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# THE MULTIVALUED PENALTY METHOD FOR EVOLUTION VARIATIONAL INEQUALITIES WITH $\lambda_0$-PSEUDOMONOTONE MULTIVALUED MAPS

# БАГАТОЗНАЧНИЙ МЕТОД ШТРАФА ДЛЯ ЕВОЛЮЦІЙНИХ ВАРІАЦІЙНИХ НЕРІВНОСТЕЙ З $\lambda_0$-ПСЕВДОМОНОТОННИМИ БАГАТОЗНАЧНИМИ ВІДОБРАЖЕННЯМИ

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*We consider evolution variational inequalities with $\lambda_0$-pseudomonotone maps. The main properties of the given maps have been investigated. By using the finite differences method, strong solvability for the class of evolution variational inequalities with $\lambda_0$-pseudomonotone map has been proved. Using the penalty method for multivalued maps, the existence of weak solutions for evolution variation inequalities on closed convex sets has been shown. The class of multivalued penalty operators has been constructed. We also consider a model example to illustrate the given theory.*

*Розглядаються еволюційні варіаційні нерівності з $\lambda_0$-псевдомонотонними відображеннями. Доведено основні властивості даних відображень. За допомогою методу скінченних різниць встановлено „сильну” розв'язність для класу еволюційних варіаційних нерівностей з $\lambda_0$-псевдомонотонними відображеннями. За допомогою методу штрафа доведено існування слабкого розв'язку для еволюційних варіаційних нерівностей на замкнених опуклих множинах. Побудовано клас багатозначних операторів штрафа. Наведено модельний приклад, який ілюструє дану теорію.*

**Introduction.** There exist many methods to study nonlinear evolution equations, including the following: Faedo – Galerkin, singular perturbations, difference approximations, nonlinear semigroups of operators and others [1, 2]. An extension of these approaches to evolutionary inclusions and variational inequalities presents a series of basic difficulties. The method of nonlinear semigroups of operators in Banach spaces for evolution inclusions was developed in works of

A. A. Tolstonogov [3], A. A. Tolstonogov and J. I. Umanskij [4], V. Barbu [2] and others. The method of singular perturbations (H. Brezis [5] and Yu. A. Dubinskiy [6]) for evolutionary inclusions was presented by A. N. Vakulenko and V. S. Mel'nik in [7–9], the method of Galerkin approximations in P. O. Kasyanov's works [10–12]. In the paper [13] the method of finite differences was extended to a class of multivalued operators for the first time.

In the present work we attempt to extend the method of difference approximations [1] to evolutionary inclusions and variational inequalities with $\lambda_0$-pseudomonotone multivalued maps. Then, by using the obtained results and the multivalued penalty method, we will prove the existence of the weak solution of an evolution variational inequality with the mentioned above maps on a convex closed subset.

The multivalued penalty method is used for the first time in this case.

**Setting of the problem.** Let $\Phi$ be a separable locally convex linear topological space, $\Phi^*$ be the space identified to its topologically conjugate such that $\Phi \subset \Phi^*$, $(f, \varphi)$ be the inner product (canonical pairing) of components $f \in \Phi^*$ and $\varphi \in \Phi$.

We consider the three spaces $\mathcal{V}, H$ and $\mathcal{V}^*$ such that

$$\Phi \subset \mathcal{V} \subset \Phi^*, \quad \Phi \subset \mathcal{H} \subset \Phi^*, \quad \Phi \subset \mathcal{V}^* \subset \Phi^*, \tag{1}$$

with continuous and dense embeddings. Let $\mathcal{H}$ be a Hilbert space (with an inner product $(h_1, h_2)_{\mathcal{H}}$ and the norm $\|h\|_{\mathcal{H}}$), $\mathcal{V}$ be a reflexive separable Banach space with a norm $\|v\|_{\mathcal{V}}$, $\mathcal{V}^*$ be its conjugate space (with respect to the bilinear form $(\cdot, \cdot)$) with the dual norm $\|f\|_{\mathcal{V}^*}$. We also consider that the pairing $(\varphi, \psi)$ of arbitrary elements $\varphi, \psi \in \Phi$ coincides with its inner product $(\varphi, \psi)_{\mathcal{H}}$ and with the pairing $\langle \varphi, \psi \rangle_{\mathcal{V}}$.

Now we assume that $\mathcal{V} = \mathcal{V}_1 \cap \mathcal{V}_2$ and $\|\cdot\|_{\mathcal{V}} = \|\cdot\|_{\mathcal{V}_1^*} + \|\cdot\|_{\mathcal{V}_2^*}$, where $(\mathcal{V}_i, \|\cdot\|_{\mathcal{V}_i})$ are some reflexive separable Banach spaces such that

$$\Phi \subset \mathcal{V}_i \subset \Phi^* \quad \text{and} \quad \Phi \subset \mathcal{V}_i^* \subset \Phi^*, \quad i = 1, 2,$$

with dense and continuous embeddings. The space $(\mathcal{V}_i^*, \|\cdot\|_{\mathcal{V}_i^*})$ is the topologically conjugate of $(\mathcal{V}_i, \|\cdot\|_{\mathcal{V}_i})$ with respect to the bilinear form $(\cdot, \cdot)$. Then $\mathcal{V}^* = \mathcal{V}_1^* + \mathcal{V}_2^*$.

For some multivalued map $\mathcal{A} : \mathcal{V}_1 \overset{\rightarrow}{\rightarrow} \mathcal{V}_1^*$ with non-empty convex closed (in the corresponding topology) bounded values, some convex lower semicontinuous functional $\varphi : \mathcal{V}_2 \to \mathbb{R}$, some linear densely defined operator $\Lambda : D(\Lambda; \mathcal{V}, \mathcal{V}^*) \subset \mathcal{V} \to \mathcal{V}^*$, and some closed convex set $K \subset \subset \mathcal{V}$, we consider the following problem on solvability of the next evolution variation inequality in the space $\mathcal{V}$ :

$$(\Lambda v, v - u) + [\mathcal{A}(u), v - u]_+ + \varphi(v) - \varphi(u) \geq (f, v - u) \quad \forall v \in K \cap D(\Lambda; \mathcal{V}, \mathcal{V}^*), \tag{2}$$

$$u \in K, \tag{3}$$

where $f \in \mathcal{V}^*$ is an arbitrary fixed element,

$$[\mathcal{A}(u), v]_+ = \sup_{d(u) \in \mathcal{A}(u)} (d(u), v) \quad \forall u, v \in \mathcal{V}.$$

**Main assumptions.** We assume that

a) the set $\Phi$ is dense in the space $(\mathcal{V} \cap \mathcal{V}^*, \|v\|_{\mathcal{V}} + \|v\|_{\mathcal{V}^*})$.

***Remark 1.*** [1, p. 241]. From a) it follows that

$$\mathcal{V} \cap \mathcal{V}^* \subset \mathcal{H}.$$

***Remark 2.*** If $\mathcal{V} \subset \mathcal{H}$ then it is possible not to introduce $\Phi$ but, at once, identify $\mathcal{H}$ with $\mathcal{H}^*$, in order to get the following relation of continuous and dense embeddings:

$$\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}^*. \tag{4}$$

**Definition 1.** *The family of the maps* $\{G(s)\}_{s\geq 0}$ *refers to a continuous semigroup on some Banach space* $X$, *if*

$\forall s \geq 0\; G(s) \in \mathcal{L}(X;X),\, G(0) = \mathrm{Id},$ *where* $\mathrm{Id}\, x = x\; \forall x \in X,$

$G(s+t) = G(s) \circ G(t)\; \forall s,t \geq 0,$

$G(t)x \to x$ *weakly in* $X$ *as* $t \to 0+\; \forall x \in X.$

**The operator** $\Lambda$. Similarly to [1, p. 243, 244] we consider a continuous semigroup $\{G(s)\}_{s\geq 0}$ on $\mathcal{V}, H, V^*$, i.e., there are three semigroups defined on the spaces $\mathcal{V}$, $H$, and $\mathcal{V}^*$, respectively, that coincide on $\Phi$. We denote each of them by $\{G(s)\}_{s\geq 0}$. We also assume that

b) $\{G(s)\}_{s\geq 0}$ is a non-expanding semigroup on $\mathcal{H}$, i.e.,

$$\|G(s)\|_{\mathcal{L}(\mathcal{H};\mathcal{H})} \leq 1 \quad \forall s \geq 0.$$

We denote by $-\Lambda$ the infinitesimal generator of the semigroup $\{G(s)\}_{s\geq 0}$ with the definition domain $D(\Lambda;\mathcal{V})$ (respectively, $D(\Lambda;\mathcal{H})$ or $D(\Lambda;\mathcal{V}^*)$) in $\mathcal{V}$ (respectively, in $\mathcal{H}$ or in $\mathcal{V}^*$). Due to [14] (Theorem 13.35) such a generator exists, moreover, it is a densely defined closed linear operator in the space $\mathcal{V}$ (respectively, in $\mathcal{H}$ or in $\mathcal{V}^*$).

Let $\{G^*(s)\}_{s\geq 0}$ be the semigroup conjugated to $\{G(s)\}_{s\geq 0}$. We assume that it operates respectively in $\mathcal{V}$, in $\mathcal{H}$, and in $\mathcal{V}^*$. Let also $-\Lambda^*$ be the infinitesimal generator for the semigroup $\{G^*(s)\}_{s\geq 0}$ with definition domains $D(\Lambda^*;\mathcal{V})$ in $\mathcal{V}$, $D(\Lambda^*;\mathcal{H})$ in $\mathcal{H}$ and $D(\Lambda^*;\mathcal{V}^*)$ in $\mathcal{V}^*$. The operator $\Lambda^*$ in $\mathcal{H}$ (respectively, in $\mathcal{V}$ or in $\mathcal{V}^*$) is the conjugate of the operator $\Lambda$ in $\mathcal{H}$ (respectively, in $\mathcal{V}$ or in $\mathcal{V}^*$) in the sense of unbounded operator theory.

**Lemma 1** [1, p. 243]. *The sets* $D(\Lambda;\mathcal{V}^*) \cap \mathcal{V}$ *and* $D(\Lambda^*;\mathcal{V}^*) \cap \mathcal{V}$ *are dense in the space* $\mathcal{V}$.

Now we define $\Lambda$ as an unbounded operator, which operates from $\mathcal{V}$ into $\mathcal{V}^*$ with the definition domain $D(\Lambda;\mathcal{V},\mathcal{V}^*)$. Let us fix

c) $D(\Lambda;\mathcal{V},\mathcal{V}^*) = \{v \in \mathcal{V}$ | the form $w \to (v, \Lambda^* w)$ is continuous on $D(\Lambda^*;\mathcal{V}^*) \cap \mathcal{V}$ in the topology induced from $\mathcal{V}\}$.

Then there is a unique element $\xi_v \in \mathcal{V}^*$ such that $(v, \Lambda^* w) = (\xi_v, w)$. If $v \in D(\Lambda;\mathcal{V}^*) \cap \mathcal{V}$, then $\xi_v = \Lambda v$. Thus we can fix, in the general case, $\xi_v = \Lambda v$ and get that

$$(v, \Lambda^* w) = (\Lambda v, w) \qquad \forall w \in D(\Lambda^*;\mathcal{V}^*) \cap \mathcal{V}. \tag{5}$$

If we consider, in the linear space $D(\Lambda;\mathcal{V},\mathcal{V}^*)$, the norm $\|v\|_{\mathcal{V}} + \|\Lambda v\|_{\mathcal{V}^*}$, then we obtain a Banach space. Let us similarly define the space $D(\Lambda^*;\mathcal{V},\mathcal{V}^*)$.

***Remark 3*** [1, p. 244]. If $\mathcal{V} \subset H$, then

$$D(\Lambda; \mathcal{V}, \mathcal{V}^*) = \mathcal{V} \cap D(\Lambda; \mathcal{V}^*) \quad \text{and} \quad D(\Lambda^*; \mathcal{V}, \mathcal{V}^*) = \mathcal{V} \cap D(\Lambda^*; \mathcal{V}^*).$$

In the case where $\mathcal{H}$ does not contain $\mathcal{V}$, we assume that

d) $\mathcal{V} \cap D(\Lambda; \mathcal{V}^*)$ is dense in $D(\Lambda; \mathcal{V}, \mathcal{V}^*)$, $\mathcal{V} \cap D(\Lambda^*; \mathcal{V}^*)$ is dense in $D(\Lambda^*; \mathcal{V}, \mathcal{V}^*)$.

***Remark 4*** [1, p. 244]. We have

$$(\Lambda v, v) \geq 0 \quad \forall v \in D(\Lambda; \mathcal{V}, \mathcal{V}^*), \qquad (\Lambda^* v, v) \geq 0 \quad \forall v \in D(\Lambda^*; \mathcal{V}, \mathcal{V}^*).$$

**The main classes of multivalued maps.** Let $X$ be some Banach space, $X^*$ be its topologically conjugate,

$$\langle \cdot, \cdot \rangle_X : X^* \times X \to \mathbb{R}$$

be the duality form on $X$. For every nonempty subset $B \subset X^*$ let us consider its *-weak closed convex hull $\overset{*}{\overline{\text{co}}}(B) := \text{cl}_{X^*_w}(\text{co}(B))$.

