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EXISTENCE OF PERIODIC SOLUTIONS FOR THE PERIODICALLY FORCED SIR MODEL*

ІСНУВАННЯ ПЕРІОДИЧНИХ РОЗВ'ЯЗКІВ ДЛЯ МОДЕЛІ SIR З ПЕРІОДИЧНОЮ ІНФЕКЦІЄЮ

G. Katriel

Ort Braude College
Karmiel, Israel
e-mail: katriel@braude.ac.il

We prove that the seasonally-forced SIR model with a T-periodic forcing has a periodic solution with period T whenever the basic reproductive number $\mathcal{R}_0 > 1$. The proof uses Leray–Schauder degree theory.

Доведено, що для моделі SIR з T-періодичною інфекцією існує періодичний розв'язок з періодом T, якщо основне репродуктивне число $\mathcal{R}_0 > 1$. При цьому використано теорію Лере–Шаудера.

1. Introduction. The periodically forced SIR model

$$S' = \mu(1 - S) - \beta(t)SI, \tag{1}$$

$$I' = \beta(t)SI - (\gamma + \mu)I, \tag{2}$$

$$R' = \gamma I - \mu R, \tag{3}$$

and variants of it, are extensively used to model seasonally recurrent diseases [1, 2, 4, 5, 7–14, 16]. Here S, I, R are the fractions of the population which are susceptible, infective, and recovered, μ denotes the birth and death rate, γ the recovery rate, and $\beta(t)$, which we assume is a positive continuous T-periodic function, is the seasonally-dependent transmission rate (so that T is the yearly period). Since $S(t)$, $I(t)$, $R(t)$ are fractions of the population we require

$$S(t) + I(t) + R(t) = 1. \tag{4}$$

Note that by adding the equations (1)–(3) we have $(S(t) + I(t) + R(t))' = 0$, so that if the initial conditions satisfy $S(0) + I(0) + R(0) = 1$, (4) will hold for all t.

When simulating this model numerically, it is observed that:

(i) If $\mathcal{R}_0 \le 1$, where

$$\mathcal{R}_0 = \frac{\bar\beta}{\gamma + \mu},$$

$$\bar\beta = \frac{1}{T}\int_0^T \beta(t)\,dt,$$

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then all solutions tend to the disease-free equilibrium $S_0 = 1$, $I_0 = 0$, $R_0 = 0$. This fact can be rigorously proved, see [15].

(ii) If $\mathcal{R}_0 > 1$, then depending on the values of the parameters, one observes convergence to T-periodic orbits, or to nT-periodic orbits with $n > 1$ (subharmonics), or chaotic behavior.

A fundamental question, that is addressed here, is the *existence* of a T-periodic solution of the system. We demand, of course that the components $S(t)$, $I(t)$, $R(t)$ of this solution will be positive. Obviously when $\mathcal{R}_0 \leq 1$, a positive periodic solution cannot exist, because such a solution would not converge to the disease-free equilibrium. We will prove, however, that the following theorem.

Theorem 1. *Whenever $\mathcal{R}_0 > 1$, there exists at least one T-periodic solution $(S(t), I(t), R(t))$ of (1)–(3), all of whose components are positive and satisfy (4).*

Thus, when T-periodic behavior is not observed in simulations, this is not due to the fact that such a solution does not exist, but rather to the fact that all T-period solutions are unstable. Such unstable periodic solutions cannot be found by direct simulation, but they can be computed by numerical continuation techniques.

Despite the fact that the existence of a T-periodic solution of the T-periodically forced SIR model is a fundamental issue, the only paper in the literature of which we are aware to have dealt with this question is the recent paper of Jódar, Villanueva and Arenas [9]. They treated a more general system than we do here (including loss of immunity and allowing other coefficients besides $\beta(t)$ to be T-periodic). Restricting their existence result to the case of the SIR model (1)–(3), they proved, using Mawhin's continuation theorem, that a T-periodic solution exists whenever the condition

$$\min_{t \in \mathbb{R}} \beta(t) > \gamma + \mu \tag{5}$$

holds. Note that the condition (5) implies that $\bar{\beta} > \gamma + \mu$, that is $\mathcal{R}_0 > 1$, but it is a stronger condition. Theorem 1 uses only the condition $\mathcal{R}_0 > 1$, so that together with the fact noted above, that when $\mathcal{R}_0 \leq 1$ a T-periodic solution does *not* exist, we have that $\mathcal{R}_0 > 1$ is a *necessary and sufficient* condition for the existence of a T-periodic solution with positive components.

