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CLASSIFICATION AND EXISTENCE OF NONOSCILLATORY SOLUTIONS OF SECOND-ORDER NEUTRAL DELAY DYNAMIC EQUATIONS ON TIME SCALES*

КЛАСИФІКАЦІЯ ТА ІСНУВАННЯ НЕКОЛИВНИХ РОЗВ'ЯЗКІВ ДИНАМІЧНИХ РІВНЯНЬ ДРУГОГО ПОРЯДКУ З НЕЙТРАЛЬНИМ ЗАПІЗНЕННЯМ НА ЧАСОВІЙ ШКАЛІ

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In this paper, we give a classification of nonoscillatory solutions of the second-order neutral delay dynamic equation on time scales

$$[x(t) - c(t)x(t-\tau)]^{\Delta\Delta} + f(t, x(g_1(t)), \ldots, x(g_m(t))) = 0, \quad t \in T.$$

Some existence results for each kind of nonoscillatory solutions are also established.

Наведено класифікацію неколивних розв'язків динамічних рівнянь другого порядку з нейтральним запізненням на часовій шкалі

$$[x(t) - c(t)x(t-\tau)]^{\Delta\Delta} + f(t, x(g_1(t)), \ldots, x(g_m(t))) = 0, \quad t \in T,$$

а також доведено існування неколивних розв'язків кожного типу.

1. Introduction. The theory of time scales was introduced by Hilger [10] in 1988 in order to unify continuous and discrete analysis. Recently, the study of dynamic equations on time scales has received a lot of attention. The general idea is to prove a result for a dynamic equation where the domain of the unknown function is a so-called time scale, which is a special case of a measure chain. By choosing the time scale to be the set of real numbers, the general results yields a result concerning a differential equation. On the other hand, by choosing the time scale to be the set of integers, the same general result yields a result for difference equations. However, since there are many other time scales than just the set of real numbers or the set of integers, one has a much more general results. The monographs by Bohner and Peterson [4, 5] and the survey on dynamic equations on time scales by Agarwal, Bohner, O'Regan and Peterson [2] summarized some important work in this area. Recently, the oscillation of dynamic equations on time scales has received much attention (see [6–8, 12–18]). But the classification and existence of nonoscillatory solutions of the delay dynamic equations on time scales received much less attention.

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In this paper, we consider the second-order neutral delay dynamic equation on time scales

$$[x(t) - c(t)x(t-\tau)]^{\Delta\Delta} + f(t, x(g_1(t)), \ldots, x(g_m(t))) = 0, \tag{1.1}$$

where $t \in [t_0, \infty) = T_0 \subseteq T$. With respect to (1.1), throughout we shall assume the following:

(H_1) $\tau > 0$, $c(t) \in C_{rd}(T_0, \mathbf{R}_+), \mathbf{R}_+ = [0, \infty)$, there exists $\delta \in (0, 1]$ such that $c(t) \leq 1 - \delta$ for $t \in T_0$.

(H_2) $g_i \in C_{rd}(T_0, \mathbf{R}_+)$, and $\lim_{t\to\infty} g_i(t) = \infty$, $i = 1, 2, \ldots, m$.

(H_3) $f : T_0 \times \mathbf{R}^m \to \mathbf{R}$ is right-dense continuous on T_0 and continuous with respect to the last m arguments, $y_1 f(t, y_1, \ldots, y_m) > 0$ for $y_1 y_i > 0$, $i = 2, \ldots, m$. Moreover,

$$|f(t, x_1, \ldots, x_m)| \geq |f(t, y_1, \ldots, y_m)|$$

when $|y_i| \leq |x_i|$ and $x_i y_i > 0$, $i = 1, 2, \ldots, m$.

For convenience, we set

$$y(t) = x(t) - c(t)x(t-\tau). \tag{1.2}$$

In Section 3, we will study the existence and asymptotic behavior of nonoscillatory solutions of equation (1.1). More precisely, we give a classification of nonoscillatory solutions of equation (1.1) according to their asymptotic behavior. Moreover, we established some existence results for each kind of nonoscillatory solutions of equation (1.1). In particular, we obtain two necessary and sufficient conditions for the existence of nonoscillatory solutions of (1.1).