For each multivalued map $A$ we introduce its *upper* and *lower function of support*,

$$[A(y), \omega]_+ = \sup_{d \in A(y)} \langle d, w \rangle_X, \quad [A(y), \omega]_- = \inf_{d \in A(y)} \langle d, w \rangle_X,$$

where $y, \omega \in X$. We also consider its *upper* and *lower* norms,

$$\|\mathcal{A}(y)\|_+ = \sup_{d \in \mathcal{A}(y)} \|d\|_{X^*}, \quad \|\mathcal{A}(y)\|_- = \inf_{d \in \mathcal{A}(y)} \|d\|_{X^*}.$$

The main properties of the given maps are considered in the papers [15–17]. Further we will denote with $C_v(X^*)$ the class of all nonempty convex subsets *-weakly compact in $X^*$.

The next properties take place.

**Proposition 1** [18, p. 58]. *Let $\mathcal{A}, \mathcal{B} : X \to C_v(X^*)$. Then for arbitrary $y, v, v_1, v_2 \in X$ we have the following:*

*1) the functional $X \ni v \to [\mathcal{A}(y), v]_+$ is convex positively homogeneous and lower semi-continuous;*

*2)* $[\mathcal{A}(y), v_1 + v_2]_+ \leq [\mathcal{A}(y), v_1]_+ + [\mathcal{A}(y), v_2]_+$, $[\mathcal{A}(y), v_1 + v_2]_- \geq [\mathcal{A}(y), v_1]_- + [\mathcal{A}(y), v_2]_-$, $[\mathcal{A}(y), v_1 + v_2]_+ \geq [\mathcal{A}(y), v_1]_+ + [\mathcal{A}(y), v_2]_-$, $[\mathcal{A}(y), v_1 + v_2]_- \leq [\mathcal{A}(y), v_1]_+ + [\mathcal{A}(y), v_2]_-$;

*3)* $[\mathcal{A}(y) + \mathcal{B}(y), v]_+ = [\mathcal{A}(y), v]_+ + [\mathcal{B}(y), v]_+$, $[\mathcal{A}(y) + \mathcal{B}(y), v]_- = [\mathcal{A}(y), v]_- + [\mathcal{B}(y), v]_-$;

*4)* $[\mathcal{A}(y), v]_+ \leq \|\mathcal{A}(y)\|_+ \|v\|_X$, $[\mathcal{A}(y), v]_- \leq \|\mathcal{A}(y)\|_- \|v\|_X$;

*5) the functional $\|\cdot\|_+ : C_v(X^*) \to \mathbb{R}_+$ defines a norm on $C_v(X^*)$;*

*6) the functional $\|\cdot\|_- : C_v(X^*) \to \mathbb{R}_+$ satisfies the conditions*

$\bar{0} \in \mathcal{A}(y) \quad \Leftrightarrow \quad \|\mathcal{A}(y)\|_- = 0,$

$\|\alpha \mathcal{A}(y)\|_- = |\alpha| \|\mathcal{A}(y)\|_- \quad \forall \alpha \in \mathbb{R}, y \in X,$

$\|\mathcal{A}(y) + \mathcal{B}(y)\|_- \leq \|\mathcal{A}(y)\|_- + \|\mathcal{B}(y)\|_-;$

*7)* $\|\mathcal{A}(y) + \mathcal{B}(y)\|_+ \geq \Big| \|\mathcal{A}(y)\|_+ - \|\mathcal{B}(y)\|_- \Big|$, $\|\mathcal{A}(y) - \mathcal{B}(y)\|_- \geq \|\mathcal{A}(y)\|_- - \|\mathcal{B}(y)\|_+$,

$d_H\big(A(y), B(y)\big) \geq \big| \|A(y)\|_{+(-)} - \|B(y)\|_{+(-)} \big|$, *where $d_H(\cdot, \cdot)$ is the Hausdorff metric;*

*8)* $d \in \mathcal{A}(y) \quad \Leftrightarrow \quad \forall \omega \in X \ [\mathcal{A}(y), \omega]_+ \geq \langle d, w \rangle_X$.

***Remark 5.*** Further $y_n \rightharpoonup y$ in $X$ will mean that $y_n$ weakly converges to $y$ in the Banach space $X$. If the space $X$ is not reflexive, then $y_n \rightharpoonup y$ in $X^*$ will mean that $y_n$ *-weakly converges to $y$ in the Banach space $X^*$.

Now we consider the main classes of maps of $\lambda_0$-pseudomonotone type.

**Definition 2.** *The multivalued map $\mathcal{A} : X \rightrightarrows X^*$ is:*

*$\lambda$-pseudomonotone, if for every sequence $\{y_n\}_{n\geq 0} \subset X$ such that $y_n \rightharpoonup y_0$ in $X$, from the inequality*

$$\varlimsup_{n\to\infty} \langle d_n, y_n - y_0\rangle_X \leq 0, \tag{6}$$

*where $d_n \in \overset{*}{\overline{\text{co}}}\,\mathcal{A}(y_n)$, $n \geq 1$, it follows the existence of subsequences*

$$\{y_{n_k}\}_{k\geq 1} \subset \{y_n\}_{n\geq 1} \quad \textit{and} \quad \{d_{n_k}\}_{k\geq 1} \subset \{d_n\}_{n\geq 1}$$

*such that*

$$\varliminf_{k\to\infty} \langle d_{n_k}, y_{n_k} - w\rangle_X \geq [\mathcal{A}(y), y_0 - w]_- \quad \forall w \in X; \tag{7}$$

*$\lambda_0$-pseudomonotone on $X$, if for every sequence $\{y_n\}_{n\geq 0} \subset X$ :*

$$y_n \rightharpoonup y_0 \quad in \quad X, \qquad d_n \rightharpoonup d_0 \quad in \quad X^*,$$

*where $d_n \in \overset{*}{\overline{\text{co}}}\,\mathcal{A}(y_n)$, $n \geq 1$, from the inequality (6) it follows existence of subsequences*

$$\{y_{n_k}\}_{k\geq 1} \subset \{y_n\}_{n\geq 1} \quad \textit{and} \quad \{d_{n_k}\}_{k\geq 1} \subset \{d_n\}_{n\geq 1}$$

*such that inequality (7) is true;*

*+-coercive, if*

$$\|y\|_X^{-1}\, [\mathcal{A}(y), y]_+ \to +\infty \quad as \quad \|y\|_X \to +\infty;$$

*monotone, if $\forall y_1, y_2 \in X$*

$$[\mathcal{A}(y_1), y_1 - y_2]_- \geq [\mathcal{A}(y_2), y_1 - y_2]_+;$$

*locally bounded, if for every fixed $y \in X$ there exist constants $m > 0$ and $M > 0$ such that*

$$\|A(\xi)\|_+ \leq M \quad for\ all \quad \xi \in X : \|y - \xi\|_X \leq m;$$

*locally finite-dimensionally bounded, if its restriction to arbitrary finite-dimensional subspace of $X$ is locally bounded.*

*A multivalued map $A : X \rightrightarrows X^*$ satisfies*

*condition $(\Pi)$, if for an arbitrary $k > 0$, a bounded set $B \subset X$, $y_0 \in X$ and some $d \in \overset{*}{\overline{\text{co}}}\,\mathcal{A}$ such that*

$$\langle d(y), y - y_0\rangle_X \leq k_1 \quad \forall y \in B,$$

*there exists* $C > 0$ *such that*

$$\|d(y)\|_{X^*} \leq C \quad \forall y \in B;$$

*property* $(\kappa)$*, if for an arbitrary set* $D$ *bounded in* $X$ *there exists* $c \in \mathbb{R}_+$ *such that*

$$[\mathcal{A}(v), v]_+ \geq -c\|v\|_X \quad \forall v \in D.$$

We remark that every bounded multivalued map and every monotone multivalued operator with bounded values, including subdifferential maps, satisfy property $(\kappa)$.

***Remark 6.*** The idea of passage to subsequences in the Definition 2 was adopted by us from I. V. Skripnik's work [19].

**The main properties of $\lambda_0$-pseudomonotone maps**.

***Remark 7.*** It is obvious that every $\lambda$-pseudomonotone map is $\lambda_0$-pseudomonotone. For bounded maps defined in reflexive Banach spaces the converse implication is true too.

***Proof.*** Let $A : X \overset{\rightarrow}{\rightarrow} X^*$ be a $\lambda_0$-pseudomonotone map, $y_n \to y$ weakly in $X$ and let (6) hold, where $d_n \in \overset{*}{\overline{\text{co}}}\, A(y_n)$. Boundedness of the operator $A$ immediately implies boundedness of $\overline{\text{co}}\, A$ and, hence, boundedness of the sequence $\{d_n\}$ in $X^*$. Consequently, there exists a subsequence $\{d_m\} \subset \{d_n\}$ and, respectively, $\{y_m\} \subset \{y_n\}$ such that $d_m \to d$ weakly in $X^*$ and, at the same time,

$$\overline{\lim_{m\to\infty}} \langle d_m, y_m - v\rangle_X \leq \overline{\lim_{n\to\infty}} \langle d_n, y_n - v\rangle_X \leq 0.$$

Since the operator $A$ is $\lambda_0$-pseudomonotone, there exists one more subsequence (we use for it the same notations) for which

$$\overline{\lim_{m\to\infty}} \langle d_m, y_m - v\rangle_X \leq [A(y), y - v]_- \quad \forall v \in X,$$

and this proves our statement. (Let us pay our attention that in the classical definitions (without passing to subsequences, this statement is complicated!).

In the work of F. Browder and P. Hess [20] a class of the generous pseudomonotone operators was introduced. An operator $A : X \overset{\rightarrow}{\rightarrow} X^*$ is called *generous pseudomonotone* if

1) $A(y) \in C_v(X^*)$ and $A(y)$ is bounded in $X^*$ $\forall y \in D(A)$;

2) for every pair of sequences $\{y_n\}_{n\geq 1} \subset X$ and $\{d_n\}_{n\geq 1} \subset X^*$ such that $d_n \in A(y_n)$, $y_n \to y$ weakly in $X$, $d_n \to d$ *-weakly in $X^*$ and inequality (6) implies that $d \in A(y)$ and $\langle d_n, y_n\rangle_X \to \langle d, y\rangle_X$.

**Proposition 2.** *Every generous pseudomonotone operator is* $\lambda_0$*-pseudomonotone.*

***Proof.*** Let $y_n \to y$ weakly in $X$, $A(y_n) \ni d_n \to d$ *-weakly in $X^*$ and let (6) hold. Then, in view of generous pseudomonotony, $\langle d_n, y_n\rangle_X \to \langle d, y\rangle_X$ and $d \in A(y)$, consequently,

$$\underline{\lim_{n\to\infty}} \langle d_n, y_n - v\rangle_X = \langle d, y - v\rangle_X \geq [A(y), y - v]_- \quad \forall v \in X.$$

The proposition is proved.

The converse implication is not true. However the next statement is true.

**Proposition 3.** *Let $A : X \rightrightarrows X^*$ be a $\lambda_0$-pseudomonotone operator. Then the next property takes place:*

*if $y_n \to y$ weakly in $X$, $\overset{*}{\overline{\text{co}}}\, A(y_n) \ni d_n \to d$ *-weakly in $X^*$ and inequality (6) is true, then there exist subsequences $\{y_m\} \subset \{y_n\}$ and $\{d_m\} \subset \{d_n\}$ such that $\langle d_m, y_m \rangle_X \to \langle d, y \rangle_X$ and, in particular, $d \in \overset{*}{\overline{\text{co}}}\, A(y)$.*

***Proof.*** Let $\{y_n\}$, $\{d_n\}$ be the required sequences. Hence we can choose subsequences $\{y_m\}$, $\{d_m\}$ such that the inequality (7) is true. Setting $v = y$ we obtain that $\langle d_m, y_m - y \rangle_X \to 0$ or $\langle d_m, y_m \rangle_X \to \langle d, y \rangle_X$. In particular,

$$\langle d, y - v \rangle_X = \lim_{m\to\infty} \langle d_m, y_m - v \rangle_X \geq [A(y), y - v]_- \quad \forall v \in X.$$

From here and from the Proposition 1 we obtain that $d \in \overset{*}{\overline{\text{co}}}\, A(y)$.

The proposition is proved.

