UDC 517.9

EXISTENCE OF SOLUTIONS FOR A CLASS OF FOURTH-ORDER MULTIPOINT BOUNDARY-VALUE PROBLEMS ON TIME SCALES*

ІСНУВАННЯ РОЗВ'ЯЗКІВ ДЛЯ ОДНОГО КЛАСУ БАГАТОТОЧКОВИХ ГРАНИЧНИХ ЗАДАЧ ЧЕТВЕРТОГО ПОРЯДКУ НА ЧАСОВІЙ ШКАЛІ

Zengji Du, Zhengmin Fu

School Math. Sci., Xuzhou Normal Univ.
Xuzhou, Jiangsu 221116
e-mail: duzengji@163.com
xiaomin_5832614@163.com

Lingju Kong

Univ. Tennessee at Chattanooga
Chattanooga, TN 37403, USA
e-mail: lingju-kong@utc.edu

This paper deals with the existence of solutions for the dynamic equation on time scales

$$u^{\triangle\triangle\triangle\triangle}(t) = f(t, u(\sigma(t)), u^{\triangle\triangle}(t)), \quad t \in [0,1]_T,$$

with the multipoint boundary conditions

$$u(0) = 0, \quad u(\sigma(1)) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad u^{\triangle\triangle}(0) = 0, \quad u^{\triangle\triangle}(\sigma(1)) = \sum_{j=1}^{n-2} b_j u^{\triangle\triangle}(\eta_j),$$

where T is a time scale, $[0,1]_T = \{t \in T \mid 0 \le t \le 1\}$, $a_i > 0$, $i = 1, 2, \ldots, m-2$, $b_j > 0$, $j = 1, 2, \ldots, n-2$, $0 < \xi_1 < \xi_2 < \ldots < \xi_{m-2} < \rho(1)$, $0 < \eta_1 < \eta_2 < \ldots < \eta_{n-2} < \rho(1)$. *The existence result is given by using Green's function, the method of upper and lower solutions, and the monotone iterative technique. We also give an example to illustrate our result.*

Розглянуто проблему існування розв'язків для динамічного рівняння на часовій шкалі

$$u^{\triangle\triangle\triangle\triangle}(t) = f(t, u(\sigma(t)), u^{\triangle\triangle}(t)), \quad t \in [0,1]_T,$$

з багатоточковими граничними умовами

$$u(0) = 0, \quad u(\sigma(1)) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad u^{\triangle\triangle}(0) = 0, \quad u^{\triangle\triangle}(\sigma(1)) = \sum_{j=1}^{n-2} b_j u^{\triangle\triangle}(\eta_j),$$

де T — часова шкала, $[0,1]_T = \{t \in T | 0 \le t \le 1\}$, $a_i > 0$, $i = 1, 2, \ldots, m-2$, $b_j > 0$, $j = 1, 2, \ldots, n-2$, $0 < \xi_1 < \xi_2 < \ldots < \xi_{m-2} < \rho(1)$, $0 < \eta_1 < \eta_2 < \ldots < \eta_{n-2} < \rho(1)$. *Існування доведено з використанням функції Гріна, методу верхніх та нижніх розв'язків і техніки монотонної ітерації. Наведено приклад для ілюстрації рузультатів.*

* This project is supported by the National Natural Science Foundation of China (11071205) and NSF of Jiangsu Province (BK2008119gundong) and the NSF of the Education Department of Jiangsu Province.

1. Introduction. Throughout this paper, we assume that the reader has a basic knowledge of time scales and time scale notations that were first introduced by Hilger [1] and later refined in the monographs of Bohner and Peterson [2, 3]. In recent years, various boundary-value problems (BVPs, for short) on time scales have been studied by many authors, (see, for example, [2–9] and the references therein). Let T be a time scale with $0, 1 \in T$. This paper deals with the following fourth-order dynamic equation:

$$u^{\triangle\triangle\triangle\triangle}(t) = f(t, u(\sigma(t)), u^{\triangle\triangle}(t)), \quad t \in [0, 1]_T, \tag{1.1}$$

with the multipoint boundary conditions

$$\begin{gathered} u(0) = 0, \quad u(\sigma(1)) = \sum_{i=1}^{m-2} a_i u(\xi_i), \\ u^{\triangle\triangle}(0) = 0, \quad u^{\triangle\triangle}(\sigma(1)) = \sum_{j=1}^{n-2} b_j u^{\triangle\triangle}(\eta_j), \end{gathered} \tag{1.2}$$

where T is a time scale, $a_i > 0$, $i = 1, 2, \ldots, m-2$, $b_j > 0$, $j = 1, 2, \ldots, n-2$, $0 < \xi_1 < < \xi_2 < \ldots < \xi_{m-2} < \rho(1)$, $0 < \eta_1 < \eta_2 < \ldots < \eta_{n-2} < \rho(1)$, $0 < \sum_{i=1}^{m-2} a_i \xi_i < \sigma(1)$, $0 < \sum_{j=1}^{n-2} b_j \eta_j < \sigma(1)$.