2. Preliminaries. To understand the delay dynamic equations on time scales we need some preliminary definitions (see [4]).

Let T be a *time scale* (i.e., a closed subset of the real numbers $\mathbf{R}$) with $\sup T = \infty$. We assume throughout that T has the topology that it inherits from the standard topology on the real numbers $\mathbf{R}$.

Definition 2.1. *For $t \in T$ we define the forward jump operator $\sigma : T \to T$ by*

$$\sigma(t) := \inf\{s \in T : s > t\}$$

while the backwards jump operator $\rho : T \to T$ is defined by

$$\rho(t) := \sup\{s \in T : s < t\}.$$

If $\sigma(t) > t$, we say that t is right-scattered, while if $\rho(t) < t$ we say that t is left-scattered. Points that are right-scattered and left-scattered at the same time are called isolated. Also, if $t < \sup T$ and $\sigma(t) = t$, then t is called right-dense, and if $t > \inf T$ and $\rho(t) = t$, then t is called left-dense. Points that are right-dense and left-dense at same time are called dense.

Definition 2.2. *Define the interval in T*

$$[a, b] := \{t \in T \quad \textit{such that} \quad a \leq t \leq b\}.$$

Open intervals and half-open intervals etc. are defined accordingly. Note that $[a, b]^K = [a, b]$ if b is left-dense and $[a, b]^K = [a, b) = [a, \rho(b)]$ if b is left-scattered.

Definition 2.3. *Assume $f : T \to \mathbf{R}$ and let $t \in T$ (if $t = \sup T$ assume t is not left-scattered), then we define $f^{\Delta}(t)$ to be the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood U of t (i.e., $U = (t-\delta, t+\delta) \cap T$ for some $\delta > 0$) such that*

$$|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]| \leq \epsilon|\sigma(t) - s|, \quad \textit{for all} \quad s \in U.$$

We call $f^{\Delta}(t)$ the delta (or Hilger) derivative of f at t.

It can be shown that if $f : T \to \mathbf{R}$ is continuous at $t \in T$ and t is right-scattered, then

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

If t is right-dense, then

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{t - s}.$$

Lemma 2.1. *Assume $g : T \to R$ to be differentiable and $g^{\Delta}(t) \geq 0$. Then $g(t)$ is nondecreasing.*

Definition 2.4. *We say $f : T \to \mathbf{R}$ is right-dense continuous on T provided it is continuous at all right-dense points and at points that are left-dense and right-scattered we just assume the left hand limit exists (and is finite). We denote this by $f \in C_{rd}(T, \mathbf{R})$.*

Lemma 2.2. *Assume $f : T \to R$ to be differentiable at t, then f is continuous at t.*

Definition 2.5. *If $F^{\Delta}(t) = f(t)$, then we define an integral by*

$$\int_a^t f(\tau)\Delta\tau := F(t) - F(a).$$

Definition 2.6. *If $a \in T$ and $f \in C_{rd}([a, \infty), \mathbf{R})$, then we define the improper integral by*

$$\int_a^{\infty} f(t)\Delta t := \lim_{b \to \infty} \int_a^b f(t)\Delta t$$

provided this limit exists, and we say that the improper integral converges in this case. If this limit does not exist, then we say that the improper integral diverges.

Lemma 2.3. *Let $a \in T^K$, $b \in T$ and assume $f : T \times T^K \to \mathbf{R}$ is continuous at (t, t), where $t \in T^K$ with $t > a$. Also assume that $f^{\Delta}(t, \cdot)$ is rd-continuous on $[a, \sigma(t)]$. Suppose that for each $\varepsilon > 0$ there exists a neighborhood U of t, independent of $\tau \in [a, \sigma(t)]$, such that*

$$|f(\sigma(t), \tau) - f(s, \tau) - f^{\Delta}(t, \tau)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|, \quad \textit{for all} \quad s \in U,$$

where f^{Δ} denotes the derivative of f with respect to the first variable. Then

(i) $g(t) := \int_a^t f(t, \tau)\Delta\tau$ *implies* $g^{\Delta}(t) = \int_a^t f^{\Delta}(t, \tau)\Delta\tau + f(\sigma(t), t)$;

(ii) $h(t) := \int_t^b f(t,\tau)\Delta\tau$ *implies* $h^{\Delta}(t) = \int_t^b f^{\Delta}(t,\tau)\Delta\tau - f(\sigma(t),t)$.