**Proposition 4.** *Let $A : X \rightrightarrows X^*$ be a $\lambda$-pseudomonotone map with bounded values. Then the next property holds:*

*if $y_n \to y$ weakly in $X$, $d_n \in \overset{*}{\overline{\text{co}}}\, A(y_n)$, and inequality (6) is true, then there exist subsequences $\{y_m\}$, $\{d_m\}$ such that for every $v \in X$ there exists $\zeta(v) \in \overset{*}{\overline{\text{co}}}\, A(y)$ such that*

$$\varliminf_{m\to\infty} \langle d_m, y_m - v \rangle_X \geq \langle \zeta(v), y - v \rangle_X\,. \tag{8}$$

***Proof.*** Let $y_n \to y$ weakly in $X$, $d_n \in \overset{*}{\overline{\text{co}}}\, A(y_n)$, and inequality (6) be true. Then, passing to the corresponding subsequences, we come to

$$\varliminf_{m\to\infty} \langle d_m, y_m - v \rangle_X \geq [A(y), y - v]_- \quad \forall\, v \in X. \tag{9}$$

The next proposition is an alternative of the generous Weierstrass theorem [21].

**Lemma 2.** *Let $X$ be a Banach space, $K \subset X^*$ be *-closed set in $X^*$, $L : X^* \to \overline{R} = R \cup \{+\infty\}$ be lower *-semicontinuous functional. Moreover, let one of the following conditions hold:*

$$\textit{the set } K \textit{ is bounded,}$$
$$\lim_{\|v\|_{X^*}\to\infty} L(v) = +\infty.$$

*Then the functional $L$ is bounded from below on $K$, it reaches on $K$ its infimum $m$ and the set*

$$E = \{v \in K |\, L(v)\} = m.$$

*is *-compact in $X^*$.*

The proof is similar to the proof of the Theorem 9.3 from [21].

Now let us continue the proof of the Proposition 4. The set $\overset{*}{\overline{\text{co}}}\, A(y)$ is *-weakly closed and bounded; the functional $X^* \ni w \mapsto \langle w, y - v \rangle_X$ is *-weakly lower semicontinuous for every

$v \in X$. Then in virtue of Lemma 2 there exists $\zeta(v) \in \overset{*}{\overline{\text{co}}}\, A(y)$ such that $[A(y), y - v]_- = \langle \zeta(v), y - v \rangle_X$. From here and from the inequality (9) we get (8).

The proposition is proved.

**Definition 3.** *The multivalued map $A : X \to C_v(X^*)$ satisfies the property $(\mathcal{M})$, if from $\{y_n\}_{n \geq 0} \subset X$, $d_n \in A(y_n)$, $n \geq 1$, such that*

$$y_n \rightharpoonup y_0 \quad \text{in} \quad X, \quad d_n \rightharpoonup d_0 \quad \text{in} \quad X^*, \quad \text{and} \quad \overline{\lim_{n\to\infty}} \langle d_n, y_n \rangle_X \leq \langle d_0, y_0 \rangle_X$$

*it follows that $d_0 \in A(y_0)$.*

**Definition 4.** *The operator $L : D(L) \subset X \to X^*$ is called maximal monotone if it is monotone and from the inequality*

$$\langle w - L(u), v - u \rangle_X \geq 0 \quad \forall u \in D(L),$$

*it follows that $v \in D(L)$ and $L(v) = w$.*

**Definition 5.** *The set*

$$\partial\varphi(v) = \{p \in X^* \| \langle p, u - v \rangle_X \leq \varphi(u) - \varphi(v)\ \forall u \in X\} =$$

$$= \{p \in X^* \mid \langle p, u - v \rangle_X \leq [\partial\varphi(v), u - v]_+\ \forall u \in X\}$$

*refers to the subdifferential map of the functional $\varphi : X \to \mathbb{R}$ at the point $v \in X$.*

**Proposition 5.** *Let a Banach space $X$ be reflexive, $A : X \rightrightarrows X^*$ be a $\lambda_0$-pseudomonotone operator, and the map $B : X \rightrightarrows X^*$ have the following properties:*

*1) the map $\overset{*}{\overline{\text{co}}}\, B : X \rightrightarrows X^*$ is compact, i.e., it maps sets bounded in $X$ into sets precompact in $X^*$;*

*2) the graph $\overset{*}{\overline{\text{co}}}\, B$ is closed in $X \times X^*$ with respect to the weak topology in $X$ and the strong one in $X^*$.*

*Then the map $C = A + B$ is $\lambda_0$-pseudomonotone.*

***Proof.*** Let $y_n \to y$ weakly in $X$, $d_n \in \overset{*}{\overline{\text{co}}}\, C(y_n)$, $d_n \to d$ *-weakly in $X^*$, and inequality (6) hold.

Since the operator $B : X \rightrightarrows X^*$ is compact, $\overset{*}{\overline{\text{co}}}\, C = \overset{*}{\overline{\text{co}}}\, A + \overset{*}{\overline{\text{co}}}\, B$ and, hence, $d_n = d_n' + d_n''$ for some $d_n' \in \overset{*}{\overline{\text{co}}}\, A(y_n)$, $d_n'' \in \overset{*}{\overline{\text{co}}}\, B(y_n)$.

In virtue of property 1) we get that, for some subsequence $\{y_k\} \subset \{y_n\}$, $d_k'' \to d''$ strongly in $X^*$ for some $d'' \in \overset{*}{\overline{\text{co}}}\, B(y)$ (Condition 2)). Hence, $d_k'' \to d''$ *-weakly in $X^*$. So $d_k' \to d' = d - d''$ *-weakly in $X^*$.

From inequality (6), passing to a subsequence $\{y_m\} \subset \{y_k\}$, we find

$$0 \geq \overline{\lim_{n\to\infty}} \langle d_n, y_n - y \rangle_X \geq \overline{\lim_{k\to\infty}} \langle d_k, y_k - y \rangle_X \geq$$

$$\geq \overline{\lim_{k\to\infty}} \left\langle d_k', y_k - y \right\rangle_X + \underline{\lim_{k\to\infty}} \left\langle d_k'', y_k - y \right\rangle_X \geq$$

$$\geq \overline{\lim_{m\to\infty}} \left\langle d_m', y_m - y \right\rangle_X + \lim_{m\to\infty} \left\langle d_m'', y_m - y \right\rangle_X .$$

Then

$$\overline{\lim_{m\to\infty}} \left\langle d'_m, y_m - y \right\rangle_X \le 0.$$

Now, if we pass again to subsequences (we use the same notations) in virtue of $\lambda_0$-pseudomonotony of $A$, we obtain

$$\underline{\lim_{m\to\infty}} \langle d'_m, y_m - v\rangle_X \ge [A(y), y-v]_- \quad \forall v \in X,$$

$$\underline{\lim_{m\to\infty}} \langle d_m, y_m - v\rangle_X = \underline{\lim_{m\to\infty}} \left\langle d'_m, y_m - v\right\rangle_X + \lim_{m\to\infty} \left\langle d''_m, y_m - v\right\rangle_X \ge$$

$$\ge [A(y), y-v]_- + \left\langle d'', y-v\right\rangle_X \ge [C(y), y-v]_- \quad \forall v \in X.$$

The proposition is proved.

**Proposition 6.** *Let $A : X \rightrightarrows X^*$ be a $\lambda_0$-pseudomonotone operator, there be a reflexive Banach space $Y$ such that $X$ is compactly and densely embedded in $Y$, $\overset{*}{\overline{\text{co}}}\, B : Y \rightrightarrows Y^*$ be a locally bounded demiclosed multivalued map (i.e., gr $\overset{*}{\overline{\text{co}}}\, B$ is closed in $Y \times Y^*$ with respect to the strong topology on $Y$ and the *-weak topology on $Y^*$). Then $C = A + B$ is a $\lambda_0$-pseudomonotone map.*

***Proof.*** Let $y_n \to y$ weakly in $X$, $d_n \in \overset{*}{\overline{\text{co}}}\, C(y_n)$, $d_n \to d$ *-weakly in $X^*$, and inequality (6) hold.

The operator $\overset{*}{\overline{\text{co}}}\, B$ is locally bounded, i.e., $\forall\, y \in X\ \exists\, N > 0$ and $\varepsilon > 0$ such that

$$\| \overset{*}{\overline{\text{co}}}\, B(\xi)\|_+ \le N, \quad \text{as} \quad \|\xi - y\|_X \le \varepsilon.$$

It is obvious that every locally bounded operator is bounded-value. Therefore

$$\overset{*}{\overline{\text{co}}}\, C(y) = \overset{*}{\overline{\text{co}}}\, A(y) + \overset{*}{\overline{\text{co}}}\, B(y)$$

and $d_n = d'_n + d''_n$ for some $d'_n \in \overset{*}{\overline{\text{co}}}\, A(y_n)$ and $d''_n \in \overset{*}{\overline{\text{co}}}\, B(y_n)$. Since $y_n \to y$ strongly in $Y$, in virtue of the locally boundedness of $\overset{*}{\overline{\text{co}}}\, B$ it follows that the sequence $\{d''_n\}$ is bounded in $Y^*$. Hence there is a subsequence $\{d''_m\} \subset \{d''_n\}$ such that $d''_m \to d''$ *-weakly $Y^*$. Under the conditions of the proposition, the embedding operator $I^* : Y^* \to X^*$ is continuous, so $I^*$ remains continuous in *-weak topologies [22] too. Hence, $d''_m \to d''$ *-weakly in $X^*$ and, finally, $d'_m = d_m - d''_m \to d' = d - d''$ *-weakly in $X^*$.

From inequality (6), passing to a subsequence $\{y_k\} \subset \{y_m\}$, we find

$$0 \ge \overline{\lim_{n\to\infty}} \langle d_n, y_n - y\rangle_X \ge \overline{\lim_{m\to\infty}} \langle d_m, y_m - y\rangle_X \ge$$

$$\ge \overline{\lim_{m\to\infty}} \left\langle d'_m, y_m - y\right\rangle_X + \underline{\lim_{m\to\infty}} \left\langle d''_m, y_m - y\right\rangle_Y \ge$$

$$\ge \overline{\lim_{k\to\infty}} \left\langle d'_k, y_k - y\right\rangle_X + \lim_{k\to\infty} \left\langle d''_k, y_k - y\right\rangle_Y .$$

In virtue of the compact embedding $X \subset Y$ we have that $y_k \to y$ strongly in $Y$. Moreover, the sequence $\{d''_k\}$ is bounded in $Y^*$, therefore,

$$\langle d''_k, y_k - y\rangle_Y \to 0.$$

Then we obtain

$$\overline{\lim_{k\to\infty}} \left\langle d'_k, y_k - v\right\rangle_X \le 0;$$

hence after the transition to a subsequence, due to the $\lambda_0$-pseudomonotonicity of $A$, we obtain

$$\underline{\lim}_{m_k\to\infty} \left\langle d'_{m_k}, y_{m_k} - v\right\rangle_X \ge [A(y), y - v]_- \quad \forall\, v \in X.$$

Further, since the operator $\overset{*}{\overline{\text{co}}}\, B$ is demiclosed, $d'' \in \overset{*}{\overline{\text{co}}}\, B(y)$ and

$$\underline{\lim}_{m_k\to\infty} \langle d_{m_k}, y_{m_k} - v\rangle_X = \underline{\lim}_{m_k\to\infty} \left\langle d'_{m_k}, y_{m_k} - v\right\rangle_X +$$

$$+ \lim_{m_k\to\infty} \left\langle d''_{m_k}, y_{m_k} - v\right\rangle_Y \ge [A(y), y - v]_- + \left\langle d'', y - v\right\rangle_Y \ge$$

$$\ge [C(y), y - v]_- \quad \forall\, v \in X.$$

The proposition is proved.

**Definition 6.** *The operator $A : X \rightrightarrows X^*$ is radially continuous from above, if $\forall\, x, h \in X$*

$$\overline{\lim_{t\to+0}}\, [A(x + th), h]_+ \le [A(x), h]_+\,;$$

*radially semi-continuous, if $\forall\, x, h \in X$ the next inequality takes place*

$$\overline{\lim_{t\to+0}}\, [A(x + th), h]_- \le [A(x), h]_+$$

*(it is clear that every radially continuous from above multivalued operator is radially semi-continuous).*

**Proposition 7.** *Let $A : X \rightrightarrows X^*$ be an upper semi-continuous multivalued operator which operates from a Banach space $X$ with the strong topology into $X^*$ with the topology $\sigma(X^*; X)$. Then $A$ is radially semi-continuous.*

***Proof.*** Every upper semi-continuous multivalued map which operates from $X$ with the strong topology into $X^*$ with the topology $\sigma(X^*; X)$ is hemicontinuous from above [23], i.e., from $x_n \to x$ strongly in $X$ it follows that

$$\overline{\lim_{n\to\infty}}\, [A(x_n), v]_+ \le [A(x), v]_+ \quad \forall\, v \in X.$$

It is necessary to note that operator an hemicontinuous from above is radially continuous from above, so it is radially semi-continuous.

The proposition is proved.