Throughout this work, we assume

$$H = \sigma(1) - \sum_{i=1}^{m-2} a_i \xi_i > 0, \quad N = \sigma(1) - \sum_{j=1}^{n-2} b_j \eta_j > 0. \tag{1.3}$$

Recently, there is much attention focused on the existence of solution for fourth-order BVPs (see [5, 6, 10–14]). In particular, Feng et al. [10] applied the lower and upper solution method to study the existence and uniqueness for the following fourth-order two-point BVP:

$$\begin{gathered} x''''(t) - f(t, x(t), x'(t), x''(t), x'''(t)) = 0, \quad t \in (0, 1), \\ x(0) = x'(1) = 0, \\ ax''(0) - bx'''(0) = 0, \quad cx''(1) - dx'''(1) = 0, \end{gathered}$$

where $a, b, c, d \geq 0$, $\rho = ad + bc + ac > 0$. Wang and Sun [5] applied the Schauder fixed point theorem and obtained some existence criteria for positive solutions of the following dynamic equation on time scales:

$$u^{\triangle\triangle\triangle\triangle}(t) - f(t, u(t), u^{\triangle\triangle}(t)) = 0, \quad t \in [a, \rho^2(b)]_T,$$

$$u(0) = A, \quad u(\sigma^2(b)) = B, \quad u^{\triangle\triangle}(a) = 0, \quad u^{\triangle\triangle}(b) = 0.$$

Zhang et al. [11] employed the method of upper and lower solutions to establish the existence of positive solutions to fourth-order BVP

$$y^{(4)}(t) = f(t, y(t)),$$
$$y(0) = y(1) = 0,$$
$$ay''(\xi_1) - by'''(\xi_1) = 0, \quad cy''(\xi_2) - dy'''(\xi_2) = 0.$$

Recently, Pang and Bai [6] generalized the problem discussed in [11] to the following fourth-order BVP on time scales:

$$u^{\triangle\triangle\triangle\triangle}(t) = f(t, u(\sigma(t)), u^{\triangle\triangle}(t)), \quad t \in [0,1]_T,$$
$$u(0) = u(\sigma(1)) = 0,$$
$$\alpha u^{\triangle\triangle}(\xi_1) - \beta u^{\triangle\triangle\triangle}(\xi_1) = 0, \quad \gamma u^{\triangle\triangle}(\xi_2) - \eta u^{\triangle\triangle\triangle}(\xi_2) = 0.$$

Motivated by the work of the above papers, the purpose of this article is to study the existence of solution for BVP (1.1), (1.2). We first establish the form of the solution and furthermore give the Green's function for associated BVP. The BVP (1.1), (1.2) and the method used in the paper are different from other fourth-order BVP on time scales, so our results are new. The tools we mainly used are the method of upper and lower solutions and the monotone iterative technique.

The paper is organized as follows. Section 2 states some definitions and a lemma which are important to obtain our main result. Section 3 is devoted to the existence result of BVP (1.1) and (1.2). Section 4 gives an example to illustrate our main result.

2. Preliminary. For the convenience of readers, we first present some definitions and some lemmas that are important to the proofs of our main results.

Definition 2.1. *A function $\varphi(t) \in C^4[0,1]_T$ is an upper solution of BVP (1.1), (1.2), provided that*

$$\varphi^{\triangle\triangle\triangle\triangle}(t) \geq f(t, \varphi(\sigma(t)), \varphi^{\triangle\triangle}(t)), \quad t \in [0,1]_T,$$
$$\varphi(0) \geq 0, \quad \varphi(\sigma(1)) \geq \sum_{i=1}^{m-2} a_i \varphi(\xi_i),$$
$$\varphi^{\triangle\triangle}(0) \leq 0, \quad \varphi^{\triangle\triangle}(\sigma(1)) \leq \sum_{j=1}^{n-2} b_j \varphi^{\triangle\triangle}(\eta_j).$$

Definition 2.2. *A function $\psi(t) \in C^4[0,1]_T$ is called a lower solution of BVP (1.1), (1.2) if*

$$\psi^{\triangle\triangle\triangle\triangle}(t) \le f(t, \psi(\sigma(t)), \psi^{\triangle\triangle}(t)), \quad t \in [0,1]_T,$$

$$\psi(0) \le 0, \; \psi(\sigma(1)) \le \sum_{i=1}^{m-2} a_i \psi(\xi_i),$$

$$\psi^{\triangle\triangle}(0) \ge 0, \quad u^{\triangle\triangle}(\sigma(1)) \ge \sum_{j=1}^{n-2} b_j \psi^{\triangle\triangle}(\eta_j).$$

Lemma 2.1. *If $h(t) \in C[0,1]_T$, then the problem*

$$\begin{aligned} & u^{\triangle\triangle}(t) = h(t), \\ & u(0) = 0, \quad u(\sigma(1)) = \sum_{i=1}^{m-2} a_i u(\xi_i), \end{aligned} \tag{2.1}$$

has the unique solution

$$u(t) = -\int_0^{\sigma(1)} G_1(t,s) h(s) \triangle s,$$

where

$$G_1(t,s) = \begin{cases} s \in [0, \xi_1] & : \dfrac{1}{H} \begin{cases} \sigma(s)\left[(\sigma(1) - t) + \sum\limits_{i=1}^{m-2} a_i (t - \xi_i)\right], & s \le t, \\ t\left[(\sigma(1) - \sigma(s)) + \sum\limits_{i=1}^{m-2} a_i(\sigma(s) - \xi_i)\right], & t \le s, \end{cases} \\ \begin{array}{l} s \in [\xi_{i-1}, \xi_i], \\ 2 \le i \le m-2 \end{array} & : \dfrac{1}{H} \begin{cases} \sigma(s)(\sigma(1) - t) + \sum\limits_{j=i}^{m-2} a_j (t - \xi_i)\sigma(s) + \\ \quad + \sum\limits_{j=1}^{i-1} a_j \xi_j (t - \sigma(s)), & s \le t, \\ t\left[(\sigma(1) - \sigma(s)) + \sum\limits_{j=i}^{m-2} a_j(\sigma(s) - \xi_j)\right], & t \le s, \end{cases} \\ s \in [\xi_{m-2}, \sigma(1)] & : \dfrac{1}{H} \begin{cases} \sigma(s)(\sigma(1) - t) + \sum\limits_{i=1}^{m-2} a_i \xi_i (t - \sigma(s)), & s \le t, \\ t(\sigma(1) - \sigma(s)), & t \le s, \end{cases} \end{cases} \tag{2.2}$$

where H is as in (1.3).