Definition 2.7. *A solution of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory.*

3. Main results. First we show some lemmas which will be useful for the main results of this section.

Lemma 3.1. *Let* $x(t)$ *be an eventually positive (or negative) solution of (1.1). If* $\lim_{t\to\infty} x(t) = 0$, *then* $y(t)$ *is eventually negative (or positive) and* $\lim_{t\to\infty} y(t) = 0$. *If* $\lim_{t\to\infty} x(t) = 0$ *fails, then* $y(t)$ *is eventually positive (or negative).*

Proof. Let $x(t)$ be an eventually positive solution of (1.1). From (1.1), $y^{\Delta\Delta}(t) < 0$ eventually. Thus $y^{\Delta}(t)$ is decreasing and $y^{\Delta}(t) > 0$ or $y^{\Delta}(t) < 0$ eventually. Also, $y(t) > 0$ or $y(t) < 0$ eventually. If $\lim_{t\to\infty} x(t) = 0$, from (1.2) we have $\lim_{t\to\infty} y(t) = 0$. Since $y(t)$ is monotonic, so $\lim_{t\to\infty} y^{\Delta}(t) = 0$, which implies that $y^{\Delta}(t) > 0$. Therefore, $y(t) < 0$ eventually. If $\lim_{t\to\infty} x(t) = 0$ fail, then $\limsup_{t\to\infty} x(t) > 0$. We show that $y(t) > 0$ eventually. If not, then $y(t) < 0$ eventually. If $x(t)$ is unbounded, then there exists a sequence $\{t_n\}$ such that $\lim_{n\to\infty} t_n = \infty$, $x(t_n) = \max_{t_0\le t\le t_n} x(t)$ and $\lim_{n\to\infty} x(t_n) = \infty$. From (1.2), we have

$$y(t_n) = x(t_n) - c(t_n)x(t_n-\tau) \ge x(t_n)(1-c(t_n)). \tag{3.1}$$

Thus $\lim_{n\to\infty} y(t_n) = \infty$, which is a contradiction. If $x(t)$ is bounded, then there exists a sequence $\{t_n\}$ such that $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} x(t_n) = \limsup_{t\to\infty} x(t)$. Since the sequences $\{c(t_n)\}$ and $\{x(t_n-\tau)\}$ are bounded, there exist convergent subsequences. Without loss of generality, we may assume that $\lim_{n\to\infty} x(t_n-\tau)$ and $\lim_{n\to\infty} c(t_n)$ exist. Hence

$$0 \ge \lim_{n\to\infty} y(t_n) = \lim_{n\to\infty}(x(t_n) - c(t_n)x(t_n-\tau)) \ge \limsup_{t\to\infty} x(t)(1-\lim_{n\to\infty} c(t_n)) > 0$$

which is a contradiction again. Therefore, $y(t) > 0$ eventually. A similar proof can be given if $x(t) < 0$ eventually.

Lemma 3.2. *Assume that* $\lim\limits_{t\to\infty} c(t) = c \in [0,1)$, *and* $x(t)$ *is an eventually positive (or negative) solution of (1.1). If* $\lim_{t\to\infty} y(t) = a \in R$, *then* $\lim_{t\to\infty} x(t) = \dfrac{a}{1-c}$. *If* $\lim_{t\to\infty} y(t) = \infty$ *(or* $-\infty$*), then* $\lim_{t\to\infty} x(t) = \infty$ *(or* $-\infty$*).*