Furthers, we denote by $\Phi_0$ the class of all functions $C : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ such that for every $r \geq 0$ the real-valued function $C(r, \cdot)$ is continuous on $\mathbb{R}_+$ and

$$t^{-1} C(r_1, tr_2) \to 0 \quad \text{as} \quad t \to 0+ \quad \forall r_1, r_2 > 0.$$

**Definition 7.** *A multivalued map $A : X \overset{\to}{\to} X^*$ is*

*an operator with semibounded variation (s.b.v.), if $\forall R > 0$ for any $y_1, y_2 \in X$ such that $\|y_i\|_X \leq R$, $i = 1, 2$, the next inequality fulfills:*

$$[A(y_1), y_1 - y_2]_- \geq [A(y_2), y_1 - y_2]_+ - C\left(R; \|y_1 - y_2\|_X^{'}\right), \tag{10}$$

*where $C \in \Phi_0$, $\|\cdot\|_X^{'}$ is the compact norm on $X$;*

*an operator with l-s.b.v., if instead of (10), the following inequality is true:*

$$[A(y_1), y_1 - y_2]_- \geq [A(y_2), y_1 - y_2]_- - C(R; \|y_1 - y_2\|_X^{'}).$$

**Proposition 8.** *Let $A = A_0 + A_1 : X \overset{\to}{\to} X^*$, where $A_0 : X \overset{\to}{\to} X^*$ is a monotone map, and the multivalued operator $A_1 : X \overset{\to}{\to} X^*$ has the following properties:*

*1) there is a linear normalized space $Y$ in which $X$ is compactly and densely embedded;*

*2) the operator $A_1 : Y \overset{\to}{\to} Y^*$ is unequivocal and locally polynomial, i.e., $\forall R > 0$ there exists a natural $n = n(R)$ and a polynomial*

$$P_R(t) = \sum_{0 < \alpha \leq n} \lambda_\alpha(R) t^\alpha$$

*with continuous factors $\lambda_\alpha(R) \geq 0$ such that*

$$\|A_1(y_1) - A_1(y_2)\|_+ \leq P_R\left(\|y_1 - y_2\|_Y\right) \quad \forall \|y_i\|_Y \leq R, \quad i = 1, 2.$$

*Then $A$ is an operator with s.b.v.*

**Proposition 9.** *Let in Proposition 8 the operator $A_0 : X \overset{\to}{\to} X^*$ be $l$-monotone, i.e.,*

$$[A_0(y_1), y_1 - y_2]_- \geq [A_0(y_2), y_1 - y_2]_- \quad \forall y_1, y_2 \in X,$$

*and instead of the condition 2) we have the following:*

*2′) the multivalued map $A_1 : Y \overset{\to}{\to} Y^*$ is locally polynomial in the sense that for an arbitrary $R > 0$ there exist $n = n(R)$ and a polynomial $P_R(t)$ for which*

$$\mathrm{d}_H\left(A_1(y_1), A_1(y_2)\right) \leq P_R\left(\|y_1 - y_2\|_Y\right) \quad \forall \|y_i\|_Y \leq R, \quad i = 1, 2, \tag{11}$$

*where $d_H(\cdot, \cdot)$ is the Hausdorff metric.*

*Then $A = A_0 + A_1$ is an operator with $l$-s.b.v.*

***Proof.*** The proof is related to Proposition 9. In the case of Proposition 8 the reasonings are similar. Since

$$\forall y_1, y_2 \in X : \quad \left[A_0\left(y_1\right), y_1 - y_2\right]_- \geq \left[A_0\left(y_2\right), y_1 - y_2\right]_-,$$

we only need to estimate

$$[A_1(y_1), y_1 - y_2]_- - [A_1(y_2), y_1 - y_2]_- .$$

For each $d_1 \in A_1(y_1)$, $d_2 \in A_1(y_2)$ we find that

$$\langle d_2, y_1 - y_2\rangle_X - \langle d_1, y_1 - y_2\rangle_X = \langle d_2, y_1 - y_2\rangle_Y -$$

$$- \langle d_1, y_1 - y_2\rangle_Y \le \|d_1 - d_2\|_{Y^*} \|y_1 - y_2\|_Y ,$$

hence

$$[A_1(y_2), y_1 - y_2]_- - [A_1(y_1), y_1 - y_2]_- \le \operatorname{dist}(A_1(y_1), A_1(y_2))\|y_1 - y_2\|_Y.$$

From here and from estimate (11), since $\|y_i\|_X \le R$ (then $\|y_i\|_Y \le \hat{R}$), $i = 1, 2$, we obtain

$$[A_1(y_1), y_1 - y_2]_- \ge [A_1(y_2), y_1 - y_2]_- - C\left(\hat{R}; \|y_1 - y_2\|_X'\right),$$

where $\|\cdot\|_X' = \|\cdot\|_Y$, $C(R,t) = P_R(t)\,t$.

It is now easy to check that $C \in \Phi_0$.

The proposition is proved.

**Proposition 10.** *Let one of the following conditions be fulfilled:*

*1) $A : X \rightrightarrows X^*$ is a radially semi-continuous operator with s.b.v.;*

*2) $A : X \rightrightarrows X^*$ is a multivalued operator radially continuous from above with $l$-s.b.v. and with compact values.*

*Then $A$ is a $\lambda_0$-pseudomonotone multivalued map.*

***Proof.*** Let $y_n \to y$ weakly in X, $\overset{*}{\overline{\text{co}}}\, A(y_n) \ni d_n \to d$ $*$-weakly in $X^*$, and inequality (6) be true.

At first we assume that the condition 1) is true. Hence, by using s.b.v. property for $A$, we get

$$\langle d_n, y_{n-v}\rangle_X \ge [A(y_n), y_n - v]_- \ge [A(v), y_n - v]_+ - C(R; \|y_n - v\|_X') \quad \forall v \in X.$$

The functional $X \ni w \mapsto [A(v), w]_+$ is convex and lower semi-continuous. So it is weakly lower semi-continuous. Therefore, if we substitute in the last inequality $v = y$ and pass to the limit as $n \to \infty$, from the properties of the function $C$, we obtain that $\underline{\lim}_{n\to\infty} \langle d_n, y_n - y\rangle_X \ge 0$, i.e., $\langle d_n, y_n - y\rangle_X \to 0$.

For each $h \in X$ and $\tau \in [0,1]$ let us put $w(\tau) = \tau h + (1-\tau)y$. Then

$$\langle d_n, y_n - w(\tau)\rangle_X \ge [A(w(\tau)), y_n - w(\tau)]_+ - C\left(R; \|y_n - w(\tau)\|_X'\right)$$

and, passing to limit as $n \to +\infty$,

$$\tau \underline{\lim}_{n\to\infty} \langle d_n, y - h\rangle_X \ge \tau [A(w(\tau)), y - h]_+ - C\left(R; \tau\|y - h\|_X'\right).$$

If we divide the obtained inequality by $\tau$ and pass to the limit as $\tau \to +0$, using the radial semi-continuity and the properties of the function $C$, we get

$$\forall h \in X: \quad \underline{\lim}_{n\to\infty} \langle d_n, y-h\rangle_X \geq \underline{\lim}_{\tau\to+0} [A(w(\tau)), y-h]_+ +$$

$$+ \lim_{\tau\to+0} \frac{1}{\tau} C\left(R; \tau \|y-h\|_X^{'}\right) \geq [A(y), y-h]_- .$$

In virtue of $\langle d_n, y_n - y\rangle_X \to 0$ we have that

$$\underline{\lim}_{n\to\infty} \langle d_n, y_n-h\rangle_X = \underline{\lim}_{n\to\infty} \langle d_n, y-h\rangle_X \geq [A(y), y-h]_- \quad \forall h \in X,$$

which proves the first statement of the proposition.

Now let us consider the main distinctive feature of the second statement. From $l$-s.b.v. of the multivalued operator $A$, we have

$$\underline{\lim}_{n\to\infty} \langle d_n, y_n-v\rangle_X \geq \underline{\lim}_{n\to\infty} [A(y_n), y_n-v]_- \geq$$

$$\geq \underline{\lim}_{n\to\infty} [A(v), y_n-v]_- - C\left(R; \|y-v\|_X^{'}\right). \tag{12}$$

Let us estimate the first component in the right-hand side of (12). At first, we prove that the functional

$$X \ni h \mapsto [A(v), h]_-$$

is weakly lower semi-continuous for every $v \in X$. In fact, let $z_n \to z$ weakly in $X$. Then for every $n = 1, 2, \ldots$ there exists $\xi_n \in \overset{*}{\text{co}}\, A(v)$ such that

$$[A(v), z_n]_- = \langle \xi_n, z_n\rangle_X .$$

From the sequence $\{\xi_n; z_n\}$ let us pass to a subsequence $\{\xi_m; z_m\}$ such that

$$\underline{\lim}_{n\to\infty} [A(v), z_n]_- = \underline{\lim}_{n\to\infty} \langle \xi_n, z_n\rangle_X = \lim_{m\to\infty} \langle \xi_m, z_m\rangle_X .$$

In virtue of compactness of the set $\overset{*}{\overline{\text{co}}}\, A(v)$, we get that $\xi_m \to \xi$ strongly in $X^*$. Hence, $\xi \in$ $\in \overset{*}{\overline{\text{co}}}\, A(v)$ and

$$\underline{\lim}_{n\to\infty} [A(v), z_n]_- = \lim_{m\to\infty} \langle \xi_m, z_m\rangle_X = \langle \xi, z\rangle_X \geq [A(v), z]_- ,$$

which proves the weak lower semicontinuity.

From the estimate (12) it follows that

$$\underline{\lim}_{n\to\infty} \langle d_n, y_n-v\rangle_X \geq \underline{\lim}_{n\to\infty} [A(y_n), y_n-v]_- \geq$$

$$\geq [A(v), y-v]_- - C\left(R; \|y-v\|_X^{\prime}\right).$$

If we fix, in the last inequality, $v = y$ we obtain that $\langle d_n, y_n - y\rangle_X \to 0$. Therefore,

$$\varliminf_{n\to\infty} \langle d_n, y_n - v\rangle_X \geq [A(v), y - v]_- - C\left(R; \|y - v\|'_X\right) \quad \forall v \in X.$$

If we put, in the last inequality, $v = tw + (1-t)y$, where $w \in X, t \in [0,1]$, then, by dividing the result by $t$ and passing to the limit as $t \to +0$, radial continuity from above of $A$ gives that

$$\varliminf_{n\to\infty} \langle d_n, y_n - w\rangle_X \geq [A(y), y - w]_- \qquad \forall w \in X.$$

The proposition is proved.

Now let $X$ be some Banach space such that $X = X_1 \cap X_2$, where $X_1$, $X_2$ are reflexive Banach spaces, densely and continuously embedded in some locally convex linear topological space $Y$.

**Definition 8.** *A pair of multivalued maps $A : X_1 \to 2^{X_1^*}$ and $B : X_2 \to 2^{X_2^*}$ is s-mutually bounded, if for every $M > 0$ there exists $K(M) > 0$ such that from*

$$\|y\|_X \leq M \quad \textit{and} \quad \langle d_1(y), y\rangle_{X_1} + \langle d_2(y), y\rangle_{X_2} \leq M$$

*it follows that*

$$\textit{or} \quad \|d_1(y)\|_{X_1^*} \leq K(M), \quad \textit{or} \quad \|d_2(y)\|_{X_2^*} \leq K(M)$$

*for some selectors $d_1 \in A$ and $d_2 \in B$.*

**Lemma 3**. *Let $A : X_1 \to C_v(X_1^*)$ and $B : X_2 \to C_v(X_2^*)$ be s-mutually bounded $\lambda_0$-pseudomonotone on $X_1$ and respectively on $X_2$ multivalued maps. Then $C := A + B : X \to C_v(X^*)$ is a $\lambda_0$-pseudomonotone on $X$ multivalued map.*

***Remark 8.*** If the pair $(A; B)$ is not $s$-mutually bounded, then the given lemma is true only for $\lambda$-pseudomonotone (respectively, on $X_1$ and on $X_2$) multivalued maps.

***Remark 9.*** It is obvious that if $A : X_1 \to 2^{X_1^*}$ or $B : X_2 \to 2^{X_2^*}$ is a bounded map, then the pair $(A; B)$ is $s$-mutually bounded.

***Proof.*** At first we check that $\forall y \in X$ $C(y) \in C_v(X^*)$. The convexity of $C(y)$ follows from the same property of $A(y)$ and $B(y)$. In virtue of Mazur's theorem, it is enough to prove that the set $C(y)$ is weakly closed. Let $g$ be a frontier point of $C(y)$ in the topology $\sigma(X^*; X^{**}) = \sigma(X^*; X)$ (the space $X$ is reflexive due to reflexivity of $X_1$ and $X_2$). Then

$$\exists \{g_m\}_{m\geq 1} \subset C(y) : \quad g_m \to g \quad \text{weakly in} \quad X^* \quad \text{as} \quad m \to \infty.$$

From here, since the maps $A$ and $B$ have bounded values, thanks to the Banach – Alaoglu theorem we may consider that for every $m \geq 1$ there exists $v_m \in A(y)$ and $w_m \in B(y)$ such that $v_m + w_m = g_m$ and passing (if it is necessary) to subsequences we obtain

$$v_m \rightharpoonup v \quad \text{in} \quad X_1^* \quad \text{and} \quad w_m \rightharpoonup w \quad \text{in} \quad X_2^*$$

for some $v \in A(y)$ and $w \in B(y)$. Hence, $g = v + w \in C(y)$. So, the set $C(y)$ is weakly closed in $X^*$.