Proof. Integrating the first equation of (2.1) over the interval $[0,r]_T$ for $r \in [0,1]_T$, we have

$$u^{\triangle}(r) = u^{\triangle}(0) + \int_0^r h(s) \triangle s.$$

Now, integrating the above equation over the interval $[0,t]_T$ for $t\in[0,1]_T$ and using [15] (Lemma 3), it follows that

$$u(t) = u(0) + u^{\triangle}(0)t + \int_0^t (t-\sigma(s))h(s)\triangle s. \tag{2.3}$$

Combining this with boundary condition (2.1) we conclude that

$$u^{\triangle}(0) = \frac{1}{H}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - \sigma(1))h(s)\triangle s - \frac{1}{H}\int_0^{\sigma(1)} (\sigma(1)-\sigma(s))h(s)\triangle s. \tag{2.4}$$

Substituting this equation into (2.3), we have

$$u(t) = \int_0^t (t-\sigma(s))h(s)\triangle s + \frac{t}{H}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - \sigma(s))h(s)\triangle s -$$

$$-\frac{t}{H}\int_0^{\sigma(1)} (\sigma(1)-\sigma(s))h(s)\triangle s.$$

Suppose $0 < t \le \xi_1$, then the unique solution of (2.1) can be written as

$$u(t) = -\int_0^t (\sigma(s)-t)h(s)\triangle s - \frac{t}{H}\left[\int_0^t (\sigma(1)-\sigma(s))h(s)\triangle s + \int_t^{\xi_1} (\sigma(1)-\sigma(s))h(s)\triangle s + \right.$$

$$\left. + \sum_{i=2}^{m-2} \int_{\xi_{i-1}}^{\xi_i} (\sigma(1)-\sigma(s))h(s)\triangle s + \int_{\xi_{m-2}}^{\sigma(1)} (\sigma(1)-\sigma(s))h(s)\triangle s\right] -$$

$$-\frac{t}{H}\left[\sum_{i=1}^{m-2} a_i \int_0^t (\sigma(s)-\xi_i)h(s)\triangle s + \sum_{i=1}^{m-2} a_i \int_t^{\xi_1} (\sigma(s)-\xi_i)h(s)\triangle s + \right.$$

$$\left. + \sum_{i=2}^{m-2} a_i \int_{\xi_1}^{\xi_i} (\sigma(s)-\xi_i)h(s)\triangle s\right] =$$

$$= -\frac{1}{H}\left[\int_0^t \sigma(s)\left[(\sigma(1)-t) + \sum_{i=1}^{m-2} a_i(t-\xi_i)\right] h(s)\triangle s\right] -$$

$$-\frac{1}{H}\left[\int_t^{\xi_1} t\left[\sum_{i=1}^{m-2} a_i(\sigma(s)-\xi_i) + (\sigma(1)-\sigma(s))\right] h(s)\triangle s\right] -$$

$$- \frac{1}{H} \left[\sum_{i=2}^{m-2} \int_{\xi_{i-1}}^{\xi_i} t \left[(\sigma(1) - \sigma(s)) + \sum_{j=i}^{m-2} a_j (\sigma(s) - \xi_j) \right] h(s) \triangle s \right] -$$

$$- \frac{1}{H} \left[\int_{\xi_{m-2}}^{\sigma(1)} t(\sigma(1) - \sigma(s)) h(s) \triangle s \right] = - \int_0^{\sigma(1)} G_1(t,s) h(s) \triangle s.$$

Suppose $\xi_{r-1} \le t \le \xi_r$, $2 \le r \le m-2$, then the unique solution of (2.1) can be written as

$$u(t) = - \left[\int_0^{\xi_1} (\sigma(s) - t) h(s) \triangle s + \sum_{j=2}^{r-1} \int_{\xi_{j-1}}^{\xi_j} (\sigma(s) - t) h(s) \triangle s + \int_{\xi_{r-1}}^{t} (\sigma(s) - t) h(s) \triangle s \right] -$$

$$- \frac{t}{H} \left[\int_0^{\xi_1} (\sigma(1) - \sigma(s)) h(s) \triangle s + \sum_{j=2}^{r-1} \int_{\xi_{j-1}}^{\xi_j} (\sigma(1) - \sigma(s)) h(s) \triangle s + \right.$$

$$+ \int_{\xi_{r-1}}^{t} (\sigma(1) - \sigma(s)) h(s) \triangle s + \int_t^{\xi_r} (\sigma(1) - \sigma(s)) h(s) \triangle s +$$

$$\left. + \sum_{j=r+1}^{m-2} \int_{\xi_{j-1}}^{\xi_j} (\sigma(1) - \sigma(s)) h(s) \triangle s + \int_{\xi_{m-2}}^{\sigma(1)} (\sigma(1) - \sigma(s)) h(s) \triangle s \right] -$$

$$- \frac{t}{H} \left[\sum_{j=1}^{m-2} a_j \int_0^{\xi_1} (\sigma(s) - \xi_j) h(s) \triangle s + \sum_{i=2}^{r-1} \sum_{j=i}^{m-2} a_j \int_{\xi_{i-1}}^{\xi_i} (\sigma(s) - \xi_j) h(s) \triangle s + \right.$$

$$+ \sum_{j=r}^{m-2} a_j \int_{\xi_{r-1}}^{t} (\sigma(s) - \xi_j) h(s) \triangle s + \sum_{j=r}^{m-2} a_j \int_t^{\xi_r} (\sigma(s) - \xi_j) h(s) \triangle s +$$