Proof. Let $x(t)$ be an eventually positive solution of (1.1). Then $x(t) \ge y(t)$ eventually. If $\lim_{t\to\infty} y(t) = \infty$, then $\lim_{t\to\infty} x(t) = \infty$. Now we consider the case that $\lim_{t\to\infty} y(t) = a \in R$. Thus $y(t)$ is bounded which implies that $x(t)$ is bounded (see (3.1)). Therefore, there exists a sequence $\{t_n\}$ such that $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} x(t_n) = \limsup_{t\to\infty} x(t)$. As before, without loss of generality, we may assume that $\lim_{n\to\infty} c(t_n)$ and $\lim_{n\to\infty} x(t_n-\tau)$ exist. Hence

$$a = \lim_{n\to\infty} y(t_n) = \lim_{n\to\infty} x(t_n) - \lim_{n\to\infty} c(t_n) \lim_{n\to\infty} x(t_n-\tau) \ge \limsup_{t\to\infty} x(t)(1-c)$$

i.e.,

$$\frac{a}{1-c} \ge \limsup_{t\to\infty} x(t). \tag{3.2}$$

On the other hand, there exists $\{t'_n\}$ such that $\lim_{n\to\infty} x(t'_n) = \liminf_{t\to\infty} x(t)$. Without loss of generality, we assume that $\lim_{n\to\infty} c(t'_n)$ and $\lim_{n\to\infty} x(t'_n - \tau)$ exist. Hence

$$a = \lim_{n\to\infty} y(t'_n) = \lim_{n\to\infty} x(t'_n) - \lim_{n\to\infty} c(t'_n) \lim_{n\to\infty} x(t'_n - \tau) \le \liminf_{t\to\infty} x(t)(1-c)$$

or

$$\frac{a}{1-c} \le \liminf_{t\to\infty} x(t). \tag{3.3}$$

Combining (3.2) and (3.3) we obtain $\lim_{t\to\infty} x(t) = \dfrac{a}{1-c}$. A similar proof can be given if $x(t) < 0$.

We are now ready to prove the following results.

Theorem 3.1. *Assume that* $\lim_{t\to\infty} c(t) = c \in [0,1)$. *Let* $x(t)$ *be a nonoscillatory solution of (1.1). Let* E *denote the set of all nonoscillatory solution of (1.1), and define*

$$E(0,0,0) = \{x(t) \in E : \lim_{t\to\infty} x(t) = 0, \lim_{t\to\infty} y(t) = 0, \lim_{t\to\infty} y^{\Delta}(t) = 0\},$$

$$E(b,a,0) = \{x(t) \in E : \lim_{t\to\infty} x(t) = b = \frac{a}{1-c}, \lim_{t\to\infty} y(t) = a, \lim_{t\to\infty} y^{\Delta}(t) = 0\},$$

$$E(\infty,\infty,0) = \{x(t) \in E : \lim_{t\to\infty} x(t) = \infty, \lim_{t\to\infty} y(t) = \infty, \lim_{t\to\infty} y^{\Delta}(t) = 0\},$$

$$E(\infty,\infty,d) = \{x(t) \in E : \lim_{t\to\infty} x(t) = \infty, \lim_{t\to\infty} y(t) = \infty, \lim_{t\to\infty} y^{\Delta}(t) = d \ne 0\}.$$

Then

$$E = E(0,0,0) \cup E(b,a,0) \cup E(\infty,\infty,0) \cup E(\infty,\infty,d).$$

Proof. Without loss of generality, let $x(t)$ be an eventually positive solution of (1.1). If $\lim_{t\to\infty} x(t) = 0$, by Lemma 3.1, $\lim_{t\to\infty} y(t) = 0$ and $\lim_{t\to\infty} y^{\Delta}(t) = 0$, i.e., $x(t) \in E(0,0,0)$. If $\lim_{t\to\infty} x(t) = 0$ fails, then by Lemma 3.1, $y(t) > 0$ eventually, and it is easy to see that $y^{\Delta}(t) > 0$, $y^{\Delta\Delta}(t) < 0$ eventually. If $\lim_{t\to\infty} y(t) = a > 0$ exists, then $\lim_{t\to\infty} y^{\Delta}(t) = 0$, by Lemma 3.2, and we have $\lim_{t\to\infty} x(t) = \dfrac{a}{1-c} = b$, i.e., $x(t) \in E(b,a,0)$. If $\lim_{t\to\infty} y(t) = \infty$, then by Lemma 3.2 $\lim_{t\to\infty} x(t) = \infty$. Since $y^{\Delta\Delta}(t) < 0$ and $y^{\Delta}(t) > 0$, we have $\lim_{t\to\infty} y^{\Delta}(t) = d$, where $d = 0$ or $d > 0$. Then either $x(t) \in E(\infty,\infty,0)$, or $x(t) \in$ $\in E(\infty,\infty,d)$.