Now let $y_n \rightharpoonup y_0$ in $X$ (from here it follows that $y_n \rightharpoonup y_0$ in $X_1$ and $y_n \rightharpoonup y_0$ in $X_2$), $C(y_n) \ni d(y_n) \rightharpoonup d_0$ in $X^*$ and inequality (6) is true. Hence, for some

$$d_A(y_n) \in A(y_n) \quad \text{and} \quad d_B(y_n) \in B(y_n), \quad d_A(y_n) + d_B(y_n) = d(y_n),$$

since the pair $(A; B)$ is $s$-mutually bounded, and from the estimate

$$\langle d(y_n), y_n \rangle_X = \langle d_A(y_n) + d_B(y_n), y_n \rangle_X = \langle d_A(y_n), y_n \rangle_{X_1} + \langle d_B(y_n), y_n \rangle_{X_2} \le k$$

it follows that either $\|d_A(y_n)\|_{X_1^*} \le C$ or $\|d_B(y_n)\|_{X_2^*} \le C$. Then, due to the reflexivity of $X_1$ and $X_2$, passing (if it is necessary) to a subsequence, we obtain

$$d_A(y_n) \rightharpoonup d_0' \text{ in } X_1^* \quad \text{and} \quad d_B(y_n) \rightharpoonup d_0'' \text{ in } X_2^*.$$

From inequality (6) it follows that

$$\varliminf_{n\to\infty} \langle d_B(y_n), y_n - y_0 \rangle_{X_2} + \varlimsup_{n\to\infty} \langle d_A(y_n), y_n - y_0 \rangle_{X_1} \le \varlimsup_{n\to\infty} \langle d(y_n), y_n - y_0 \rangle_X \le 0,$$

or, symmetrically

$$\varliminf_{n\to\infty} \langle d_A(y_n), y_n - y_0 \rangle_{X_1} + \varlimsup_{n\to\infty} \langle d_B(y_n), y_n - y_0 \rangle_{X_2} \le \varlimsup_{n\to\infty} \langle d(y_n), y_n - y_0 \rangle_X \le 0.$$

Let us consider the last inequality. It is obvious, that there is a subsequence $\{y_m\}_m \subset \{y_n\}_{n\ge 1}$ such that

$$\begin{aligned} 0 &\ge \varlimsup_{n\to\infty} \langle d_B(y_n), y_n - y_0 \rangle_{X_2} + \varliminf_{n\to\infty} \langle d_A(y_n), y_n - y_0 \rangle_{X_1} \ge \\ &\ge \varlimsup_{m\to\infty} \langle d_B(y_m), y_m - y_0 \rangle_{X_2} + \lim_{m\to\infty} \langle d_A(y_m), y_m - y_0 \rangle_{X_1}. \end{aligned} \tag{13}$$

From here we obtain

$$\text{either} \quad \lim_{m\to\infty} \langle d_A(y_m), y_m - y_0 \rangle_{X_1} \le 0 \quad \text{or} \quad \varlimsup_{m\to\infty} \langle d_B(y_m), y_m - y_0 \rangle_{X_2} \le 0.$$

Without loss of generality we assume that

$$\lim_{m\to\infty} \langle d_A(y_m), y_m - y_0 \rangle_{X_1} \le 0.$$

Then, in virtue of $\lambda_0$-pseudomonotony of $A$ on $X_1$, there exists $\{y_{m_k}\}_{k\ge 1}$ in $\{y_m\}_m$ such that

$$\lim_{k\to\infty} \langle d_A(y_{m_k}), y_{m_k} - v \rangle_{X_1} \ge [A(y_0), y_0 - v]_- \quad \forall v \in X_1.$$

If we put $v = y_0$ in the last relation we obtain that

$$\langle d_A(y_{m_k}), y_{m_k} - y_0 \rangle_{X_1} \to 0 \quad \text{as} \quad k \to +\infty.$$

Then, due to (13),

$$\overline{\lim_{k\to\infty}} \langle d_B(y_{m_k}), y_{m_k} - y_0\rangle_{X_2} \leq 0.$$

So, in virtue of $\lambda_0$-pseudomonotony of $B$ on $X_2$, passing to a subsequence $\{y_{m'_k}\} \subset \{y_{m_k}\}_{k\geq 1}$, we will find

$$\underline{\lim_{k\to\infty}} \langle d_B(y_{m'_k}), y_{m'_k} - w\rangle_{X_2} \geq [B(y_0), y_0 - w]_- \quad \forall w \in X_2,$$

and, finally

$$\underline{\lim_{k\to\infty}} \langle d(y_{m'_k}), y_{m'_k} - x\rangle_X \geq \underline{\lim_{k\to\infty}} \langle d_A(y_{m'_k}), y_{m'_k} - x\rangle_{X_1} +$$

$$+ \underline{\lim_{k\to\infty}} \langle d_B(y_{m'_k}), y_{m'_k} - x\rangle_{X_2} \geq [A(y_0), y_0 - x]_- +$$

$$+ [B(y_0), y_0 - x]_- = [C(y_0), y_0 - x]_- \quad \forall x \in X.$$

The lemma is proved.

**Lemma 4.** *Let $A : X_1 \rightrightarrows X_1{}^*$, $B : X_2 \rightrightarrows X_2{}^*$ be +-coercive maps satisfying property $(\kappa)$. Then the multivalued map $C := A + B : X \rightrightarrows X^*$ is +-coercive too.*

***Proof.*** We obtain this statement arguing by contradiction. Let $\{x_n\}_{n\geq 1}$ with $x_n \neq \bar{0}$ and $\|x_n\|_X = \|x_n\|_{X_1} + \|x_n\|_{X_2} \to +\infty$ as $n \to +\infty$, but

$$\sup_{n\geq 1} \frac{[C(x_n), x_n]_+}{\|x_n\|_X} < +\infty.$$

Let also

$$\gamma_A(r) := \inf_{\|v\|_{X_1}=r} \frac{[A(v), v]_+}{\|v\|_{X_1}}, \quad \gamma_B(r) := \inf_{\|w\|_{X_2}=r} \frac{[B(w), w]_+}{\|w\|_{X_2}}, \quad r > 0.$$

We remark that $\gamma_A(r) \to +\infty$, $\gamma_B(r) \to +\infty$ as $r \to +\infty$.

In case $\|x_n\|_{X_1} \to +\infty$ as $n \to \infty$ and $\|x_n\|_{X_2} \leq c\, \forall n \geq 1$ we get

$$\frac{[A(x_n), x_n]_+}{\|x_n\|_X} \geq \gamma_A(\|x_n\|_{X_1}) \frac{\|x_n\|_{X_1}}{\|x_n\|_X} \to +\infty \quad \text{as} \quad \|x_n\|_{X_1} \to +\infty$$

and, moreover,

$$\frac{[B(x_n), x_n]_-}{\|x_n\|_X} \geq -c_1 \frac{\|x_n\|_{X_2}}{\|x_n\|_X} \to 0 \quad \text{as} \quad n \to \infty,$$

where $c_1 \in \mathbb{R}_+$ is a constant as in property $(\kappa)$ with

$$D = \{x \in X_2 \mid \|x\|_{X_2} \leq c\}.$$

Consequently,

$$\frac{[C(x_n), x_n]_+}{\|x_n\|_X} = \frac{[A(x_n), x_n]_+}{\|x_n\|_X} + \frac{[B(x_n), x_n]_+}{\|x_n\|_X} \to +\infty \quad \text{as} \quad n \to \infty.$$

We have a contradiction with the boundedness of the left-hand side of the above expression.

If $\|x_n\|_{X_1} \le c$ and $\|x_n\|_{X_2} \to \infty$ as $n \to \infty$, the reasoning is the same.

When $\|x_n\|_{X_1} \to \infty$ and $\|x_n\|_{X_2} \to \infty$ as $n \to \infty$, and we get the contradiction

$$+\infty > \sup_{n\ge 1} \frac{[C(x_n), x_n]_+}{\|x_n\|_X} \ge \gamma_A(\|x_n\|_{X_1}) \frac{\|x_n\|_{X_1}}{\|x_n\|_{X_1} + \|x_n\|_{X_2}} +$$

$$+ \gamma_B(\|x_n\|_{X_2}) \frac{\|x_n\|_{X_2}}{\|x_n\|_{X_1} + \|x_n\|_{X_2}} \ge \min\{\gamma_A(\|x_n\|_{X_1}), \, \gamma_B(\|x_n\|_{X_2})\} \to +\infty.$$

The lemma is proved.

**The approximation of evolution inclusions by stationary.** For multivalued maps $\mathcal{A} : \mathcal{V}_1 \to C_v(\mathcal{V}_1^*)$ and $\mathcal{B} : \mathcal{V}_2 \to C_v(\mathcal{V}_2^*)$, for the nonbounded operator $\Lambda$ that maps from $D(\Lambda; \mathcal{V}, \mathcal{V}^*) \subset \mathcal{V}$ into $\mathcal{V}^*$, we consider the next problem:

$$u \in D(\Lambda; \mathcal{V}, \mathcal{V}^*), \tag{14}$$

$$\Lambda u + \mathcal{A}(u) + \mathcal{B}(u) \ni f, \tag{15}$$

where $f \in \mathcal{V}^*$ is arbitrary and fixed.

**Theorem 1.** *Let the next conditions be true:*

$\mathcal{A} : \mathcal{V}_1 \to C_v(\mathcal{V}_1^*)$ *is a bounded multivalued map* $\lambda_0$*-pseudomonotone on* $\mathcal{V}_1$;

$\mathcal{B} : \mathcal{V}_2 \to C_v(\mathcal{V}_2^*)$ *is a locally finite-dimensional bounded multivalued operator* $\lambda_0$*-pseudomonotone on* $\mathcal{V}_2$ *that satisfies condition* $(\Pi)$;

*the operator* $\Lambda$ *satisfies all conditions listed above, including conditions b) and d);*

$f \in \mathcal{V}^*$ *is such that for some* $R > 0$

$$[\mathcal{A}(u) + \mathcal{B}(u) - f, u]_+ \ge 0 \quad \forall u \in \mathcal{V} : \quad \|u\|_{\mathcal{V}} = R. \tag{16}$$

*Then there exists* $u \in \mathcal{V}$ *that satisfies (14) and (15).*

***Remark 10.*** If $\mathcal{V} \subset \mathcal{H}$, inclusion (14) implies that $u \in \mathcal{V} \cap D(\Lambda; \mathcal{V}^*)$.

***Proof.*** *The approximate solutions.* A natural approximation of inclusion (15) is

$$\frac{I - G(h)}{h} u_h + \mathcal{A}(u_h) + \mathcal{B}(u_h) \ni f, \quad h > 0. \tag{17}$$

Although, if $\mathcal{V}$ is not included in $\mathcal{H}$, problem (17), has no solutions in general, and it is necessary to modify the given inclusion in an appropriate way. Let us choose a sequence $\theta_h \in (0,1)$ such that

$$\frac{1 - \theta_h}{h} \to 0 \quad \text{as} \quad h \to 0. \tag{18}$$

Note that we put $\theta_h = 1$ as $\mathcal{V} \subset \mathcal{H}$. Further, let

$$\Lambda_h = \frac{I - \theta_h G(h)}{h}.$$

So, we replace (17) with the inclusion

$$\Lambda_h u_h + \mathcal{A}(u_h) + \mathcal{B}(u_h) \ni f. \tag{19}$$

**Lemma 5.** *Problem (19) has a solution $u_h \in \mathcal{V} \cap \mathcal{H}$ such that $\|u_h\|_{\mathcal{V}} \le R$.*

***Proof.*** Let us introduce a new map,

$$\mathcal{D}_h = \Lambda_h + \mathcal{A} : \mathcal{H} \cap \mathcal{V}_1 \to C_v(\mathcal{H} + \mathcal{V}_1^*),$$

and consider the following inclusion:

$$\mathcal{D}_h(u_h) + \mathcal{B}(u_h) \ni f.$$

Now we prove existence of a solution $u_h \in \mathcal{V} \cap \mathcal{H}$ for the given inclusion such that $\|u_h\|_{\mathcal{V}} \le$ $\le R$. The given statement follows from [24] (Theorem 2.1) with $V = \mathcal{H} \cap \mathcal{V}_1$, $W = \mathcal{V}_2$, $A = \mathcal{D}_h$, $B = \mathcal{B}$, $L \equiv \overline{0}$, $D(L) = V$, $f = f$, $R = R$ and using the following lemma.

