(s)$ ensure the extremum of the functional (9) for arbitrary δa and $\delta z|_{s=a}$, then the same obviously remains true for $\delta a = 0$ and $\delta x|_{s=a} = 0$. Hence the main lemma of calculus of variations allows us to conclude that the extremum may be achieved only on integral curves of the Euler system of equations

$$\begin{aligned} &\Phi_z - \frac{d}{ds}\Phi_{z'} = 0, \\ &\Phi_r - \frac{d}{ds}\Phi_{r'} = 0, \qquad s \in [0, a], \end{aligned} \tag{12}$$

$$\left[\Phi_z - \frac{d}{ds}\Phi_{z'} + \frac{\partial\varphi}{\partial z}\left(\Phi_r - \frac{d}{ds}\Phi_{r'}\right)\right]_{r=\varphi(z)} = 0, \qquad s \in [a, s_0]. \tag{13}$$

Making simple transformations and keeping the relations (4) in mind, one can show that the Euler equations (12) are nothing but the equilibrium equations (3) for the shell in free domain, while equation (13) turns into the first equilibrium equation for the shell in the contact domain (6).

Collecting non-integral terms in the variations (10) and (11) and taking into account the arbitrariness of the variations δa and $\delta z|_{s=a}$ yields the equations for shell generatrix at the point of its transition from the domain $\psi(z,r) < 0$ to the boundary $\psi(z,r) = 0$:

$$\left[\Phi - \widetilde{\Phi} - z'\Phi_{z'} + \widetilde{z}'\widetilde{\Phi}_{z'} + \widetilde{r}'\widetilde{\Phi}_{r'} - r'\Phi_{r'}\right]_{s=a} = 0,$$

$$\left[\Phi_{z'} - \widetilde{\Phi}_{z'} + \frac{\partial \varphi}{\partial z}\left(\Phi_{r'} - \widetilde{\Phi}_{r'}\right)\right]_{s=a} = 0.$$

These equations are satisfied if we set

$$z'(a) = \widetilde{z}'(a), \qquad r'(a) = \widetilde{r}'(a).$$

As follows from the theory of sufficient conditions of extremum, the case where, at the point of transition from the free domain to the boundary $\psi(z,r) = 0$, the conditions of continuity of derivatives are fulfilled is the most important and significant one and takes place in most applied problems [6].

Let us show that the conditions obtained are sufficient for the actual determination of the extremal curve, provided that it exists. Let

$$\langle f(s)\rangle = \lim_{s\to a-0} f(s) - \lim_{s\to a+0} f(s),$$

let $z = z(s,\alpha_1,\alpha_2,\alpha_3,\alpha_4)$, $r = r(s,\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ be the general solution of the Euler equations (12) in the free domain, and let $z = z(s,\beta_1,\beta_2)$ be the general solution of the equation (13) in the contact domain. Thus, for determination of the extremal curve we must find four constants α_i, $i = 1,4$, and two constants β_i, $i = 1,2$. The boundary conditions (7) allow us to exclude three arbitrary constants. The conditions

$$\langle z\rangle = \langle r\rangle = \langle z'\rangle = 0$$

allow us to obtain additional relations for the other three constants. It seems that we lack one constant to ensure the continuity of derivative of the function $r(s)$ at the transition through the point $s = a$. But since the position of the boundary $s = a$ is unknown, the condition

$$\langle r'\rangle = 0$$

can be considered as a necessary requirement for the determination of the point $s = a$. This fact will be used below for the construction of the approximate solution of the problem.

The question of the behavior of higher derivatives of solutions of this problem while passing through the point $s = a$ is very important, so let us investigate it in detail.