In the following we shall show some existence results for each kind of nonoscillatory solution of Eq. (1.1).

Theorem 3.2. *Assume that* $\lim_{t\to\infty} c(t) = c \in [0,1)$. *Then Eq.(1.1) has a nonoscillatory solution* $x(t) \in E(b,a,0)(b \ne 0, a \ne 0)$ *if and only if*

$$\int\limits_{t_0}^{\infty} \sigma(u)|f(u,b_1,\ldots,b_1)|\Delta u < \infty \quad \textit{for some} \quad b_1 \ne 0. \tag{3.4}$$

Proof. *Necessity.* Without loss of generality, let $x(t) \in E(b,a,0)$ be an eventually positive solution of (1.1). By Theorem 3.1 we know that $b > 0,\ a > 0$. From (1.1) and (1.2) we have

$$y^{\Delta\Delta}(t) = -f(t, x(g_1(t)), \ldots, x(g_m(t))).$$

Integrating it from s to ∞ for $s \geq t_0$ we obtain

$$y^{\Delta}(s) = \int\limits_s^{\infty} f(u, x(g_1(u)), \ldots, x(g_m(u)))\Delta u.$$

Integrating it from t_1 to t for t_1 sufficiently large, we get

$$y(t) = y(t_1) + \int\limits_{t_1}^{t} (\sigma(u) - t_1) f(u, x(g_1(u)), \ldots, x(g_m(u)))\Delta u +$$

$$+ \int\limits_{t}^{\infty} (t - t_1) f(u, x(g_1(u)), \ldots, x(g_m(u)))\Delta u.$$

Since $\lim_{u\to\infty} x(g_i(u)) = b > 0,\ i = 1, 2, \ldots, m$, there exists an $t_1 \geq t_0$ such that $x(g_i(u)) \geq \dfrac{b}{2}$ for $t \geq t_1$. Hence we have

$$\int\limits_{t_1}^{t} (\sigma(u) - t_1) \left| f\left(u, \frac{b}{2}, \ldots, \frac{b}{2}\right)\right| \Delta u < y(t) - y(t_1)$$

which implies that (3.4) holds.

Sufficiency. Set $b_1 > 0$ and $A > 0$ so that $A < (1-c)b_1$. From (3.4) there exists a sufficiently large t_1 so that for $t \geq t_1$ we have $t - \tau \geq t_0$ and $g_i(t) \geq t_0,\ i = 1, 2, \ldots, m$, and

$$\frac{A}{b_1} + c(t) + \frac{1}{b_1} \int\limits_{t_1}^{\infty} \sigma(u) f(u, b_1, \ldots, b_1)\Delta u \leq 1. \tag{3.5}$$

Let X denote the Banach space of all bounded rd-continuous functions $x(t)$ on $[t_0, \infty)$ with the norm $\|x(t)\| = \sup_{t \geq t_0} |x(t)| < \infty$. Define a set Ω by

$$\Omega = \{x(t) \in X | 0 \leq x(t) \leq b_1, t \geq t_0\}$$

and an operator S on Ω by

$$(Sx)(t) = \begin{cases} A + c(t)x(t-\tau) \\ + \displaystyle\int_{t_1}^{t} \sigma(u) f(u, x(g_1(u)), \ldots, x(g_m(u)))\Delta u + \\ + \displaystyle\int_{t}^{\infty} t f(u, x(g_1(u)), \ldots, x(g_m(u)))\Delta u, \quad \text{if} \quad t \geq t_1, \\ (Sx)(t_1), \quad \text{if} \quad t_0 \leq t < t_1. \end{cases} \tag{3.6}$$

Clearly, for $x(t) \in \Omega$,

$$(Sx)(t) \le A + c(t)b_1 + \int_{t_1}^{t} \sigma(u) f(u, b_1, \ldots, b_1)\Delta u + \int_{t}^{\infty} \sigma(u) f(u, b_1, \ldots, b_1)\Delta u \le$$

$$\le A + c(t)b_1 + \int_{t_1}^{\infty} \sigma(u) f(u, b_1, \ldots, b_1)\Delta u \le b_1, \quad t \ge t_1,$$

and

$$(Sx)(t) = (Sx)(t_1) \le b_1, \quad t_0 \le t \le t_1,$$

i.e., $S\Omega \subset \Omega$.