**Lemma 6.** *The operator $\mathcal{D}_h$ satisfies to the following conditions:*
*(i) $[\mathcal{D}_h(u) + \mathcal{B}(u) - f, u]_+ \ge 0 \quad \forall u \in \mathcal{V} : \ \|u\|_{\mathcal{V}} = R$;*
*(ii) $\mathcal{D}_h$ is $\lambda_0$-pseudomonotone on $\mathcal{H} \cap \mathcal{V}_1$;*
*(iii) $\mathcal{D}_h$ is bounded on $\mathcal{H} \cap \mathcal{V}_1$.*

***Proof.*** i). Since the semigroup $G(s)$ is not-expanding on $\mathcal{H}$,

$$\forall v \in \mathcal{H} \quad (\Lambda_h v, v) = \frac{1}{h}\big(v - \theta_h G(h)v, v\big) \ge$$

$$\ge \frac{1}{h}\big(\|v\|^2_{\mathcal{H}} - \theta_h \|G(s)v\|_{\mathcal{H}} \|v\|_{\mathcal{H}}\big) \ge \frac{1 - \theta_h}{h} \|v\|^2_{\mathcal{H}} \ge 0. \tag{20}$$

Due to (16), (20) and Proposition 1, we will obtain i).

iii). Boundedness of $\mathcal{D}_h$ on $\mathcal{H} \cap \mathcal{V}_1$ follows from the boundedness of $\Lambda_h$ on $\mathcal{H}$ and from the same condition for $\mathcal{A}$ on $\mathcal{V}_1$. Boundedness of $\Lambda_h$ on $\mathcal{H}$ immediately follows from the definition of $\Lambda_h$ and from estimate b).

ii). Let us prove $\lambda_0$-pseudomonotonicity of $\mathcal{D}_h$ on $\mathcal{H} \cap \mathcal{V}_1$. To this end, we will use Lemma 3 with $A = \Lambda_h$ on $V = \mathcal{H}$ and $B = \mathcal{A}$ on $W = \mathcal{V}_1$. From here, due to $\lambda_0$-pseudomonotonicity and boundedness of $\mathcal{A}$ on $\mathcal{V}_1$, it is enough to prove $\lambda$-pseudomonotonicity of $\Lambda_h$ on $\mathcal{H}$. Let us prove this. We assume that

$$y_n \rightharpoonup y \quad \text{in} \quad \mathcal{H} \quad \text{and} \quad \overline{\lim_{n\to\infty}} (\Lambda_h y_n, y_n - y) \le 0.$$

Then, from estimate (20) we have

$$\underline{\lim}_{n\to\infty} (\Lambda_h y_n, y_n - y) \ge \underline{\lim}_{n\to\infty} (\Lambda_h y_n - \Lambda_h y, y_n - y) + \underline{\lim}_{n\to\infty} (\Lambda_h y, y_n - y) \ge 0 + 0 = 0.$$

Hence, $\lim_{n\to\infty}(\Lambda_h y_n, y_n - y) = 0$. Furthermore, for arbitrary $u \in \mathcal{H}$, $s > 0$ let $w := y + s(u - y)$. Then, for every $n \geq 1$,

$$s(\Lambda_h y_n, y - u) \geq -(\Lambda_h y_n, y_n - y) + (\Lambda_h w, y_n - y) - s(\Lambda_h w, u - y),$$

$$s \varliminf_{n\to\infty}(\Lambda_h y_n, y - u) \geq -s(\Lambda_h w, u - y) \quad \text{and} \quad \varliminf_{n\to\infty}(\Lambda_h y_n, y - u) \geq -(\Lambda_h w, u - y).$$

Now let $s \to 0+$. Then

$$\varliminf_{n\to\infty}(\Lambda_h y_n, y - u) \geq -(\Lambda_h y, u - y) = (\Lambda_h y, y - u)$$

and

$$\varliminf_{n\to\infty}(\Lambda_h y_n, y_n - u) \geq \varliminf_{n\to\infty}(\Lambda_h y_n, y_n - y) + \varliminf_{n\to\infty}(\Lambda_h y_n, y - u) \geq (\Lambda_h y, y - u) \quad \forall u \in \mathcal{H}.$$

Thus we obtain the required statement.

Lemma 6 is proved.

Lemma 5 is proved.

*Boundary transition on* $h$. From Lemma 5, for arbitrary $h > 0$, it follows that there exist $u_h \in \mathcal{H} \cap \mathcal{V}$, $d'_h \in \mathcal{A}(u_h)$ and $d''_h \in \mathcal{B}(u_h)$ such that

$$\Lambda_h u_h + d'_h + d''_h = f, \tag{21}$$

$$\|u_h\|_{\mathcal{V}} \leq R \quad \text{as} \quad h > 0. \tag{22}$$

From estimate (22) and boundedness of the multivalued operator $\mathcal{A}$ on $\mathcal{V}_1$, it follows that

$$\mathcal{A}(u_h) \quad \text{are uniformly bounded in} \quad \mathcal{V}_1^* \quad \text{as} \quad h > 0. \tag{23}$$

Now we prove that

$$d''_h \quad \text{are uniformly bounded in} \quad \mathcal{V}_2^* \quad \text{as} \quad h > 0. \tag{24}$$

At first, from equality (21), estimate (22), boundedness of the map $\mathcal{A}$, and Proposition 1, we obtain that for $h > 0$

$$\sup_{h>0}(d''_h, u_h) = \sup_{h>0}(f, u_h) + \sup_{h>0}(-d'_h, u_h) + \sup_{h>0}(-\Lambda_h u_h, u_h) \leq$$

$$\leq \|f\|^*_{\mathcal{V}} \sup_{h>0}\|u_h\|_{\mathcal{V}} + \sup_{h>0}\|\mathcal{A}(u_h)\|_+ \sup_{h>0}\|u_h\|_{\mathcal{V}} < +\infty.$$

Hence, due to condition $(\Pi)$ for $\mathcal{B}$, estimate (24) follows.

From equality (21), estimates (22) – (24), and the Banach – Alaoglu theorem, it follows that there are elements $u \in \mathcal{V}$, $d' \in \mathcal{V}_1^*$, $d'' \in \mathcal{V}_2^*$, $\chi \in \mathcal{V}^*$, and subsequences $\{u_{h_n}\}_{n\geq 1} \subset \{u_h\}_{h>0}$,

$\{d'_{h_n}\}_{n\geq 1} \subset \{d'_h\}_{h>0}$, $\{d''_{h_n}\}_{n\geq 1} \subset \{d''_h\}_{h>0}$ $(0 < h_n \searrow 0+)$, which we further denote by $\{u_h\}_{h>0}$, $\{d'_h\}_{h>0}$, $\{d''_h\}_{h>0}$, respectively, such that

$$u_h \rightharpoonup u \quad \text{in} \quad \mathcal{V},\ d'_h \rightharpoonup d' \quad \text{in} \quad \mathcal{V}_1^*,\ d_h \rightharpoonup d'' \quad \text{in} \quad \mathcal{V}_2^*,\ L_h u_h \rightharpoonup \chi \quad \text{in} \quad \mathcal{V}^*.$$

From here, in particular, it follows that

$$v_h := d'_h + d''_h \rightharpoonup d' + d'' =: w \quad \text{in} \quad \mathcal{V}^*. \tag{25}$$

Now let us consider the following map: $\mathcal{C}(v) = \mathcal{A}(v) + \mathcal{B}(v) : \mathcal{V} \to C_v(\mathcal{V}^*)$ and prove that it satisfies property $(\mathcal{M})$. For this purpose, it is enough to show $\lambda_0$-pseudomonotonicity of $\mathcal{C}$ on $\mathcal{V}$. Indeed, if $\mathcal{C}$ is $\lambda_0$-pseudomonotone on $\mathcal{V}$ and $\{y_n\}_{n\geq 0} \subset \mathcal{V}$, $d_n \in \mathcal{C}(y_n)$, $n \geq 1$, are such that

$$y_n \rightharpoonup y_0 \quad \text{in} \quad \mathcal{V}, \quad d_n \rightharpoonup d_0 \quad \text{in} \quad \mathcal{V}^* \quad \text{and} \quad \overline{\lim_{n\to\infty}}\,(d_n, y_n) \leq (d_0, y_0),$$

then

$$\overline{\lim_{n\to\infty}}\,(d_n, y_n - y_0) \leq \overline{\lim_{n\to\infty}}\,(d_n, y_n) + \overline{\lim_{n\to\infty}}\,(d_n, -y_0) \leq (d_0, y_0) - (d_0, y_0) = 0.$$

Hence, due to $\lambda_0$-pseudomonotonicity of $\mathcal{C}$ on $\mathcal{V}$, it follows that there are subsequences $\{y_{n_k}\}_{k\geq 1} \subset \{y_n\}_{n\geq 1}$, $\{d_{n_k}\}_{k\geq 1} \subset \{d_n\}_{n\geq 1}$ such that

$$\forall w \in \mathcal{V} \quad \underline{\lim_{k\to\infty}}\,(d_{n_k}, y_{n_k} - w) \geq [\mathcal{C}(y_0), y_0 - w]_-.$$

From here,

$$[\mathcal{C}(y_0), y_0 - w]_- \leq \underline{\lim_{k\to\infty}}\,(d_{n_k}, y_{n_k} - w) \leq \overline{\lim_{n\to\infty}}\,(d_n, y_n - w) \leq (d_0, y_0 - w) \quad \forall w \in \mathcal{V}.$$

Hence $d_0 \in \mathcal{C}(y_0)$. Thus $\mathcal{C}$ satisfies condition $(\mathcal{M})$ on $\mathcal{V}$.

In virtue of Lemma 3, due to $\lambda_0$-pseudomonotonicity of $\mathcal{A}$ on $\mathcal{V}_1$ and $\mathcal{B}$ on $\mathcal{V}_2$, thanks to boundedness of $\mathcal{A}$ on $\mathcal{V}_1$, we obtain that $\mathcal{C}$ is $\lambda_0$-pseudomonotone on $\mathcal{V}$. Therefore, it satisfies condition $(\mathcal{M})$ on $\mathcal{V}$.

Now we will use the last fact to prove that $u \in \mathcal{V}$ is a solution of problem (14), (15). Let $v$ be an arbitrary element from $\mathcal{V} \cap D(\Lambda^*; \mathcal{V}^*)$. From (21) and (25) it follows that

$$(u_h, \Lambda_h^* v) + (v_h, v) = (f, v), \tag{26}$$

but

$$\Lambda_h^* v = \frac{I - G(h)^*}{h} v + \frac{I - \theta_h}{h} G(h)^* v,$$

and, due to (18), $\Lambda_h^* v \to \Lambda^* v$ in $\mathcal{V}^*$. So, if we pass to the limit in (26) as $h \searrow 0+$, then we obtain

$$(u, \Lambda^* v) + (w, v) = (f, v),\ \forall v \in \mathcal{V} \cap D(\Lambda^*; \mathcal{V}^*).$$

Then, due to b) and c), $u \in D(\Lambda, \mathcal{V}, \mathcal{V}^*)$, $\Lambda u = \chi$,

$$\Lambda u + w = f,$$

and the theorem will be proved if we show that

$$w \in \mathcal{C}(u). \tag{27}$$

On the other hand, in virtue of (21) and (25), for every $v \in \mathcal{V} \cap D(\Lambda; \mathcal{V}^*) \subset \mathcal{H}$ we obtain

$$(v_h, u_h - v) = (f, u_h - v) - (\Lambda_h v, u_h - v) - (\Lambda_h(u_h - v), u_h - v) \leq$$

$$\leq (f, u_h - v) - (\Lambda_h v, u_h - v),$$

since $\Lambda_h \geq 0$ in $\mathcal{L}(\mathcal{H}, \mathcal{H})$. From here,

$$\overline{\lim_{h \searrow 0+}} (v_h, u_h) \leq (w, v) - (f, u - v) - (\Lambda v, u - v) \quad \forall v \in \mathcal{V} \cap D(\Lambda; \mathcal{V}^*).$$

But, due to assumption (5), the same inclusion is fulfilled $\forall v \in D(\Lambda; \mathcal{V}, \mathcal{V}^*)$, and if we put $v = u$, in the last inequality we obtain

$$\overline{\lim_{h \searrow 0+}} (v_h, u_h) \leq (w, u),$$

hence inclusion (27) follows since $\mathcal{C}$ is an operator of type $(\mathcal{M})$.

The theorem is proved.