$$\left. + \sum_{i=r+1}^{m-2} \sum_{j=i}^{m-2} a_j \int_{\xi_{i-1}}^{\xi_i} (\sigma(s) - \xi_j) h(s) \triangle s \right] =$$

$$= -\frac{1}{H} \int_0^{\xi_1} \sigma(s) \left[(\sigma(1) - t) + \sum_{i=1}^{m-2} a_i (t - \xi_i) \right] h(s) \triangle s -$$

$$- \frac{1}{H} \sum_{i=2}^{r-1} \int_{\xi_{i-1}}^{\xi_i} \left[\sigma(s)(\sigma(1) - t) + \sum_{j=i}^{m-2} a_j (t - \xi_j) \sigma(s) + \sum_{j=1}^{i-1} a_j \xi_j (t - \sigma(s)) \right] h(s) \triangle s -$$

$$-\frac{1}{H}\int\limits_{\xi_{r-1}}^{t}\left[\sigma(s)(\sigma(1)-t)+\sum_{j=r}^{m-2}a_j(t-\xi_j)\sigma(s)+\sum_{j=1}^{r-1}a_j\xi_j(t-\sigma(s))\right]h(s)\triangle s-$$

$$-\frac{1}{H}\int\limits_{t}^{\xi_r}t\left[(\sigma(1)-\sigma(s))+\sum_{j=r}^{m-2}a_j(\sigma(s)-\xi_j)\right]h(s)\triangle s-$$

$$-\frac{1}{H}\left[\sum_{i=r+1}^{m-2}\int\limits_{\xi_{i-1}}^{\xi_i}t[(\sigma(1)-\sigma(s)+\sum_{j=i}^{m-2}a_j(\sigma(s)-\xi_j)]h(s)\triangle s\right]-$$

$$-\frac{1}{H}\left[\int\limits_{\xi_{m-2}}^{\sigma(1)}t(\sigma(1)-\sigma(s))h(s)\triangle s\right]=-\int\limits_{0}^{\sigma(1)}G_1(t,s)h(s)\triangle s.$$

Suppose $\xi_{m-2}\le t\le\sigma(1)$, then the unique solution of (2.1) can be written as

$$u(t)=-\left[\int\limits_{0}^{\xi_1}(\sigma(s)-t)h(s)\triangle s+\sum_{j=2}^{m-2}\int\limits_{\xi_{j-1}}^{\xi_j}(\sigma(s)-t)h(s)\triangle s+\int\limits_{\xi_{m-2}}^{t}(\sigma(s)-t)h(s)\triangle s\right]-$$

$$-\frac{t}{H}\left[\int\limits_{0}^{\xi_1}(\sigma(1)-\sigma(s))h(s)\triangle s+\sum_{j=2}^{m-2}\int\limits_{\xi_{j-1}}^{\xi_j}(\sigma(1)-\sigma(s))h(s)\triangle s+\right.$$

$$\left.+\int\limits_{\xi_{m-2}}^{t}(\sigma(1)-\sigma(s))h(s)\triangle s+\int\limits_{t}^{\sigma(1)}(\sigma(1)-\sigma(s))h(s)\triangle s\right]-$$

$$-\frac{t}{H}\sum_{i=1}^{m-2}a_i\int\limits_{0}^{\xi_i}(\sigma(s)-\xi_i)h(s)\triangle s=$$

$$=-\frac{1}{H}\int\limits_{0}^{\xi_1}\sigma(s)\left[(\sigma(1)-t)+\sum_{j=1}^{m-2}a_j(t-\xi_j)\right]h(s)\triangle s-$$

$$-\frac{1}{H}\sum_{i=2}^{m-2}\int\limits_{\xi_{i-1}}^{\xi_i}\left[\sigma(s)(\sigma(1)-t)+\sum_{j=i}^{m-2}a_j(t-\xi_j)\sigma(s)+\sum_{j=1}^{i-1}a_j\xi_j(t-\sigma(s))\right]h(s)\triangle s-$$

$$-\frac{1}{H}\int\limits_{\xi_{m-2}}^{t}\left[\sigma(s)(\sigma(1)-t)+\sum_{j=1}^{m-2}a_j\xi_j(t-\sigma(s))\right]h(s)\triangle s-$$

$$-\frac{1}{H}\int\limits_{t}^{\sigma(1)} t(\sigma(1)-\sigma(s))h(s)\triangle s = -\int\limits_{0}^{\sigma(1)} G_1(t,s)h(s)\triangle s.$$

Then the unique solution of (2.1) can be written as $u(t) = -\int_0^{\sigma(1)} G_1(t,s)h(s)\triangle s$, for $t \in$ $\in [0,1]_T$.

Lemma 2.1 is proved.

We observe that condition $0 < \sum_{i=1}^{m-2} a_i\xi_i < \sigma(1)$ implies $G_1(t,s) \geq 0$ for any $(t,s) \in$ $\in [0,1]_T \times \in [0,1]_T$.

Similar to Lemma 2.1, we give the following lemma.