The equilibrium equations for the shell in contact and free domains and the conditions of continuity of first derivatives of functions $z(s)$ and $r(s)$ yield the relation

$$\left\langle \frac{dT_1}{ds}\right\rangle = 0, \qquad \left\langle \frac{1}{R_1}\right\rangle = \left(\frac{Q_R}{T_1}\right)_{s=a}. \tag{14}$$

Furthermore, the continuity of $\frac{dT_1}{ds}$ implies the continuity of $\frac{d\lambda_1}{ds}$. We can obtain relations for the second derivatives of functions $z(s)$ and $r(s)$:

$$\frac{d^2z}{ds^2} = -\frac{\lambda_1}{R_1}\frac{dr}{ds} - \frac{r}{R_2}\frac{d\lambda_1}{ds}, \qquad \frac{d^2r}{ds^2} = \frac{1}{\lambda_1}\left(\frac{dr}{ds}\frac{d\lambda_1}{ds} - \frac{r\lambda_1^3}{R_1R_2}\right). \tag{15}$$

Then, on the basis of formulae (14) and (15), we obtain

$$\left\langle\frac{d^2z}{ds^2}\right\rangle = -\left(\lambda_1\frac{dr}{ds}\frac{Q_R}{T_1}\right)_{s=a}, \qquad \left\langle\frac{d^2r}{ds^2}\right\rangle = -\left(\frac{r\lambda_1^2Q_R}{T_1R_2}\right)_{s=a}. \tag{16}$$

Using the repeated differentiation of equilibrium equations, elasticity conditions and expressions (15) and taking into account the relations (16), we can conclude that, except for some particular cases, the higher order derivatives of the functions $z(s)$ and $r(s)$ have discontinuities of the first kind on the line of transition from free domain to contact one.

Thus, we may consider the problem about the contact interaction of nonlinearly deformed pliant shell with rigid surface of rotation as the conjugation problem for the systems of nonlinear differential equations (3) and (6), where the conjugation point $s = a$ is not known beforehand.

Let us seek for solutions for functions $z(s)$ and $r(s)$ in the form

$$z(s) = \begin{cases} \sum_{k=1}^{2} x_k u_k(s) + \sum_{k=3}^{m} x_k u_k(s), & 0 \le s \le a; \\ \sum_{k=1}^{2} x_k u_k(s) + \sum_{k=3}^{m} x_{k+2m-2} u_k(s), & a \le s \le s_0, \end{cases}$$
$$r(s) = \begin{cases} \frac{s}{a}\varphi\left(x(a)\right) + \sum_{k=1}^{m} x_{k+m} v_k(s), & 0 \le s \le a; \\ \varphi\left(x(s)\right), & a \le s \le s_0. \end{cases} \tag{17}$$

In these expressions, the coordinate functions $u_k(s)$ for $k = 1, 2$ are defined for all $s \in [0, s_0]$ and each of them obeys the restriction $u_k(s_0) = 0$. The systems of functions $u_k(s)$ for $k > 2$ have different representations on the intervals $[0, a]$ and $[a, s_0]$. Moreover, they must obey some restrictions at the ends of the intervals.

Namely, for $s \in [0, a]$, they must satisfy the boundary conditions $u_k(a) = \left.\frac{du_k}{ds}\right|_{s=a} = 0$, while for $s \in [a, s_0]$, the conditions $u_k(a) = \left.\frac{du_k}{ds}\right|_{s=a} = u_k(s_0) = 0$ must hold. Similarly, the coordinate functions $v_k(s)$, which are defined on the interval $0 \le s \le a$, must satisfy the condition $v_k(a) = 0$. The constants x_k, $k = \overline{1, 3m-2}$, are to be determined later. In order to make the reasonable choice of coordinate functions, we expand the required solutions in Taylor series in a vicinity of the point $s = 0$ and use the repeated differentiation of equilibrium equations and geometrical relations describing the shell for the determination of the coefficients of the expansion.