Our technique for proving Theorem 1 relies on nonlinear functional analysis, for which we refer to the textbooks [3, 19]. Reformulating the problem as one of solving an equation in an infinite dimensional space of periodic functions, we define a homotopy between the periodically forced problem and the autonomous problem in which $\beta(t)$ is replaced by the mean $\bar{\beta}$. The autonomous problem has an endemic equilibrium, which is a trivial periodic solution. We then employ Leray–Schauder degree theory to continue this solution along the homotopy. The challenge here lies in the fact that there we always have a trivial periodic solution, given by the disease-free equilibrium, which lies on the boundary of the relevant domain D in the functional space, which requires us to construct a smaller domain $U \subset D$ excluding the trivial solution, and to show that the conditions for applying the Leray–Schauder theory hold for the domain U.

We note that our proof of Theorem 1 is easily extended to give the same result for the SIR model, which includes loss of immunity

$$S' = \alpha S + \mu(1-S) - \beta(t)SI, \tag{6}$$

$$I' = \beta(t)SI - (\gamma+\mu)I, \tag{7}$$

$$R' = \gamma I - (\mu+\alpha)R. \tag{8}$$

We present the proof for the SIR model ($\alpha = 0$) in order to avoid notational clutter.

In Section 2 we prove Theorem 1. In the discussion Section 3 we mention some other works providing rigorous mathematical results on forced SIR models, beyond numerical simulation.

2. Proof of Theorem 1. Since R does not appear in (1), (2) and can be determined from (4) once S and I are known, the equation (3) can be ignored, and it suffices to proved the existence of a periodic solution of (1), (2) satisfying

$$S(t) > 0, I(t) > 0, S(t) + I(t) < 1 \quad \forall t, \tag{9}$$

where the third condition is equivalent to $R(t) = 1 - I(t) - S(t) > 0$.

We decompose $\beta(t)$ as

$$\beta(t) = \bar{\beta} + \beta_0(t), \qquad \int\limits_0^T \beta_0(t)\,dt = 0.$$

Setting, for $\lambda \in [0,1]$,

$$\beta_\lambda(t) = \bar{\beta} + \lambda\beta_0(t), \tag{10}$$

we consider the system

$$S' = \mu(1-S) - \beta_\lambda(t)SI, \tag{11}$$

$$I' = \beta_\lambda(t)SI - (\gamma+\mu)I, \tag{12}$$

which is homotopy between an unforced system with $\beta_0(t) = \bar{\beta}$ and our system (1), (2), which corresponds to $\lambda = 1$. For $\lambda = 0$, (11), (12) has exactly two periodic solutions, which are constant, given by

$$S_0 = 1, \quad I_0 = 0, \tag{13}$$

$$S^* = \frac{\gamma+\mu}{\bar{\beta}}, \quad I^* = \mu\left(\frac{1}{\gamma+\mu} - \frac{1}{\bar{\beta}}\right). \tag{14}$$

We note that (S_0, I_0) (the disease-free equilibrium) is in fact a (trivial) periodic solution of (11), (12) for *all* λ. Our aim is to continue the solution (S^*, I^*) with respect to λ in order to prove the existence of a periodic solution for $\lambda = 1$. To this end, we now reformulate the problem in a functional-analytic setting, which will enable us to employ degree theory.

We rewrite (11), (12) as

$$S' + \mu S = \mu - \beta_\lambda(t)SI, \tag{15}$$

$$I' + (\gamma + \mu)I = \beta_\lambda(t)SI. \tag{16}$$

Let X, Y be the Banach spaces

$$X = \{(S, I) | S, I \in C^1(\mathbb{R}), S(t + T) = S(t), I(t + T) = I(t)\},$$

$$Y = \{(S, I) | S, I \in C^0(\mathbb{R}), S(t + T) = S(t), I(t + T) = I(t)\}.$$

Define the linear operator $L : X \to Y$ by

$$L(S, I) = (S' + \mu S, I' + (\gamma + \mu)I)$$

and the nonlinear operator $N : Y \to Y$

$$N_\lambda(S, I) = (\mu - \beta_\lambda(t)SI, \beta_\lambda(t)SI).$$

Then the periodic problem for (15), (16) can be rewritten as

$$L(S, I) = N_\lambda(S, I). \tag{17}$$

It is easy to check that L is invertible, that is the equations $S' + \mu S = f$ and $I' + (\gamma + \mu)I = g$ have unique C^1 T-periodic solutions S, I for any $f, g \in Y$, and the mapping $L^{-1} : Y \to X$ given by $L^{-1}(f, g) = (S, I)$ is bounded. We can thus rewrite (17) as