Lemma 2.2. *If $h(t) \in C[0,1]_T$, we have the Green's function of*

$$\begin{aligned} &u^{\triangle\triangle}(t) = h(t), \\ &u(0) = 0, \quad u(\sigma(1)) = \sum_{j=1}^{n-2} b_j u(\eta_j) \end{aligned} \tag{2.5}$$

is

$$G_2(t,s) = \begin{cases} s \in [0,\eta_1] \quad : \dfrac{1}{N}\begin{cases} \sigma(s)\left[(\sigma(1)-t) + \sum\limits_{j=1}^{n-2} b_j(t-\eta_j)\right], & s \leq t, \\ t\left[(\sigma(1)-\sigma(s)) + \sum\limits_{j=1}^{n-2} b_j(\sigma(s)-\eta_j)\right], & t \leq s, \end{cases} \\ \begin{matrix} s \in [\eta_{j-1},\eta_j] \\ 2 \leq j \leq n-2 \end{matrix} \quad : \dfrac{1}{N}\begin{cases} \sigma(s)(\sigma(1)-t) + \sum\limits_{k=j}^{n-2} b_k(t-\eta_k)\sigma(s) + \\ + \sum\limits_{k=1}^{j-1} b_k\eta_k(t-\sigma(s)), & s \leq t, \\ t\left[(\sigma(1)-\sigma(s)) + \sum\limits_{k=j}^{n-2} b_k(\sigma(s)-\eta_k)\right], & t \leq s, \end{cases} \\ s \in [\eta_{n-2},\sigma(1)] \ : \dfrac{1}{N}\begin{cases} \sigma(s)(\sigma(1)-t) + \sum\limits_{j=1}^{n-2} b_j\eta_j(t-\sigma(s)), & s \leq t, \\ t(\sigma(1)-\sigma(s)), & t \leq s, \end{cases} \end{cases} \tag{2.6}$$

where N is as in (1.3).

Lemma 2.3. *If* $u(t) \in C^4[0,1]_T$ *satisfies*

$$u^{\triangle\triangle\triangle\triangle}(t) \geq 0, \quad t \in [0,1]_T,$$
$$u(0) \geq 0, \quad u(\sigma(1)) \geq \sum_{i=1}^{m-2} a_i u(\xi_i), \tag{2.7}$$
$$u^{\triangle\triangle}(0) \leq 0, \quad u^{\triangle\triangle}(\sigma(1)) \leq \sum_{j=1}^{n-2} b_j u^{\triangle\triangle}(\eta_j),$$

then $u(t) \geq 0$, $u^{\triangle\triangle}(t) \leq 0$.

Proof. Let

$$u^{\triangle\triangle\triangle\triangle}(t) = h(t),$$
$$u(0) = x_0, \quad u(\sigma(1)) - \sum_{i=1}^{n-2} a_i u(\xi_i) = x_1, \tag{2.8}$$
$$u^{\triangle\triangle}(0) = x_2, \quad u^{\triangle\triangle}(\sigma(1)) - \sum_{j=1}^{n-2} b_j u^{\triangle\triangle}(\eta_j) = x_3,$$

where $x_0 \geq 0$, $x_1 \geq 0$, $x_2 \leq 0$, $x_3 \leq 0$ and $h(t) \geq 0.$
Let $u^{\triangle\triangle}(t) = w(t)$, then, from (2.8),

$$w^{\triangle\triangle}(t) = h(t),$$
$$w(0) = x_2, \quad w(\sigma(1)) - \sum_{j=1}^{n-2} b_j w(\eta_j) = x_3.$$

We obtain by Lemma 2.1 that

$$w(t) = -\int_0^{\sigma(1)} G_2(t,s)h(s)\triangle s + \frac{\sum_{j=1}^{n-2} b_j t - t + \sigma(1) - \sum_{j=1}^{n-2} b_j\eta_j}{N}\, x_2 + \frac{x_3}{N}\, t$$

and (2.8) changes to

$$u^{\triangle\triangle}(t) = -\int_0^{\sigma(1)} G_2(t,s)h(s)\triangle s + \frac{\sum_{j=1}^{n-2} b_j t - t + \sigma(1) - \sum_{j=1}^{n-2} b_j\eta_j}{N}\, x_2 + \frac{x_3}{N}\, t,$$
$$u(0) = x_0, \quad u(\sigma(1)) - \sum_{i=1}^{m-2} a_i u(\xi_i) = x_1.$$

By Lemma 2.1, we obtain that

$$u(t) = \int_0^{\sigma(1)} G_1(t,\xi) \int_0^{\sigma(1)} G_2(t,s)h(s)\triangle s\triangle\xi - \int_0^{\sigma(1)} G_1(t,\xi)R(\xi)\triangle\xi +$$

$$+ \frac{\sum_{i=1}^{m-2} a_i t - t + \sigma(1) - \sum_{i=1}^{m-2} a_i\xi_i}{H}\, x_0 + \frac{x_1}{H}\, t,$$

where

$$R(t) = \frac{\sum_{j=1}^{n-2} b_j t - t + \sigma(1) - \sum_{j=1}^{n-2} b_j\eta_j}{N}\, x_2 + \frac{x_3}{N}\, t.$$

On the other hand, the assumptions of the lemma implies that $R(t) < 0, G_1(t,s), G_2(t,s) \geq \geq 0$ for any $(t,s) \in [0,1]_T \times [0,1]_T$. Thus, $u(t) \geq 0$ and $u^{\triangle\triangle}(t) \leq 0$.

Lemma 2.3 is proved.

3. Main result.

Theorem 3.1. *Assume that $\varphi(t)$ and $\psi(t)$ are upper and lower solutions to BVP (1.1), (1.2), respectively, and satisfy*

$$\psi(t) \leq \varphi(t) \quad \textit{and} \quad \psi^{\triangle\triangle}(t) \geq \varphi^{\triangle\triangle}(t), \tag{3.1}$$

and that $f : [0,1]_{\mathrm{T}} \times R \times R \longrightarrow R$ is continuous and satisfies

$$\begin{aligned} &f(t,u_1,v) - f(t,u_2,v) \leq 0, \quad \psi(t) \leq u_1 \leq u_2 \leq \varphi(t), \quad v \in R, \quad t \in [0,1]_T, \\ &f(t,u,v_1) - f(t,u,v_2) \geq 0, \quad \varphi^{\triangle\triangle}(t) \leq v_1 \leq v_2 \leq \psi^{\triangle\triangle}(t), \quad u \in R, \quad t \in [0,1]_T. \end{aligned} \tag{3.2}$$

Then, there exist two function sequences $\{\varphi_n(t)\}$ and $\{\psi_n(t)\}$ that converge uniformly to the solutions of the BVP (1.1), (1.2).