Define a series of sequences $\{x_k(t)\}$, $k \in \mathbf{N}_0$, as

$$x_0(t) = 0,$$
$$x_k(t) = (Sx_{k-1})(t), \quad k \in \mathbf{N}, \quad t \ge t_0. \tag{3.7}$$

By induction, we can prove that

$$0 \le x_k(t) \le x_{k+1}(t) \le b_1, \quad t \ge t_0, \quad k \in \mathbf{N_0}.$$

Then there exists $x(t) \subset \Omega$ such that $\lim_{t\to\infty} x_k(t) = x(t)$, $t \ge t_0$.

In the following, we shall show that

$$\lim_{k\to\infty} \int_{t}^{\infty} t f(u, x_k(g_1(u)), \ldots, x_k(g_m(u)))\Delta u = \int_{t}^{\infty} t f(u, x(g_1(u)), \ldots, x(g_m(u)))\Delta u.$$

In fact, by (3.4), for any $\varepsilon > 0$ there exists $t_1 \ge t_0$ such that

$$\int_{t_1}^{\infty} \sigma(u) f(u, b_1, \ldots, b_1)\Delta u < \varepsilon.$$

Thus, for $t_2 \ge t_1$ we get

$$\left| \int_{t}^{t_2} t f(u, x_k(g_1(u)), \ldots, x_k(g_m(u)))\Delta u - \int_{t}^{\infty} t f(u, x_k(g_1(u)), \ldots, x_k(g_m(u)))\Delta u \right| =$$

$$= \left| \int_{t_2}^{\infty} t f(u, x_k(g_1(u)), \ldots, x_k(g_m(u)))\Delta u \right| \le \int_{t_2}^{\infty} \sigma(u) f(u, x_k(g_1(u)), \ldots, x_k(g_m(u)))\Delta u \le$$

$$\le \int_{t_2}^{\infty} \sigma(u) f(u, b_1, \ldots, b_1)\Delta u < \varepsilon.$$

Hence, $\int_t^{t_2} tf(u, x_k(g_1(u)), \ldots, x_k(g_m(u)))\Delta u \to \int_t^{\infty} tf(u, x_k(g_1(u)), \ldots, x_k(g_m(u)))\Delta u$ uniformly for $k \in N$ as $t_2 \to \infty$. Therefore,

$$\lim_{k\to\infty} \int_t^{\infty} tf(u, x_k(g_1(u)), \ldots, x_k(g_m(u)))\Delta u =$$

$$= \lim_{k\to\infty} \lim_{t_2\to\infty} \int_t^{t_2} tf(u, x_k(g_1(u)), \ldots, x_k(g_m(u)))\Delta u =$$

$$= \lim_{t_2\to\infty} \lim_{k\to\infty} \int_t^{t_2} tf(u, x_k(g_1(u)), \ldots, x_k(g_m(u)))\Delta u =$$

$$= \lim_{t_2\to\infty} \int_t^{t_2} tf(u, x(g_1(u)), \ldots, x(g_m(u)))\Delta u =$$

$$= \int_t^{\infty} tf(u, x(g_1(u)), \ldots, x(g_m(u)))\Delta u.$$

Let $k \to \infty$. Then (3.7) gives

$$x(t) = \begin{cases} A + c(t)x(t-\tau) + \int_{t_1}^{t} \sigma(u) f(u, x(g_1(u)), \ldots, x(g_m(u)))\Delta u + \\ + \int_t^{\infty} tf(u, x(g_1(u)), \ldots, x(g_m(u)))\Delta u, \quad \text{if} \quad t \geq t_1, \\ x(t_1), \quad \text{if} \quad t_0 \leq t < t_1. \end{cases}$$

Clearly, $x(t) > 0$ on $[t_0, \infty)$. Therefore, $x(t)$ is a positive solution of (1.1). Since $0 < A \leq x(t) \leq \leq b_1$, from Theorem 3.1, $x(t) \in E(b, a, 0)$.