$$F_\lambda(S, I) = 0, \tag{18}$$

where $F_\lambda : Y \to X$ is given by

$$F_\lambda(S, I) = (S, I) - L^{-1} \circ N_\lambda(S, I). \tag{19}$$

Since $L^{-1} : Y \to X$ is bounded, and since, by the Arzela – Ascoli theorem, X is compactly embedded in Y, we can consider L^{-1} as a compact operator from Y to itself, and since $N : Y \to Y$ is continuous, $L^{-1} \circ N_\lambda$ is compact as an operator from Y to itself. We therefore consider (18) in the space Y, and we note that any solution in Y will in fact be in X, hence a classical solution of (15), (16). Since F_λ is a compact perturbation of the identity on Y, Leray – Schauder theory is applicable. Since we want our solution to satisfy (9), we want to solve (18) in the subset $D \subset Y$ given by

$$D = \{(S, I) \in Y \mid S(t) > 0,\ I(t) > 0,\ S(t) + I(t) < 1\}.$$

Note that for $\lambda = 0$ the solution (S^*, I^*) given by (14) lies in D. Our aim is to continue this solution in λ up to $\lambda = 1$.

We recall that the Leray – Schauder degree theory (see, e.g., [3, 19]) implies that, given a bounded open set $U \subset Y$, the existence of a solution (S, I) of (18) for all $\lambda \in [0,1]$ will be assured if the following conditions hold:

(I) $(S^*, I^*) \in U$,

(II) $\deg(F_0, (S^*, I^*)) \neq 0$,

(III) $F_\lambda(S, I) \neq 0$ for all $(S, I) \in \partial U$, $\lambda \in [0,1]$.

The most obvious choice for U would be $U = D$. However, this will not do, since (S_0, I_0) (given by (13)) satisfies $(S_0, I_0) \in \partial D$ and $F_\lambda(S_0, I_0) = 0$, so that (III) does not hold. To satisfy (III) we will need to choose U so as to exclude (S_0, I_0) from its boundary. We take U to be the open subset of D given by

$$U = \{(S, I) \in D \mid \min_{t \in \mathbb{R}} S(t) < \delta\}, \tag{20}$$

where $\delta \in (0,1)$ is fixed. Note that $(S_0, I_0) \notin \bar{U}$. We will show below that U satisfies (I) – (III) if δ is chosen so that $\delta \in (\mathcal{R}_0^{-1}, 1)$.

We first show that (S_0, I_0) is the *only* solution of (18) on ∂D.

Lemma 1. *If $(S, I) \in \partial D$ is a solution of (18) for some $\lambda \in [0,1]$, then $(S, I) = (S_0, I_0)$, as given by (13).*

Proof. Assume that $(S, I) \in \partial D$ is a solution of (15), (16). Note that $(S, I) \in \partial D$, if an only if

$$S(t) \geq 0, \quad I(t) \geq 0, \quad S(t) + I(t) \leq 1, \tag{21}$$

and at least one of the following conditions holds:

(i) there exists $t_0 \in \mathbb{R}$ so that $I(t_0) = 0$,

(ii) there exists $t_0 \in \mathbb{R}$ so that $S(t_0) = 0$,

(iii) there exists $t_0 \in \mathbb{R}$ so that $S(t_0) + I(t_0) = 1$.

We now consider each of these three cases:

(1) Assume (i) holds. Let $\tilde{S}$ be a solution of

$$\tilde{S}' = \mu(1 - \tilde{S}), \quad \tilde{S}(t_0) = S(t_0)$$

and let $\tilde{I}(t) = 0$. Then $\tilde{S}$, $\tilde{I}$ is a solution of the initial-value problem (15), (16) with initial condition

$$\tilde{S}(t_0) = S(t_0), \quad \tilde{I}(t_0) = 0.$$

By uniqueness of the solution for the initial-value problem, we conclude that $S = \tilde{S}$, $I = 0$. Thus S satisfies $S' = \mu(1 - S)$, and since the only periodic solution of this equation is $S = 1$, we conclude that $(S, I) = (1, 0)$, as we wanted to prove.

(2) Assume now that (ii) holds. Then from (15) we get $S'(t_0) = \mu$. But this implies that $S(t) < 0$ for $t < t_0$ sufficiently close to t_0, which contradicts (21). Thus this case is impossible.

(3) Assume now that (iii) holds. Moreover since we have already proven the result in the case that (i) holds, we may assume that $I(t) > 0$ for all t. Adding (11) and (12) we get

$$(S + I)'(t_0) = \mu(1 - S(t_0) - I(t_0)) - \gamma I(t_0) = -\gamma I(t_0) < 0.$$

Therefore we conclude that $S(t)+I(t) > 1$ for $t < t_0$ sufficiently close to t_0, contradicting (21). Therefore this case is impossible.