Consider the undeformed state of the class of domal shells whose generatrices cross the symmetry axis at the right angles and whose main curvatures in the pole are equal. Taking into account the symmetry conditions in a pole of deformed shell, which have the form

$$\lambda_1 = \lambda_2 = \lambda, \qquad T_1 = T_2 = T, \qquad \frac{1}{R_1} = \frac{1}{R_2} = \frac{1}{R},$$

we can show that the solutions bounded for $s \to 0$ have the structure

$$z(s) = a_1 + a_2 s^2 + a_3 s^4 + \ldots,$$

$$r(s) = b_1 s + b_2 s^3 + b_3 s^5 + \ldots.$$

On the other hand, let us choose the systems of coordinate functions with the domain of definition $[a, s_0]$ within the whole class of power functions. In view of the reasoning presented above, the systems of functions $\{u_k(s)\}$ and $\{v_k(s)\}$ take the form

$$u_k(s) = (s^2 - s_0^2)(s^2 - a^2)^{k-1}, \qquad k = 1, 2,$$

$$v_k(s) = (s^2 - a^2)s^{2k-1}, \qquad k = \overline{1, m},$$

$$u_k(s) = \begin{cases} (s^2 - a^2)^{k-1}, & 0 \le s \le a, \\ (s - s_0)(s - a)^2 s^{k-3}, & a \le s \le s_0, \end{cases} \qquad k = \overline{3, m}.$$

The direct check shows that the solutions for functions $z(s)$ and $r(s)$ in the form (17) will trivially satisfy the boundary conditions at the point $s = s_0$ and the conjugation conditions

$$\langle z \rangle = \left\langle \frac{dz}{ds} \right\rangle = \langle r \rangle = 0.$$

The finite gap in the higher orders derivatives of solutions is caused by the second terms in (17).

In fact, the suggested form of solution for function $x(s)$ is a specifically chosen superposition of two classes of expansions, which describe continuous and singular parts of the solution in question. We take only two terms in the first sum of expression for $z(s)$, which provides the continuous transition of the solution and its first derivative through the point $s = a$, since the remaining terms of this sum are linearly expressed via the elements of the second sum.

Let us substitute the expressions (17) into functional (9), which as a result, becomes a function of $3m - 2$ variables x_i, being the components of unknown vector $\vec{x}$.

Suppose that the point $s = a$ of transition of the shell generatrix from the contact domain to free one is known. Then we determine the constants x_i from the conditions of stationarity of functional (9). This yields the system of nonlinear algebraic equations

$$\vec{g}\,(\vec{x}) = 0. \tag{18}$$

The components of the $(3m - 2)$-dimensional vector function $\vec{g}$ have the form

$$g_{i+\varepsilon_i} = \int\limits_0^{s_0} \left[\mu_1 \frac{dz}{ds}\frac{du_i}{ds} - Q\lambda_2 \frac{dr}{ds} u_i\right] \xi(s) ds +$$

$$+\delta_i k_1 \int\limits_0^{a} \left[\mu_1 \frac{dr}{ds} + \mu_2 \xi(s)\right] \xi(s) ds + \int\limits_a^{s_0} \left[\mu_1 \frac{dr}{ds} r_{1i} + \mu_2 r_i\right] \xi(s) ds, \tag{19}$$

$$g_{i+m} = \int\limits_0^{a} \left[\mu_1 \frac{dr}{ds}\frac{dv_i}{ds} + \mu_2 v_i\right] \xi(s) ds, \qquad i = \overline{1, m},$$

where

$$\mu_1 = U(\lambda_1, \lambda_2)/\lambda_1, \qquad U(\lambda_1, \lambda_2) = \left(\lambda_1 - \frac{1}{\lambda_1^3 \lambda_2^2}\right)\left(1 + \Gamma\lambda_2^2\right),$$

$$\mu_2 = U(\lambda_2, \lambda_1)/\xi(s) + Q\lambda_2 \frac{dz}{ds}, \qquad k_1 = \frac{u_1(a)}{a} \left.\frac{\partial\varphi}{\partial z}\right|_{z = x_1 u_1(a)},$$

$$r_i = u_i(s) \frac{\partial\varphi}{\partial z}, \qquad r_{1i} = \frac{dr_i}{ds}, \qquad \delta_i = \begin{cases} 1 & \forall\ i = 1, \\ 0 & \forall\ i = \overline{2, m}. \end{cases}$$

The quantities ε_i in (19) are equal to $2(m-1)$, if $i > 2$ and $s > 0$, and zero otherwise.