Proof. We consider the operator $S : C^2[0,1]_T \longrightarrow C^4[0,1]_T$ defined by

$$Su(t) = \int_0^{\sigma(1)} G_1(t,\xi) \int_0^{\sigma(1)} G_2(t,s) f(s, u(\sigma(s)), u^{\triangle\triangle}(s))\triangle s\triangle\xi, \tag{3.3}$$

where $G_1(t,s)$ and $G_2(t,s)$ are as in (2.2) and (2.6).
Now, we divide the proof into three steps.

Step 1. We show

$$SD \subset D, \tag{3.4}$$

where

$$D = \{h \in C^2[0,1]_T \mid \psi(t) \leq h(t) \leq \varphi(t),\ \psi^{\triangle\triangle}(t) \geq h^{\triangle\triangle}(t) \geq \varphi^{\triangle\triangle}(t)\}$$

is a nonempty bounded closed subset in $C^2[0,1]_T$.

In fact, for any $u(t) \in D$, set $r(t) = Su(t)$, then

$$r^{\triangle\triangle\triangle\triangle}(t) = f(t, u(\sigma(t)), u^{\triangle\triangle}(t)), \quad t \in [0,1]_T,$$

$$r(0) = 0, \quad r(\sigma(1)) = \sum_{i=1}^{m-2} a_i r(\xi_i),$$

$$r^{\triangle\triangle}(0) = 0, \quad r^{\triangle\triangle}(\sigma(1)) = \sum_{j=1}^{n-2} b_j r^{\triangle\triangle}(\eta_j).$$

Let $w(t) = \varphi(t) - r(t)$, from the definition of $\varphi(t)$, $\psi(t)$, D and condition (3.2) of the theorem, we have

$$w^{\triangle\triangle\triangle\triangle}(t) = (\varphi(t) - r(t))^{\triangle\triangle\triangle\triangle} \geq f(t, \varphi(\sigma(t)), \varphi^{\triangle\triangle}(t)) - f(t, u(\sigma(t)), u^{\triangle\triangle}(t)),$$

$$w(0) \geq 0, \quad w(\sigma(1)) \geq \sum_{i=1}^{m-2} a_i w(\xi_i),$$

$$w^{\triangle\triangle}(0) \leq 0, \quad w^{\triangle\triangle}(\sigma(1)) \leq \sum_{j=1}^{n-2} b_j w^{\triangle\triangle}(\eta_j).$$

In virtue of Lemma 2.3, we can obtain that $w(t) \geq 0$, and $w^{\triangle\triangle}(t) \leq 0$, so $r(t) \leq \varphi(t)$ and $r^{\triangle\triangle}(t) \geq \varphi^{\triangle\triangle}(t)$. Analogously we can prove that $\psi(t) \leq r(t)$, and $r^{\triangle\triangle}(t) \leq \psi^{\triangle\triangle}(t)$. Thus, (3.4) holds.

Step 2. Let $u_1(t) = S\eta_1(t)$, $u_2(t) = S\eta_2(t)$, where $\eta_1(t), \eta_2(t) \in D$ satisfy

$$\psi(t) \leq \eta_1(t) \leq \eta_2(t) \leq \varphi(t) \quad \text{and} \quad \psi^{\triangle\triangle}(t) \geq \eta_1^{\triangle\triangle}(t) \geq \eta_2^{\triangle\triangle}(t) \geq \varphi^{\triangle\triangle}(t),$$

we show

$$u_1(t) \leq u_2(t), \quad u_1(t)^{\triangle\triangle} \geq u_2^{\triangle\triangle}(t). \tag{3.5}$$

In fact, if we let $u_3(t) = u_1(t) - u_2(t)$, then

$$u_3^{\triangle\triangle\triangle\triangle}(t) = (u_2 - u_1)^{\triangle\triangle\triangle\triangle}(t) = f(t, \eta_2(\sigma(t)), \eta_2^{\triangle\triangle}(t)) - f(t, \eta_1(\sigma(t)), \eta_1^{\triangle\triangle}(t)) \geq 0,$$

$$u_3(0) = 0, \quad u_3(\sigma(1)) = \sum_{i=1}^{m-2} a_i u_3(\xi_i),$$

$$u_3^{\triangle\triangle}(0) = 0, \quad u_3^{\triangle\triangle}(\sigma(1)) = \sum_{j=1}^{n-2} b_j u_3^{\triangle\triangle}(\eta_j).$$

Using Lemma 2.3, we get that $u_1(t) \leq u_2(t)$ and $u_1^{\triangle\triangle}(t) \geq u_2^{\triangle\triangle}(t)$.

Step 3. Now, we define the sequence $\{\varphi_n(t)\}$ and $\{\psi_n(t)\}$ by

$$\varphi_n(t) = S\varphi_{n-1}(t), \quad \varphi_0(t) = \varphi(t),$$
$$\psi_n(t) = S\psi_{n-1}(t), \quad \psi_0(t) = \psi(t).$$