Lemma 1 is proved.

We can now show that U, defined by (20), satisfies (III).

Lemma 2. *Assume $\mathcal{R}_0 > 1$. If $\frac{1}{\mathcal{R}_0} < \delta < 1$ then, for any $\lambda \in [0,1]$ there are no solutions (S,I) of (18) with $(S,I) \in \partial U$.*

Proof. Suppose $(S,I) \in \partial U$. Then either $(S,I) \in \partial D$ or $(S,I) \in D$ and

$$\min_{t\in\mathbb{R}} S(t) = \delta. \tag{22}$$

In the first case, Lemma 1 and the fact that $(S_0, I_0) \notin \partial U$ imply that (S,I) is not a solution of (18). We therefore assume that $(S,I) \in D$ and (22) holds, which implies that

$$S(t) \geq \delta \quad \forall t. \tag{23}$$

Assume by way of contradiction that (S,I) solves (18), or equivalently (S,I) solves (15), (16). Using the assumption $(S,I) \in D$, we have that I is everywhere positive, so we can divide (16) by I, and integrate over $[0,T]$, to obtain

$$\frac{1}{T}\int_0^T \beta_\lambda(t)S(t)\,dt = \gamma + \mu. \tag{24}$$

But from (23) we get

$$\frac{1}{T}\int_0^T \beta_\lambda(t)S(t)\,dt \geq \delta\frac{1}{T}\int_0^T \beta_\lambda(t)\,dt = \delta\bar{\beta}. \tag{25}$$

By the assumption $\delta > \frac{1}{\mathcal{R}_0}$ we have $\delta\bar{\beta} > \gamma + \mu$, so that (25) implies

$$\frac{1}{T}\int_0^T \beta_\lambda(t)S(t)\,dt > \gamma + \mu,$$

contradicting (24).

Lemma 2 is proved.

To apply the Leray – Schauder degree it remains to verify that (I) and (II) hold. Since $S^* = \frac{1}{\mathcal{R}_0}$, the condition $\delta > \frac{1}{\mathcal{R}_0}$ implies $(S^*, I^*) \in U$, so (I) holds. To prove (II), it suffices to show that the Fréchet derivative $DF_0(S^*, I^*)$ is invertible. Since F is a compact perturbation of the identity so that $DF_0(S^*, I^*)$ is Fredholm, it suffices to prove that the kernel of $DF_0(S^*, I^*)$

is trivial. Indeed, let us assume that $(V,W) \in \ker(DF_0(S^*,I^*))$, and prove that $(V,W) = 0$. We have $DF_0(S^*,I^*)(V,W) = 0$, or, equivalently,

$$L(V,W) = DN_0(S^*,I^*)(V,W). \tag{26}$$

Note that

$$DN_0(S^*,I^*)(V,W) = (-\bar{\beta}(S^*W + I^*V), \bar{\beta}(S^*W + I^*V)),$$

so that (26) is equivalent to

$$\begin{pmatrix} V \\ W \end{pmatrix}' = \begin{pmatrix} -\mu\mathcal{R}_0 & -(\gamma+\mu) \\ \mu(\mathcal{R}_0 - 1) & 0 \end{pmatrix} \begin{pmatrix} V \\ W \end{pmatrix}. \tag{27}$$

The characteristic polynomial of the above matrix is

$$p(x) = x^2 + \mu\mathcal{R}_0 x + (\gamma+\mu)\mu(\mathcal{R}_0 - 1).$$

Noting that $p(0) > 0$ and that, for $\omega \in \mathbb{R}$, $\mathrm{Im}\,(p(\omega i)) = \mu\mathcal{R}_0\omega$, we see that the matrix has no imaginary or 0 eigenvalues, so that (27) has no periodic solutions except $(V,W) = (0,0)$, and the claim is proved.

We have thus proven that (I) – (III) hold, which completes the proof of Theorem 1.