**The multivalued penalty method for evolution variation inequalities with $\lambda_0$-pseudomonotone maps.** Let us again consider the operators $A$, $\Lambda$ and the convex set $K$ such that

A) the operator $\Lambda : D(\Lambda) = D(\Lambda; \mathcal{V}, \mathcal{V}^*) \subset \mathcal{V} \to \mathcal{V}^*$ satisfies all the conditions mentioned above, in particular, the b) and d);

B) $K$ is a convex closed subset from $\mathcal{V}$ such that for every $v \in K$ there exists a sequence $v_j \in K \cap D(\Lambda)$ such that $v_j \to v$ in $\mathcal{V}$ and $\overline{\lim_{j\to\infty}} (\Lambda v_j, v_j - v) \leq 0$ (see [1, p. 396]);

C) the multivalued map $A : \mathcal{V} \to C_v(\mathcal{V}^*)$ is $\lambda_0$-pseudomonotone on $\mathcal{V}$, locally finite-dimensionally bounded, satisfies the condition $(\Pi)$, and for some

$$y_0 \in K \cap D(\Lambda) \quad \frac{[A(y), y - y_0]_+}{\|y\|_{\mathcal{V}}} \to +\infty \quad \text{as} \quad \|y\|_{\mathcal{V}} \to \infty;$$

D) $\beta : \mathcal{V} \to C_v(\mathcal{V}^*)$ is a monotone, bounded, radially semicontinuous multivalued "penalty" operator that corresponds to the set $K$, i.e., $K = \{y \in \mathcal{V} \,|\, \beta(y) \ni \bar{0}\}$.

***Remark 11*** [1, p. 284]. A sufficient condition for B) is

$$G(s)K \subset K \quad \forall s \geq 0.$$

If $\bar{0} \in K$, then this condition is fulfilled.

**Theorem 2.** *Let conditions A) – D) hold, $f \in \mathcal{V}^*$ be arbitrary and fixed. Then for each $\varepsilon > 0$ the problem*

$$\Lambda y_\varepsilon + A(y_\varepsilon) + \frac{1}{\varepsilon}\beta(y_\varepsilon) \ni f, \qquad y_\varepsilon \in D(\Lambda), \tag{28}$$

*has a solution. Moreover, there is a sequence* $\{y_\varepsilon\}_\varepsilon \subset D(\Lambda)$ *such that*
*for every* $\varepsilon > 0$ $y_\varepsilon$ *is a solution of problem (28);*
*there exists a subsequence* $\{y_\tau\}_\tau \subset \{y_\varepsilon\}_\varepsilon$ *such that for some* $y \in \mathcal{V}$ $y_\tau \to y$ *weakly in* $\mathcal{V}$*;*
$y$ *is a solution of the problem*

$$(\Lambda v, v - y) + [A(y), v - y]_+ \geq (f, v - y) \quad \forall v \in K \cap D(\Lambda), \quad y \in K. \tag{29}$$

***Proof.*** By analogy with [1, p. 396] without loss of generality we can assume that $y_0 = \bar{0} \in K$. Otherwise, the maps $\tilde{A}(\cdot) = A(\cdot - y_0)$, $\tilde{f} = f - \Lambda y_0$, $\tilde{\Lambda} = \Lambda$, the set $\tilde{K} = K - y_0$ and $\tilde{y_0} = \bar{0}$ satisfy the conditions A)–D).

For every $\varepsilon > 0$ let us consider a new multivalued map,

$$A_\varepsilon(y) := A(y) + \frac{1}{\varepsilon}\beta(y), \quad y \in \mathcal{V}.$$

In virtue of the Lemma 3 and Proposition 10, $A_\varepsilon : \mathcal{V} \to C_v(\mathcal{V}^*)$ is $\lambda_0$-pseudomonotone on $\mathcal{V}$. Due to boundedness of $\beta$, thanks to condition $(\Pi)$ and local finite-dimensional boundedness for $A$ it follows that $A_\varepsilon$ is locally finite-dimensionally bounded and satisfies condition $(\Pi)$.

Now, let us use the coercivity condition. From C) it follows that there exists of $R > 0$ such that

$$[A(y) - f, y]_+ \geq 0 \quad \forall y \in \mathcal{V} : \|y\|_\mathcal{V} = R.$$

Then, for every $\varepsilon > 0$,

$$[A_\varepsilon(y) - f, y]_+ \geq [A(y) - f, y]_+ + \frac{1}{\varepsilon}[\beta(y), y - \bar{0}]_- \geq$$

$$\geq [A(y) - f, y]_+ + \frac{1}{\varepsilon}[\beta(\bar{0}), y]_+ = [A(y) - f, y]_+ \geq 0 \quad \forall \|y\|_X = R.$$

Hence, we can apply Theorem 1 for

$$\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{V},\ D(\Lambda; \mathcal{V}; \mathcal{V}^*) = D(\Lambda),\ \Lambda = \Lambda,\ \mathcal{A} \equiv \bar{0},\ \mathcal{B} = A_\varepsilon,\ f = f,\ R = R.$$

Then we obtain that for every $\varepsilon > 0$ there exists $y_\varepsilon \in \mathcal{V}$ such that
e) $y_\varepsilon$ is a solution of (28), $\|y_\varepsilon\|_\mathcal{V} \leq R$.
We remark that the constant $R$ does not depends on $\varepsilon > 0$.

From e) it follows that there exist $d_\varepsilon \in A(y_\varepsilon)$, $b_\varepsilon \in \beta(y_\varepsilon)$ such that

$$\Lambda y_\varepsilon + d_\varepsilon + \frac{1}{\varepsilon} b_\varepsilon = f. \tag{30}$$

Due to $\bar{0} \in K \cap D(\Lambda)$ and monotonicity of $\Lambda$ and $\beta$ we have

$$(d_\varepsilon, y_\varepsilon) \leq -(\Lambda y_\varepsilon, y_\varepsilon) + \frac{1}{\varepsilon}(b_\varepsilon, \bar{0} - y_\varepsilon) + (f, y_\varepsilon) \leq \|f\|_{\mathcal{V}^*} R < +\infty.$$

In virtue of property $(\Pi)$ for $A$ and from e) it follows that there exists $c_1 > 0$ such that

$$\|d_\varepsilon\|_{\mathcal{V}^*} \leq c_1 \quad \forall \varepsilon > 0. \tag{31}$$

Moreover, from (30) and (31) it follows that

$$0 \le (b_\varepsilon, y_\varepsilon) = \varepsilon(f - d_\varepsilon - \Lambda y_\varepsilon, y_\varepsilon) \le$$
$$\le \varepsilon(\|f\|_{\mathcal{V}^*} + c_1)R =: c_2 \cdot \varepsilon \to 0 \quad \text{as} \quad \varepsilon \searrow 0+. \tag{32}$$

Since $\beta$ is monotone, using (32) we get that for every $\omega \in \mathcal{V}$

$$(b_\varepsilon, \omega) \le [\beta(y_\varepsilon), \omega - y_\varepsilon]_+ + (b_\varepsilon, y_\varepsilon) \le [\beta(\omega), \omega - y_\varepsilon]_+ + c_2\varepsilon \le$$
$$\le \|\beta(\omega)\|_-(\|\omega\|_{\mathcal{V}} + R) + c_2\varepsilon.$$

Hence, due to the Banach – Steinhauss theorem there exists $c_3 > 0$ such that

$$\|b_\varepsilon\|_{\mathcal{V}^*} \le c_3 \quad \forall \varepsilon \in (0, \varepsilon_0), \tag{33}$$

for some $\varepsilon_0 > 0$.

Conditions A) imply that for every $\omega \in D(\Lambda^*)$

$$(\Lambda y_\varepsilon, \omega) = (\Lambda^*\omega, y_\varepsilon) \le \|\omega\|_{D(\Lambda^*)} R \quad \forall \varepsilon > 0.$$

Hence, there exists $c_4 > 0$ such that

$$\|\Lambda y_\varepsilon\|_{D(\Lambda^*)^*} \le c_4 \quad \forall \varepsilon > 0.$$

From here, due to equality (30) we obtain that

$$b_\varepsilon \to \bar{0} \quad \text{in} \quad D(\Lambda^*)^* \quad \text{as} \quad \varepsilon \searrow 0+. \tag{34}$$

From (30), (31) and monotonicity of $\Lambda$ it follows that

$$(\Lambda y_\varepsilon, \omega) = (\Lambda y_\varepsilon, \omega - y_\varepsilon) + (\Lambda y_\varepsilon, y_\varepsilon) \le (\Lambda \omega, \omega - y_\varepsilon) + (\|f\|_{\mathcal{V}^*} + c_1)R \le$$
$$\le \|\Lambda\omega\|_{\mathcal{V}}(\|\omega\|_{\mathcal{V}} + R) + (\|f\|_{\mathcal{V}^*} + c_1)R \quad \forall \omega \in D(\Lambda).$$

Therefore, there exists $c_5 > 0$ such that

$$\|\Lambda y_\varepsilon\|_{D(\Lambda)^*} \le c_5 \quad \forall \varepsilon > 0.$$

*Passing to limit.* From estimate e), (31), (33), convergence in (34), due to the Banach – Alaoglu theorem it follows that there exists a subsequence $\{y_\tau\}_\tau$ from $\{y_\varepsilon\}_\varepsilon$ such that for some $y \in \mathcal{V}$, $d \in \mathcal{V}^*$,

$$y_\tau \rightharpoonup y \quad \text{in} \quad \mathcal{V},\ d_\tau \rightharpoonup d \quad \text{in} \quad \mathcal{V}^*,\ b_\tau \rightharpoonup \bar{0} \quad \text{in} \quad \mathcal{V}^* \quad \text{as} \quad \tau \searrow 0+. \tag{35}$$

In virtue of Proposition 10, the map $\beta$ is $\lambda_0$-pseudomonotone on $\mathcal{V}$. Moreover, due to (32) and (35) we have

$$\lim_{\tau\searrow 0+}(b_\tau, y_\tau - y) = \overline{\lim_{\tau\searrow 0+}}(b_\tau, y_\tau - y) \le 0.$$

Hence, since we consider a subsequence, for every $\omega \in \mathcal{V}$,

$$0 = \underline{\lim_{\tau\searrow 0+}}(b_\tau, y_\tau - \omega) \ge [\beta(y), y-\omega]_-.$$

The last relation is equivalent to $\bar{0} \in \beta(y)$. Hence, in virtue of D), we obtain that

$$y \in K. \tag{36}$$

Now, let us show that

$$\overline{\lim_{\tau\searrow 0+}}(d_\tau, y_\tau - y) \le 0. \tag{37}$$

Indeed, from (30) and D) it follows that for every $v \in D(\Lambda)\cap K$

$$\begin{aligned}(d_\tau, y_\tau - v) &= \frac{1}{\varepsilon}(b_\tau, v-y_\tau) + (f, y_\tau - v) + (\Lambda y_\tau, v - y_\tau) \le \\ &\le \frac{1}{\varepsilon}[\beta(y_\tau), v-y_\tau]_+ + (f, y_\tau - v) + (\Lambda v, v - y_\tau) \le \\ &\le \frac{1}{\varepsilon}[\beta(v), v-y_\tau]_- + (f, y_\tau - v) + (\Lambda v, v - y_\tau) \le \\ &\le (f, y_\tau - v) + (\Lambda v, v - y_\tau),\end{aligned} \tag{38}$$

since $\bar{0} \in \beta(v)$. So,

$$\overline{\lim_{\tau\searrow 0+}}(d_\tau, y_\tau) \le (d, v) + (f, y-v) + (\Lambda v, v-y) \quad \forall v \in D(\Lambda)\cap K.$$

But, in virtue of B) and (36), we can choose $v_j \in K\cap D(\Lambda)$ such that $v_j \to y$ in $\mathcal{V}$ and $\overline{\lim_{j\to\infty}}(\Lambda v_j, v_j - y) \le 0$. If we put $v = v_j$ in the last relation, we obtain

$$\overline{\lim_{\tau\searrow 0+}}(d_\tau, y_\tau) \le (d, y).$$

Therefore, due to (35), inequality (37) is true.

Let us use the $\lambda_0$-pseudomonotonicity of $A$. From (35) and (37) it follows that there exist subsequences $\{y_\nu\}_\nu \subset \{y_\tau\}_\tau$ and $\{d_\nu\}_\nu \subset \{d_\tau\}_\tau$ such that

$$\underline{\lim_{\nu\searrow 0+}}(d_\nu, y_\nu - v) \ge [A(y), y-v]_- \quad \forall v \in \mathcal{V}, \tag{39}$$

in particular, from inequality (37) it follows that

$$\lim_{\nu\searrow 0+}(d_\nu, y_\nu - y) = 0.$$

Hence, due to (35), (38), and (39),

$$[A(y), y-v]_- \le (f, y-v) + (\Lambda v, v-y) \quad \forall v \in K \cap D(\Lambda),$$

that is equivalent (due to Proposition 1) to (29).

The theorem is proved.