We solve the algebraic system (18) by using the Newton iteration procedure

$$\vec{x}^{(k+1)} = \vec{x}^{(k)} - H^{-1}\left(\vec{x}^{(k)}\right) \vec{g}\left(\vec{x}^{(k)}\right), \tag{20}$$

where $H(\vec{x})$ is the Jacobi matrix of the system of functions $g_1, g_2, \ldots, g_{3m-2}$ with respect to the variables $x_1, x_2, \ldots, x_{3m-2}$.

The upper symmetric part of the nonzero elements h_{ij} of the matrix H is given by

$$h_{i+\varepsilon_i, j+\varepsilon_j} = \int\limits_0^{s_0} \left(\alpha_1 \frac{du_i}{ds}\frac{du_j}{ds} + \alpha_2 u_i u_j\right) \xi(s) ds +$$

$$+\delta_i \delta_j k_2 \int\limits_0^{a} \left(\mu_1 \frac{dr}{ds} + \mu_2 \xi(s)\right) \xi(s) ds + k_1 \int\limits_0^{a} \Big[(\gamma_1 + \xi(s)\gamma_2) \left(\delta_i \frac{du_j}{ds} + \delta_j \frac{du_i}{ds}\right) +$$

$$+\gamma_3 \xi(s) \left(\delta_i u_j + \delta_j u_i\right) + \delta_i \delta_j k_1 \left(\beta_1 + 2\xi(s)\beta_2 + \xi^2(s)\beta_3\right) \Big] \xi(s) ds +$$

$$+ \int\limits_a^{s_0} \left[\gamma_1 \left(\frac{du_i}{ds} r_{1j} + \frac{du_j}{ds} r_{1i}\right) + \gamma_2 \left(\frac{du_i}{ds} r_j + \frac{du_j}{ds} r_i\right) + \gamma_3 \left(u_i r_j + u_j r_i\right)\right] \xi(s) ds +$$

$$+ \int\limits_0^{s_0} \left[\mu_1 \frac{dr}{ds} \left(\frac{\partial^2\varphi}{\partial z^2} \frac{d}{ds}(u_i u_j) + \frac{\partial^3\varphi}{\partial z^3} \frac{dz}{ds} u_i u_j\right) + \mu_2 \frac{\partial^2\varphi}{\partial z^2} u_i u_j\right] \xi(s) ds +$$

$$+\int_a^{s_0}\left[\beta_1 r_{1i}r_{1j}+\beta_2\frac{d}{ds}(r_ir_j)+\beta_3 r_ir_j\right]\xi(s)ds,$$

$$h_{i+m,j+m}=\int_0^a\left[\beta_1\frac{dv_i}{ds}\frac{dv_j}{ds}+\beta_2\frac{d}{ds}(v_iv_j)+\beta_3 v_iv_j\right]\xi(s)ds,\qquad i=\overline{1,m};\ j=\overline{1,m},\tag{21}$$

$$h_{i,j+m}=\int_0^a\left[\gamma_1\frac{du_i}{ds}\frac{dv_j}{ds}+\gamma_2 v_j\frac{du_i}{ds}+\gamma_3 v_ju_i+\right.$$

$$\left.+\,\delta_i k_1\left(\beta_1\frac{dv_j}{ds}+\beta_2\frac{d}{ds}\left(\xi(s)v_j\right)+\beta_3\xi(s)v_j\right)\right]\xi(s)ds,\qquad i,j=\overline{1,m}.$$

In (21), we have used the following notation:

$$k_2=\frac{u_1^2(a)}{a}\left.\frac{\partial^2\varphi}{\partial z^2}\right|_{z=x_1u_1(a)},\qquad \alpha_1=\frac{z_1(\lambda_1,\lambda_2)}{\lambda_1^2}\left(\frac{dz}{ds}\right)^2+\mu_1\left(\frac{dr}{ds}\right)^2\frac{1}{\lambda_1^2},$$