From the properties of S, step 1 and step 2, we have that

$$\begin{aligned}
&\varphi(t) \geq \varphi_1(t) \geq \varphi_2(t) \geq \ldots \geq \varphi_n(t) \geq \ldots \geq \psi(t),\\
&\varphi^{\triangle\triangle}(t) \leq \varphi_1^{\triangle\triangle}(t) \leq \varphi_2^{\triangle\triangle}(t) \leq \ldots \leq \varphi_n^{\triangle\triangle}(t) \leq \ldots \leq \psi(t),\\
&\psi(t) \leq \psi_1(t) \leq \psi_2(t) \leq \ldots \leq \psi_n(t) \leq \ldots \leq \varphi(t),\\
&\psi^{\triangle\triangle}(t) \geq \psi_1^{\triangle\triangle}(t) \geq \psi_2^{\triangle\triangle}(t) \geq \ldots \geq \psi_n^{\triangle\triangle}(t) \geq \ldots \geq \varphi^{\triangle\triangle}(t).
\end{aligned} \tag{3.6}$$

Moreover, from the definition of $\varphi_n(t)$, we obtain

$$\varphi_n^{\triangle\triangle\triangle\triangle}(t) = f(t, \varphi_{n-1}(\sigma(t)), \varphi_{n-1}^{\triangle\triangle}(t)), \quad t \in [0,1]_T,$$
$$\varphi_n(0) = 0, \quad \varphi_n(\sigma(1)) = \sum_{i=1}^{m-2} a_i \varphi_n(\xi_i),$$
$$\varphi_n^{\triangle\triangle}(0) = 0, \quad \varphi_n^{\triangle\triangle}(\sigma(1)) = \sum_{j=1}^{n-2} b_j \varphi_n^{\triangle\triangle}(\eta_j).$$

Which together with the conditions (3.2) of the theorem imply that

$$f(t, \varphi(\sigma(t)), \varphi^{\triangle\triangle}(t)) \geq \ldots \geq f(t, \varphi_n(\sigma(t)), \varphi_n^{\triangle\triangle}(t)) \geq \ldots \geq f(t, \psi(\sigma(t)), \psi^{\triangle\triangle}(t)),$$
$$f(t, \varphi(\sigma(t)), \varphi^{\triangle\triangle}(t)) \geq \ldots \geq f(t, \psi_n(\sigma(t)), \psi_n^{\triangle\triangle}(t)) \geq f(t, \psi(\sigma(t)), \psi^{\triangle\triangle}(t)).$$

So we have

$$\begin{aligned}
&\varphi^{\triangle\triangle\triangle\triangle}(t) \geq \varphi_1^{\triangle\triangle\triangle\triangle}(t) \geq \varphi_2^{\triangle\triangle\triangle\triangle}(t) \geq \ldots \geq \varphi_n^{\triangle\triangle\triangle\triangle}(t) \geq \ldots \geq \psi^{\triangle\triangle\triangle\triangle}(t),\\
&\psi^{\triangle\triangle\triangle\triangle}(t) \leq \psi_1^{\triangle\triangle\triangle\triangle}(t) \leq \psi_2^{\triangle\triangle\triangle\triangle}(t) \leq \ldots \leq \psi_n^{\triangle\triangle\triangle\triangle}(t) \leq \ldots \leq \varphi^{\triangle\triangle\triangle\triangle}(t).
\end{aligned} \tag{3.7}$$

Hence, from (3.6) and (3.7) we know there exists a constant M such that

$$\begin{aligned}
&|\varphi_n(t)| \leq M, \quad |\psi_n(t)| \leq M, \quad t \in [0,1]_T, \quad n = 1, 2, \ldots, n, \ldots,\\
&|\varphi_n^{\triangle\triangle}(t)| \leq M, \quad |\psi_n^{\triangle\triangle}(t)| \leq M, \quad t \in [0,1]_T, \quad n = 1, 2, \ldots, n, \ldots,\\
&|\varphi_n^{\triangle\triangle\triangle\triangle}(t)| \leq M, \quad |\psi_n^{\triangle\triangle\triangle\triangle}(t)| \leq M, \quad t \in [0,1]_T, \quad n = 1, 2, \ldots, n, \ldots,
\end{aligned} \tag{3.8}$$

and thus

$$|\varphi_n^{\triangle\triangle\triangle}(t)| = |(\varphi_n^{\triangle\triangle}(t))^{\triangle}| = \left| \left(\int_0^1 G_2(t,s)\varphi_n^{\triangle\triangle\triangle\triangle}(s) \right)^{\triangle_t} \triangle s \right| =$$

$$= \left| \left(\int_0^1 (G_2(t,s))^{\triangle_t} \triangle s \right) \varphi_n^{\triangle\triangle\triangle\triangle}(s) \right| \le \frac{2\sigma(1)^2 - \sum_{j=1}^{n-2} b_j\eta_j}{N} M, \tag{3.9}$$

$$|\varphi_n^{\triangle}(t)| = |(\varphi_n(t))^{\triangle}| = \left| \left(\int_0^1 G_2(t,s)\varphi_n^{\triangle\triangle}(s) \right)^{\triangle_t} \triangle s \right| =$$

$$= \left| \left(\int_0^1 (G_1(t,s))^{\triangle_t} \triangle s \right) \varphi_n^{\triangle\triangle\triangle}(s) \right| \le \frac{2\sigma(1)^2 - \sum_{i=1}^{m-2} a_i\xi_i}{H} M. \tag{3.10}$$

Hence, from (3.6), (3.7) and (3.9), (3.10), we know that $\{\varphi_n(t)\}$ and $\{\psi_n(t)\}$ are uniformly bounded in $C^4[0,1]_T$.