***Corollary 1.*** *Let assumptions A), B) and D) hold, $f \in \mathcal{V}^*$ be arbitrary fixed. Moreover, let*

*E) the multivalued map $A : \mathcal{V}_1 \to C_v(\mathcal{V}_1^*)$ be $\lambda_0$-pseudomonotone on $\mathcal{V}_1$, locally finite-dimensionally bounded, A satisfy condition ($\Pi$) and, for some $y_0 \in K \cap D(\Lambda)$, $\dfrac{[A(y), y-y_0]_+}{\|y\|_{\mathcal{V}_1}} \to$ $\to +\infty$ as $\|y\|_{\mathcal{V}_1} \to \infty$;*

*F) the functional $\varphi : \mathcal{V}_2 \to \mathbb{R}$ be convex, lower semicontinuous on $\mathcal{V}_2$ and satisfy the next coercivity condition: $\dfrac{\varphi(y)}{\|y\|_{\mathcal{V}_2}} \to +\infty$ as $\|y\|_{\mathcal{V}_2} \to \infty$.*

*Then for each $\varepsilon > 0$ the problem*

$$(\Lambda y_\varepsilon, v - y_\varepsilon) + [A(y_\varepsilon), v - y_\varepsilon]_+ + \frac{1}{\varepsilon}[\beta(y_\varepsilon), v - y_\varepsilon]_+ +$$

$$+ \varphi(v) - \varphi(y_\varepsilon) \ge (f, v - y_\varepsilon) \quad \forall v \in \mathcal{V}, \quad y_\varepsilon \in D(\Lambda), \tag{40}$$

*has a solution. Moreover, there is a sequence $\{y_\varepsilon\}_\varepsilon \subset D(\Lambda)$ such that*

*for every $\varepsilon > 0$ $y_\varepsilon$ is a solution of problem (40);*

*there exists a subsequence $\{y_\tau\}_\tau \subset \{y_\varepsilon\}_\varepsilon$ such that for some $y \in \mathcal{V}$ $y_\tau \to y$ weakly in $\mathcal{V}$;*

*$y$ is a solution of problem (2), (3).*

***Proof.*** At first let us consider the multivalued map

$$B(y) = \partial\varphi(y) \in C_v(\mathcal{V}_2^*) \quad \forall y \in \mathcal{V}_2.$$

Let us check that the given map satisfies the next conditions.

*Property* ($\Pi$). Let $y_0 \in \mathcal{V}_2$, $k > 0$ and the bounded set $B \subset \mathcal{V}_2$ be arbitrary fixed. Then $\forall y \in B$ and $\forall d(y) \in \partial\varphi(y)$ such that $(d(y), y - y_0) \le k$ is fulfilled. Let $u \in \mathcal{V}_2$ be arbitrary fixed, then

$$(d(y), u) = (d(y), u + y_0 - y) + (d(y), y - y_0) \le \varphi(u + y_0) - \varphi(y) + k \le$$

$$\le \varphi(u + y_0) - \inf_{y \in B} \varphi(y) + k \equiv \text{const} < +\infty,$$

because every convex lower semicontinuous functional is bounded from below on every bounded set. Hence, by the Banach–Steinhaus theorem, there exists $N = N(y_0, k, B)$ such that $\|d(y)\|_{\mathcal{V}_2} \le N$ for all $y \in B$.

+-*Coercivity on* $\mathcal{V}_2$. Let us put $v = y_0$ in Definition 5. Then

$$\|y\|_{\mathcal{V}_2}^{-1}\,[\partial\varphi(y), y - y_0]_+ \geq \|y\|_{\mathcal{V}_2}^{-1}\varphi(y) - \|y\|_{\mathcal{V}_2}^{-1}\varphi(y_0) \to +\infty \text{ as } \|y\|_{\mathcal{V}_2} \to +\infty.$$

$\lambda_0$-*Pseudomonotony on* $\mathcal{V}_2$. Let $y_n \rightharpoonup y_0$ in $\mathcal{V}_2$, $\partial\varphi(y_n) \ni d_n \rightharpoonup d$ in $\mathcal{V}_2^*$ and inequality (6) hold. Then, due to the monotonicity of $\partial\varphi$, for each $d_0 \in \partial\varphi(y_0)$ and for all $n \geq 1$,

$$(d_n, y_n - y_0) = (d_n - d_0, y_n - y_0) + (d_0, y_n - y_0) \geq (d_0, y_n - y_0).$$

Hence,

$$\varliminf_{n\to+\infty}(d_n, y_n - y_0) \geq \varliminf_{n\to+\infty}(d_0, y_n - y_0) = 0,$$

which, together with (6), gives

$$\lim_{n\to+\infty}(d_n, y_n - y_0) = 0.$$

Thus, for every $w \in \mathcal{V}_2$,

$$\varliminf_{n\to+\infty}(d_n, y_n - w) \geq \lim_{n\to+\infty}(d_n, y_n - y_0) + \varliminf_{n\to+\infty}(d_n, y_0 - w) = (d_0, y_0 - w). \tag{41}$$

On the other hand,

$$(d_0, w - y_0) \leq \varlimsup_{n\to+\infty}(d_n, w - y_n) \leq \varphi(w) - \varliminf_{n\to+\infty}\varphi(y_n) \leq \varphi(w) - \varphi(y_0), \tag{42}$$

because every convex lower semicontinuous functional is weakly lower semicontinuous. From (42) and from Definition 5 it follows that $d_0 \in \partial\varphi(y_0)$. From here, due to Proposition 1 and inequality (41), we will obtain inequality (7) for $\mathcal{A} = \partial\varphi$ on $\mathcal{V}_2$.

So, due to Lemma 3, Lemma 4, and Remark 11, all the assumptions C) for the multivalued map

$$C(y) = A(y) + B(y), \quad y \in \mathcal{V},$$

are true. In order to finish the proof of the statement, it is enough to note that problem (40) is equivalent to problem (28). Furthermore, problem (2), (3) is equivalent to problem (29). The last one follows from Definition 5, Proposition 1, and the formula [23]

$$D_+\varphi(u; v - u) := \lim_{t\to 0+}\frac{\varphi(u + t(v - u)) - \varphi(u)}{t} = [\partial\varphi(u), v - u]_+.$$

The corollary is proved.

**The class of multivalued penalty operators.** Let $K \subset \mathcal{V}$ be a nonempty closed convex subset,

$$P_K(y) = \arg\min_{v\in K}\|y - v\|_{\mathcal{V}}, \quad y \in \mathcal{V}.$$

We consider the main convex (generally not strictly convex) lower semicontinuous functional

$$\varphi(y) = \|y - P_K y\|_{\mathcal{V}}^2, \quad y \in \mathcal{V}.$$

Let us put

$$\beta(y) = \partial\varphi(y) \in C_v(\mathcal{V}^*), \quad y \in \mathcal{V}.$$

In virtue of properties of subdifferential maps (see [23]), the multivalued operator $\beta$ is monotone, bounded, radially semicontinuous. So, it is enough to show that

$$K = \{y \in \mathcal{V} \,|\, \bar{0} \in \beta(y)\}.$$

„$\subset$". Let $y \in K$. Then $\varphi(y) = 0$ and for every $\omega \in \mathcal{V}$, $t > 0$,

$$[\beta(y), \omega]_+ = [\partial\varphi(y), \omega]_+ \leftarrow \frac{\varphi(y + t\omega) - \varphi(y)}{t} = \frac{\varphi(y + t\omega)}{t} \geq 0,$$

as $t \searrow 0+$. Hence, $\bar{0} \in \beta(y)$.

„$\supset$". Let $\bar{0} \in \beta(y)$. Then for every $\omega \in \mathcal{V}$ (in particular for $\omega \in K$),

$$0 \leq [\beta(y), \omega - y]_+ = [\partial\varphi(y), \omega - y]_+ \leq \varphi(\omega) - \varphi(y).$$

Hence, $\varphi(y) \leq 0$ and $y \in K$.

***Example.*** Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with regular boundary, $\partial\Omega$, $S = [0, T]$ be a finite time interval, $Q = \Omega \times (0; T)$, $\Gamma_T = \partial\Omega \times (0; T)$.

As the operator $\mathcal{A}$ let us take $(\mathcal{A}u)(t) = \mathcal{A}(u(t))$, where

$$\mathcal{A}(\varphi) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( \left| \frac{\partial \varphi}{\partial x_i} \right|^{p-2} \frac{\partial \varphi}{\partial x_i} \right) + |\varphi|^{p-2}\varphi, \quad \varphi \in \mathcal{D}(\Omega).$$

Further let $k \geq 1$, $A \subset \mathbb{R}^k$ be a non-empty compact set. Let us also consider a family of maps $U_\alpha : \mathbb{R}^n \to \mathbb{R}$, where $\alpha \in A$, which satisfies the next conditions:

the map $\mathbb{R}^n \times A \ni (\xi, \alpha) \to U_\alpha(\xi) \in \mathbb{R}$ is continuous;

$\mathbb{R}^n \ni \xi \to U_\alpha(\xi) \in \mathbb{R}$ is convex for all $\alpha \in A$;

there exist $a > 0$, $b > 0$ such that

$$\|\partial U_\alpha(\xi)\|_+ \leq a + b\|\xi\| \quad \forall \xi \in \mathbb{R}^n, \ \forall \alpha \in A.$$

Together with $\{U_\alpha(\xi)\}_{\alpha \in A}$, let us consider the next function:

$$U(\xi) := \max_{\alpha \in A} U_\alpha(\xi) : \mathbb{R}^n \to \mathbb{R}$$

and the multivalued map with compact values

$$A(\xi) := \{\alpha \in A \mid U_\alpha(\xi) = U(\xi)\},$$

where $x \in \Omega$, $\xi \in \mathbb{R}^n$. Let us also assume the next coercivity condition:

d) there exist constants $M, C > 0$ such that

$$U(\xi) \geq M \|\xi\|^2 + C \quad \forall \xi.$$

We consider that

$$V_1 = W^{1,p}(\Omega), \quad V_2 = H^1(\Omega), \quad H = L_2(\Omega),$$

and

$$\mathcal{V}_1 = L_p(0,T;V_1), \quad \mathcal{H} = L_2(0,T;H), \quad \mathcal{V}_2 = L_2(0,T;V_2).$$

If we put $\mathcal{V} = \mathcal{V}_1 \cap \mathcal{V}_2$ (from here $\mathcal{V}^* = L_q(0,T;V^*) + L_2(0,T;L_2(\Omega))$, where $\frac{1}{p} + \frac{1}{q} = 1$, then with $p \geq 2$ we will obtain situation (4). Since $1 < p < 2$ the usual case will be if we take $\Phi = \mathcal{D}(0,T;V)$ (see [1]).

For the operator $\Lambda$ we take the derivative operator in the sense of the scalar distributions space $\mathcal{D}^*(0,T;V^*)$,

$$D(\Lambda;\mathcal{V},\mathcal{V}^*) = \{y \in \mathcal{V} \cap \mathcal{H} \mid y' \in \mathcal{H} + \mathcal{V}^*,\ y(0) = \bar{0}\},$$

$$G(s)\varphi(t) := \{\varphi(t-s) \quad \text{as} \quad t \geq s;\ \bar{0} \quad \text{as} \quad t \leq s\}$$

(see [1, p. 291]).

By the analogy with [1, p. 293] let us also consider a family of nonempty subsets $\Gamma_1(t)$ from $\Gamma$ such that

$$\Gamma_1(t) \supset \Gamma_1(t'), \quad \text{if} \quad t \leq t',$$

$$K = \{v \in \mathcal{V} \mid v(t) \geq 0 \quad \text{on} \quad \Gamma_1(t)\}.$$

By the analogy with [1] (Chapter 2.9.5), due to Theorem 2 and Corollary 1, the next problem:

$$\frac{\partial y(t,x)}{\partial t} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial y(t,x)}{\partial x_i}\right|^{p-2}\frac{\partial y(t,x)}{\partial x_i}\right) + |y(t,x)|^{p-2}y(t,x) -$$

$$- \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\overline{\text{co}}\left(\bigcup_{\alpha \in A(\nabla y(t,x))} \partial U_\alpha(\nabla y(t,x))\right) \ni f(t,x), \quad (t,x) \in Q,$$

$$y(t,x) \geq 0 \quad \text{on} \quad \Sigma_1 = \{(t,x) \mid x \in \Gamma_1(t)\},$$

$$\frac{\partial y(t,x)}{\partial \nu} \geq 0 \quad \text{on} \quad \Sigma_1,$$

$$y(t,x)\frac{\partial y(t,x)}{\partial \nu} = 0 \quad \text{on} \quad \Sigma_1,$$

$$\frac{\partial y(t,x)}{\partial \nu} = 0 \quad \text{on} \quad \Sigma_2 = \Sigma - \Sigma_1,$$

$$y(0,x) = 0 \quad \text{on} \quad \Omega$$

has a weak solution $y \in L_2(0,T;H^1(\Omega)) \cap L_p(0,T;W^{1,p}(\Omega))$ for every

$$f \in L_2(0,T;H^{-1}(\Omega)) \cap L_q(0,T;W^{-1,q}(\Omega)).$$

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