$$\alpha_2=-\frac{\partial Q}{\partial z}\lambda_2\frac{dz}{ds},\qquad \beta_1=\frac{z_1(\lambda_1,\lambda_2)}{\lambda_1^2}\left(\frac{dr}{ds}\right)^2+\frac{\mu_1}{\lambda_1^2}\left(\frac{dz}{ds}\right)^2,\qquad \beta_2=\frac{z_2(\lambda_1,\lambda_2)}{\xi(s)\lambda_1}\frac{dr}{ds},$$

$$\beta_3=\frac{1}{\xi(s)}\left[\frac{z_1(\lambda_2,\lambda_1)}{\xi(s)}+Q\frac{dz}{ds}\right],\qquad \gamma_1=\left[\frac{z_1(\lambda_1,\lambda_2)}{\lambda_1}-\frac{\mu_1}{\lambda_1}\right]\frac{dz}{ds}\frac{dr}{ds},$$

$$\gamma_2=\frac{z_2(\lambda_1,\lambda_2)}{\xi(s)\lambda_1}\frac{dz}{ds}+Q\lambda_2,\qquad \gamma_3=\lambda_2\frac{dz}{ds}\frac{\partial Q}{\partial z},$$

$$z_1(\lambda_1,\lambda_2)=\left(1+\frac{3}{\lambda_1^4\lambda_2^2}\right)\left(1+\Gamma\lambda_2^2\right),\qquad z_2(\lambda_1,\lambda_2)=\frac{2}{\lambda_1^3\lambda_2^3}\left(1+\Gamma\lambda_1^4\lambda_2^4\right).$$

To determine the point $s=a$, we use the continuity condition for the first derivative of the function $r(s)$ at this point. This yields one more nonlinear algebraic equation for the parameter a:

$$\left.\frac{\varphi(z)}{2a}\right|_{s=a}-a\left[x_1+(a^2-1)x_2\right]\left.\frac{\partial\varphi}{\partial z}\right|_{z=a}+\sum_{k=1}^m x_{k+m}a^{2k}=0.\tag{22}$$

We seek a solution of the algebraic system (18) and the equation (22) by the method of successive approximations. For the chosen zero approximation $\vec{x}^{(0)}$, we solve equation (22) with respect to the parameter a by the chord method. Then we precise the vector $\vec{x}^{(0)}$ by iterations (20) and then return to solving equation (22). We repeat this process as many times as necessary to provide the required accuracy of the solutions of the equations (18) and (22).

Let us consider a shell which has, in the undeformed state, the shape of a circular membrane as the first example of calculation of chracteristics of pliant shells under contact and static loading. As a rigid surface of rotation S, we take the conic surface of angle $90°$. We assume that while entering the contact interaction with the considered surface the membrane is under constant pressure ($D = 0$). In all the numerical results given below the parameter Γ describing the ratio of constants in the elastic potential (5) is supposed to be equal to $0{,}1$.

The results of numerical computation of values of the functions under study and their first two derivatives at the point $s = 0{,}3$ (free domain) in the function of the number m of iterations in expansions (17) for $C = 1,6$ are given in Table 1 below.

Table 1

m	z	$-z'$	z''	r	r'	$-r''$
1	0,44501	0,29341	0,9780	0,34300	1,1062	0,37144
2	0,52587	0,45520	1,3743	0,35753	1,1278	0,62024
3	0,53301	0,43626	1,4503	0,35964	1,1391	0,60854
4	0,53251	0,43967	1,4384	0,35944	1,1377	0,61108
5	0,53251	0,43966	1,4383	0,35944	1,1377	0,61100
6	0,53251	0,43966	1,4383	0,35944	1,1377	0,61099

The similar convergence takes place for the solution $z(s)$ which was evaluated in the domain of contact of deformed shell with cone. The relative error

$$\varepsilon = \left| \frac{T_1}{R_1} + \frac{T_2}{R_2} - Q \right| / \, |Q| \,, \tag{23}$$

to within which the constructed approximate solutions satisfy the second equilibrium equation for all values of the parameter s, $0 \leq s \leq 1$, when one keeps six terms in expansions (17), does not exceed 10^{-4}.