On the other hand, let

$$M_1 = \max\left\{ \frac{2\sigma(1)^2 - \sum_{i=1}^{m-2} a_i\xi_i}{H} M, \frac{2\sigma(1)^2 - \sum_{j=1}^{n-2} b_j\eta_j}{N} M \right\}.$$

Then, for any $t_1, t_2 \in [0,1]_T$,

$$|\varphi_n(t_1) - \varphi_n(t_2)| \le M|t_1 - t_2|, \quad |\psi_n(t_1) - \psi_n(t_2)| \le M|t_1 - t_2|,$$

$$|\varphi_n^{\triangle\triangle}(t_1) - \varphi_n^{\triangle\triangle}(t_2)| \le (M_1 + M)|t_1 - t_2|,$$

$$|\psi_n^{\triangle\triangle}(t_1) - \psi_n^{\triangle\triangle}(t_2)| \le (M_1 + M)|t_1 - t_2|.$$

So, $\{\varphi_n(t)\}$, $\{\psi_n(t)\}$ and $\{\varphi_n^{\triangle\triangle}(t)\}$ and $\{\psi_n^{\triangle\triangle}(t)\}$ are equicontinuous. Thus, by applying the Arzela – Ascoli theorem on time scales [4] and (3.6) and (3.7) we know that $\{\varphi_n(t)\}$, $\{\psi_n(t)\}$ converge uniformly to solutions of the BVP (1.1), (1.2).

4. Example. Let $T = \left\{1 - \frac{1}{2}\right\}^{N_0} \cup \{1\}$, where N_0 denotes the set of all nonnegative integers. We consider the following fourth-order dynamic equation:

$$u^{\triangle\triangle\triangle\triangle}(t) = 3u^{\triangle\triangle}(t) + \left(\sin \pi t + \frac{1}{2}\right)^2, \tag{4.1}$$

with the boundary conditions

$$u(0) = 0, \quad u(\sigma(1)) = \frac{1}{4}\,u\left(\frac{1}{4}\right) + \frac{1}{3}\,u\left(\frac{1}{2}\right),$$
$$u^{\triangle\triangle}(0) = 0, \quad u^{\triangle\triangle}(\sigma(1)) = \frac{1}{5}\,u^{\triangle\triangle}\left(\frac{1}{2}\right) + \frac{1}{6}\,u^{\triangle\triangle}\left(\frac{3}{4}\right). \tag{4.2}$$

Here $m = n = 4, \xi_1 = \dfrac{1}{4}, \xi_2 = \dfrac{1}{2}, a_1 = \dfrac{1}{4}, a_2 = \dfrac{1}{3}, \eta_1 = \dfrac{1}{2}, \eta_2 = \dfrac{3}{4}, b_1 = \dfrac{1}{5}, b_2 = \dfrac{1}{6}$.

It is easily to prove that $\alpha(t) = \int_0^t (c-s)\triangle s + 2, c = \int_0^{\sigma(1)} \tau\triangle\tau, \beta(t) = 0$, are upper and lower upper solutions of (4.1), (4.2), respectively, and that all assumptions of Theorem 3.1 are fulfilled. So the BVP (4.1), (4.2) has at least a solution $u(t)$ satisfying

$$0 \le u(t) \le \int\limits_0^t (c-s)\triangle s + 2, \quad -1 \le u^{\triangle\triangle}(t) \le 0.$$

1. *Hilger S.* Analysis on measure chains–a unified approach to continuous and discrete calculus // Result. Math. — 1990. — **18**. — P. 18–56.
2. *Bohner M., Peterson A.* Dynamic equations on time scales. An introduction with applications. — Boston: Birkhäuser, 2001.
3. *Bohner M., Peterson A.* Advances in dynamic equations on time scales. — Boston: Birkhäuser, 2003.
4. *Agarwal R. P., Bohner M., Rehak P.* Half-linear dynamic equations // Nonlinear Anal. and Appl. — 2003. — **1**. — P. 1–57.
5. *Wang D. B., Sun J. P.* Existence of a solution and a positive solution of a boundary value problem for a nonlinear fourth-order dynamic equation // Nonlinear Anal. — 2008. — **69**. — P. 1817–1823.
6. *Pang Y. Y., Bai Z. B.* Upper and lower solutions method for a fourth-order four point boundary value problem on time scales // Appl. Math. Comput. — 2009. — **215**. — P. 2243–2247.
7. *Lin X. J., Du Z. J.* Positive solutions of M-point boundary value problem for a nonlinear for a second-order dynamic equation on time scales // J. Difference Equat. and Appl. — 2008. — **14** — P. 851–864.
8. *Sang Y. B., Su H., Xu F. Y.* Positive solutions of nonlinear M-point BVP for an increasing homeomorphism and homeomorphism with sign changing nonlinearity on time scales // Comput. Math. Appl. — 2009. — **58**. — P. 216–226.
9. *Hao Z. C., Liang J., Xiao T. J.* Existence results for time scale boundary value problem // J. Comput. Appl. Math. — 2009. — **197**. — P. 156–168.
10. *Feng H. F., Ji D. H., Ge W. G.* Existence and uniqueness of solutions for a fourth-order boundary-value problem // Nonlinear Anal. — 2009. — **70**. — P. 3561–3566.
11. *Zhang Q., Chen S. H., Lu J. H.* Upper and lower solution method for four-order four-point boundary value problem and topological degree // J. Comput. Appl. Math. — 2006. — **196**. — P. 387–393.
12. *Bai Z. B.* The upper and lower solution method for some fourth-order boundary value problem // Nonlinear Anal. — 2007. — **67**. — P. 1704–1709.
13. *Zhang B. G., Kong L., Sun Y., Deng X.* Existence of Positive solutions for BVPs of fourth-order difference equations // Appl. Math. Comput. — 2002. — **131**. — P. 583–591.
14. *Ma R. Y., Zhang J. H., Fu S. M.* The method of upper and lower solutions for fourth-order two-point boundary value problems // J. Math. Anal. and Appl. — 1997. — **215**. — P. 415–422.
15. *Bohner M., Stević S.* Asymptotic behavior of second-order dynamic equations // Appl. Math. Comput. — 2007. — **188**. — P. 1503–1512.

Received 24.05.11