3. Discussion. The forced SIR model is a beautiful example of a simple nonlinear dynamical system which displays complicated behaviors which are difficult to understand in intuitive terms. Moreover, these complicated behaviors are relevant to explaining the epidemiology of infectious diseases in humans, as studies comparing the behavior of the SIR and variants of it to surveillance data have shown [2, 5, 12]. We have proven the fundamental result that a T-periodic solution exists for the T-periodically forced SIR model whenever $\mathcal{R}_0 > 1$. As we have stressed, this does not mean that the dynamics of the model is periodic, since the periodic solution whose existence is proved need not be stable, although one can use standard perturbation theory to prove that the T-periodic solution *is* stable provided the seasonality parameter λ in (10) is sufficiently small. Numerical simulations show that complex dynamics — subharmonic and chaotic behavior — is very common in the forced SIR model. It is interesting to ask to what extent the complex dynamics of the forced SIR model can be rigorously understood, beyond numerical simulations. While we do not expect to be able to precisely characterize the dynamics of the model for different parameter values, it is of great interest even to be able to rigourously prove that complicated dynamics occurs for at least *some* parameter values. In this context we mention the work of H. L. Smith [17, 18], who proved that the forced SIR model can have multiple stable subharmonic oscillations in certain parameter ranges. Chaotic behavior has been rigorously established by Glendinning and Perry [6] for a variant of the forced SIR model, in which the dependence of the incidence term on I is nonlinear. For the standard SIR model (1) – (3), we are not aware of a proof of chaotic behavior. Classifying and explaining the dynamical patterns observed in simulations of the forced SIR model is still very challenging, so that, like other well-known 'simple' models such as the forced pendulum equation, the forced SIR model can serve as a stimulus and as a benchmark problem for new developments in nonlinear analysis.

1. *Aron J. L., Schwartz I. B.* Seasonality and period-doubling bifurcations in an epidemic model // J. Theor. Biol. — 1984. — **110**. — P. 665 – 679.
2. *Bolker B. M., Grenfell B. T.* Chaos and biological complexity in measles dynamics // Proc. Roy. Soc. London B. — 1993. — **251**. — P. 75 – 81.
3. *Brown R. F.* A topological introduction to nonlinear analysis. — Boston: Birkhäuser, 1993.
4. *Dietz K.* The incidence of infectious diseases under the influence of seasonal fluctuations // Lect. Notes Biomath. — New York: Springer, 1976. — **11**. — P. 1 – 15.
5. *Earn D. J. D., Rohani P., Bolker B. M., Grenfell B. T.* A simple model for complex dynamical transitions in epidemics // Science. — 2000. — **287**. — P. 667 – 670.
6. *Glendinning P., Perry L. P.* Melnikov analysis of chaos in a simple epidemiological model // J. Math. Biol. — 1997. — **35**. — P. 359 – 373.
7. *Grassly N. C., Fraser C.* Seasonal infectious disease epidemiology // Proc. Roy. Soc. London B. — 2006. — **273**. — P. 2541 – 2550.
8. *Greenman J., Kamo M., Boots M.* External forcing of ecological and epidemiological systems: a resonance approach // Physica D. — 2004. — **190**. — P. 136 – 151.
9. *Jódar L., Villanueva R. J., Arenas A.* Modeling the spread of seasonal epidemiological diseases: Theory and applications // Math. and Comput. Modelling. — 2008. — **48**. — P. 548 – 557.
10. *Katriel G., Stone L.* Attack rates of seasonal epidemics // Math. Biosci. — 2012. — **235**. — P. 56 – 65.
11. *Keeling M. J., Rohani P., Grenfell B. T.* Seasonally forced disease dynamics explored as switching between attractors // Physica D. — 2001. — **148**. — P. 317 – 335.
12. *Keeling M. J., Grenfell B. T.* Understanding the persistence of measles: reconciling theory, simulation and observation // Proc. Roy. Soc. London B. — 2002. — **269**. — P. 335 – 343.
13. *Kuznetsov Y. A., Piccardi C.* Bifurcation analysis of periodic SEIR and SIR epidemic models // J. Math. Biol. — 1994. — **32**. — P. 109 – 121.
14. *London W., Yorke J. A.* Recurrent outbreaks of measles, chickenpox and mumps. i. seasonal variation incontact rates // Amer. J. Epidemiol. — 1973. — **98**. — P. 453 – 468.
15. *Ma J., Ma Z.* Epidemic threshold conditions for seasonally forced SEIR models // Math. Biol. Eng. — 2006. — **3**. — P. 161 – 172.
16. *Olinky R., Huppert A., Stone L.* Seasonal dynamics and thresholds governing recurrent epidemics // J. Math. Biol. — 2008. — **56**. — P. 827 – 839.
17. *Smith H. L.* Subharmonic bifurcation in an S-I-R epidemic model // J. Math. Biol. — 1983. — **17**. — P. 163 – 177.
18. *Smith H. L.* Multiple stable subharmonics for a periodic epidemic model // J. Math. Biol. — 1983. — **17**. — P. 179 – 190.
19. *Zeidler E.* Nonlinear functional analysis and its applications, I. — New York: Springer, 1993.

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