Figure 2 displays the behavior of solutions $z(s)$ and $r(s)$ and their first derivatives on the whole interval of integration of initial equations ($D = 0$, $C = 1,6$). While the functions $z(s)$ and $r(s)$ are smooth, their first derivatives possess an obvious fracture at the point $s = a$.

The presented results show that the suggested variant of the Ritz method for solution of the problem of contact interaction of pliant shells of rotation with axially symmetric rigid surfaces allows us to obtain the uniform convergence of solutions and their first derivatives within their domain of definition. The main reason for this is taking into account the asymptotics of solutions for $s \to 0$, as well as their differential properties on the line of transition of the shell from contact domain to free one.

It should also be mentioned that one may use the representations of solutions of conjugation problem for systems of nonlinear differential equations with unknown point of conjugation, which are different from (17). But the advantage of the algorithm is that, while the realization of iterative procedure, described above, we usually have no problems with choosing initial approximations which ensure its convergence.

The profiles of the deformed membrane for different values of the loading parameter C with taking into account the contact interaction with the conic surface in absence of restrictions for displacements (dashed curve) are shown at Fig. 3. The vertical dashed lines indicate the points of transition of the generatrix of deformed shell from the contact domain to free one.

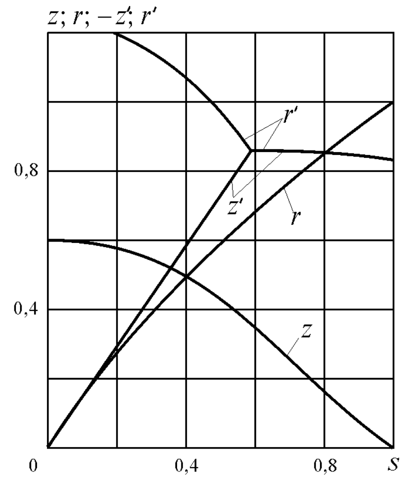

Fig. 2

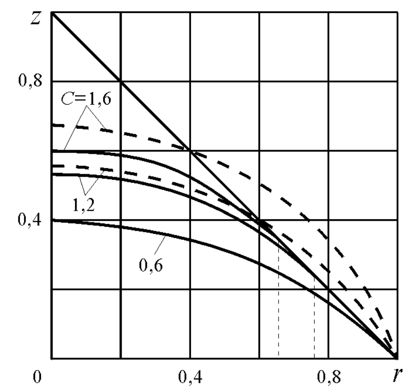

Fig. 3

The main relative lengthening and stresses for the median surface of deformed shell in function of the parameter s ($\Gamma = 0{,}1$, $D = 0$, $C = 1{,}6$) are presented at Fig. 4. The dashed lines refer to the case of free deformation of membrane. As one may observe, the presence of restricting surface S implies the considerable decrease in efforts and deformations of the median surface of the shell with respect to the case where the contact is absent.

Let us consider the example where the membrane is rigidly fixed in unstressed state on the walls of circular cylinder orthogonally to its symmetry axis. To solve this contact problem, we apply the above approach with the only difference that we take into account higher smoothness of solutions. For example, it can be easily seen from relations (16) that the second derivative of the function $z(s)$ at the point $s = a$ is continuous. At the same time the derivatives of higher orders of $z(s)$ remain discontinuous.

In this connection, let us represent the solutions for functions $z(s)$ and $r(s)$ in the form

$$
z(s) = \begin{cases} \sum\limits_{n=1}^{3} x_n u_n(s) + \sum\limits_{n=4}^{m} x_n u_n(s), & 0 \le s \le a; \\ \sum\limits_{n=1}^{3} x_n u_n(s) + \sum\limits_{n=4}^{m} x_{n+2m-3} u_n(s), & a \le s \le 1, \end{cases}
$$
$$
r(s) = \begin{cases} \dfrac{s}{a} + \sum\limits_{n=1}^{m} x_{n+m} v_n(s), & 0 \le s \le a; \\ 1, \quad a \le s \le 1, \end{cases} \tag{24}
$$

where

$$u_n(s) = \left(s^2 - 1\right)\left(s^2 - a^2\right)^{n-1}, \qquad n = 1, 2, 3,$$

$$u_n(s) = \begin{cases} \left(s^2 - a^2\right)^3 s^{2n-8}, & 0 \le s \le a; \\ \left(s^2 - 1\right)\left(s^2 - a^2\right)^{n-1}, & a \le s \le 1, \end{cases} \qquad n = \overline{4, m},$$

$$v_n(s) = \left(s^2 - a^2\right) s^{2n-1}, \qquad n = \overline{1, m}.$$

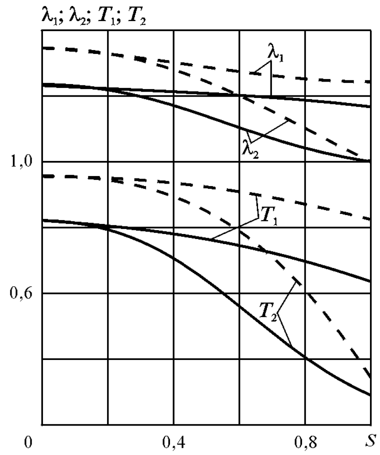

Fig. 4

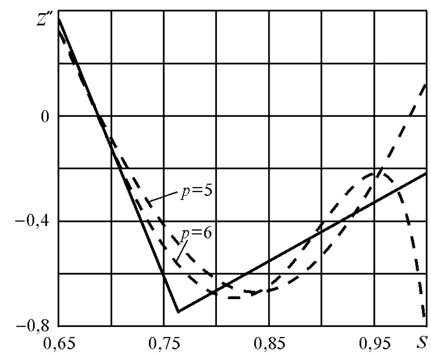

Fig. 5

Expressions (24) take into account a priori the asymptotic behavior of solutions in vicinity of the pole of deformed membrane and satisfy the conjugation conditions

$$\langle z \rangle = \left\langle \frac{dz}{ds} \right\rangle = \left\langle \frac{d^2 z}{ds^2} \right\rangle = \langle r \rangle = 0$$

which imply the existence of discontinuities of the first kind in the derivatives of higher order.

The components of the $(3m-3)$-dimensional vector function $\vec{g}$ in the case considered have the form

$$\begin{aligned} g_{i+\varepsilon_i} &= \int\limits_0^1 \left[\mu_1 \frac{dz}{ds}\frac{du_i}{ds} - Q\lambda_2 \frac{dr}{ds} u_i \right] sds, \\ g_{i+m} &= \int\limits_0^a \left[\mu_1 \frac{dr}{ds}\frac{dv_i}{ds} + \mu_2 v_i \right] sds, \qquad i = \overline{1,m}. \end{aligned} \tag{25}$$

Here, the quantities ε_i are equal to $2m-3$ if $s > a$ and $i > 3$, and to zero if at least one of these conditions is not fulfilled.

The condition of continuity of the first derivative of the function $r(s)$ at the point $s = a$ yields one more nonlinear equation for the parameter a

$$1 + 2\sum_{n=1}^{m} x_{n+m} a^{2n+1} = 0. \tag{26}$$

Since the function $z(s)$ has a discontinuity of the first kind in its third derivative, we can construct a simpler algorithm for solving the problems, in which the computation of the second derivatives of the functions $z(s)$ and $r(s)$ at the points of integration of initial equations is not required. Namely, we can seek the solutions for functions $z(s)$ in the class of continuous functions. In this case, we can seek a solution in the form

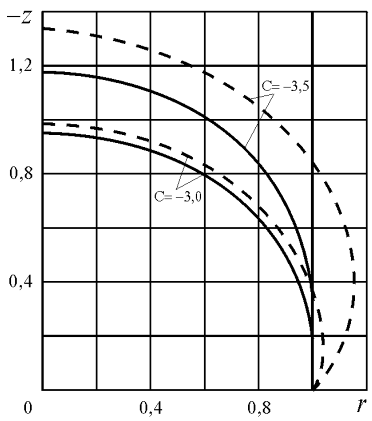

Fig. 6

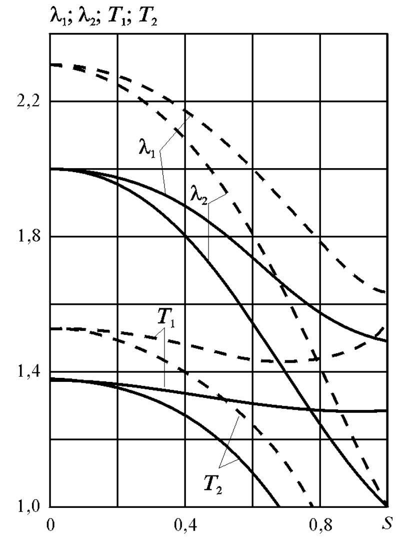

Fig. 7

$$
z(s) = \sum_{n=1}^{p} x_n u_n(s), \qquad s \in [0,1],
$$

$$
r(s) = \begin{cases} \dfrac{s}{a} + \sum\limits_{n=1}^{p} x_{n+p} v_n(s), & s \in [0,a], \\ 1, & s \in [a,1], \end{cases} \tag{27}
$$

where

$$
u_n(s) = \left(s^2 - 1\right) s^{2n-2}, \qquad v_n(s) = \left(s^2 - a^2\right) s^{2n-1}.
$$

We must also assume that $\varepsilon_i = 0$ in expressions (25) and thus reduce the problem to the simultaneous solution of $2m$ nonlinear equations (18) and equation (26).

As in the previous example, the parameter Γ in the energy function is taken to be equal to 0,11.

The numerical computation of the values of required functions and their derivatives and the quantity ε (23) at the point $s = 0{,}6$ in the function of the number m of approximations in expansions (24) are listed in Table 2. The parameters of hydrostatic loading are $C = -3{,}5$, $D = 1$. We have chosen values of m such that the expansions (24) take into account the discontinuity of the function $z(s)$.

Table 2

m	$-z$	z'	r	r'	ε
4	0,62642	1,5852	0,94180	0,70409	$3 \cdot 10^{-3}$
5	0,62617	1,5869	0,94186	0,70312	$2 \cdot 10^{-4}$
6	0,62617	1,5869	0,94186	0,70313	$3 \cdot 10^{-5}$

Figure 5 shows the behavior of the second derivative of the function $z(s)$ calculated from the fifth (sixth) approximation, when the solutions were represented in the form (24) (solid line) and in the form (27) (dashed line). The results obtained demonstrate the uniform convergence of solutions and their first two derivatives, when one takes into account the presence of discontinuities in the solutions.

The profiles of deformed shell for different values of the loading parameter C with account of contact interaction with cylindrical surface of unit radius are shown at Fig. 6. Dashed lines refer to the case of absence of restrictions on displacements. Figure 7 demonstrates the dependences of main relative lengthenings and efforts in the shell in the function of the parameter s ($C = -3,5$, $D = 1$). T_1 in the contact domain is constant, which is in agreement which the first equilibrium equation.

1. *Green A., Adkins J.* Large elastic deformations and nonlinear mechanics of continuum media. — Moscow: Mir, 1965. — 465 p. (in Russian: Translation from English).
2. *Oden J.* Finite elements of nonlinear continua. — Moscow: Mir, 1976. — 464 p. (in Russian).
3. *Trotsenko V.A.* Axially symmetric problem on equilibrium of a circular membrane under the hydrostatic load // Phys. and Techn. Appl. of Boundary Problems. — Kiev: Naukova dumka, 1978. — P. 126 — 140 (in Russian).
4. *Himmelblau D.* Applied nonlinear programming. — Moscow: Mir, 1975. — 536 p. (in Russian).
5. *Gel'fand I.M., Fomin S.V.* Calculus of variations. — Moscow: Fizmatgiz, 1961. — 228 p. (in Russian).
6. *Lavrent'ev M. and Lyusternik L.* Foundations of calculus of variations. — Moscow: ONTI, 1935. — Vol. 1, Pt 2. — 400 p. (in Russian).

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