Theorem 3.3. *Assume that* $\lim_{t\to\infty} c(t) = c \in [0, 1)$. *Then Eq.(1.1) has a nonoscillatory solution* $x(t) \in E(\infty, \infty, d)$ $(d \neq 0)$ *if and only if*

$$\int_{t_0}^{\infty} |f(u, hg_1(u), \ldots, hg_m(u))|\Delta u < \infty \quad \textit{for some} \quad h \neq 0. \tag{3.8}$$

Proof. *Necessity.* Without loss of generality, let $x(t) \in E(\infty, \infty, d)$ be an eventually positive solution of (1.1). From Theorem 3.1, we have that $d > 0$. From (1.1) and (1.2) we have

$$y^{\Delta\Delta}(t) + f(t, x(g_1(t)), \ldots, x(g_m(t))) = 0.$$

Integrating it from t_1 to t, we get

$$y^{\Delta}(t) - y^{\Delta}(t_1) + \int\limits_{t_1}^{t} f(u, x(g_1(u)), \ldots, x(g_m(u)))\Delta u = 0.$$

Since $\lim_{t\to\infty} y^{\Delta}(t) = d > 0$, we obtain

$$\int\limits_{t_1}^{\infty} f(u, x(g_1(u)), \ldots, x(g_m(u)))\Delta u < \infty \tag{3.9}$$

and there exist $d_1 > 0$ and $t_2 \geq t_1$ such that $y(t) \geq d_1 t$ for $t \geq t_2$. Therefore,

$$\int\limits_{t_1}^{\infty} f(u, x(g_1(u)), \ldots, x(g_m(u)))\Delta u \geq \int\limits_{t_1}^{\infty} f(u, y(g_1(u)), \ldots, y(g_m(u)))\Delta u \geq$$

$$\geq \int\limits_{t_1}^{\infty} f(u, d_1 g_1(u), \ldots, d_1 g_m(u))\Delta u. \tag{3.10}$$

Choosing $h = d_1$ and combining (3.9) and (3.10), we get

$$\int\limits_{t_1}^{\infty} f(u, hg_1(u), \ldots, hg_m(u))\Delta u < \infty.$$

Sufficiency. Set $h > 0$. Let $d > 0$, $B > 0$. From (3.8) there exists a sufficiently large t_1 so that for $t \geq t_1$ we have $t - \tau \geq t_0$ and $g_i(t) \geq t_0$, $i = 1, 2, \ldots, m$, and

$$\frac{d}{h} + \frac{B}{th} + c(t) + \frac{1}{h}\int\limits_{t_1}^{\infty} f(u, hg_1(u), \ldots, hg_m(u))\Delta u < 1. \tag{3.11}$$

Define a set Ω by

$$\Omega = \{z(t) \in X | d \leq z(t) \leq h\}$$

and a operator S on Ω by

$$(Sz)(t) = \begin{cases} d + \dfrac{B}{t} + c(t)\dfrac{t-\tau}{t}\, z(t-\tau) + \\ +\dfrac{1}{t}\displaystyle\int_{t_1}^{t} \sigma(u) f(u, g_1(u)z(g_1(u)), \ldots, g_m(u)z(g_m(u)))\Delta u + \\ +\displaystyle\int_{t}^{\infty} f(u, g_1(u)z(g_1(u)), \ldots, g_m(u)z(g_m(u)))\Delta u, \quad \text{if} \quad t \geq t_1, \\ (Sz)(t), \quad \text{if} \quad t_0 \leq t < t_1. \end{cases} \tag{3.12}$$

Clearly, for $z(t) \in \Omega$

$$(Sz)(t) \le d + \frac{B}{t} + c(t)h + \frac{1}{t}\int\limits_{t_1}^{t} \sigma(u) f(u, hg_1(u), \ldots, hg_m(u))\Delta u +$$

$$+ \int\limits_{t}^{\infty} f(u, hg_1(u), \ldots, hg_m(u))\Delta u \le d + \frac{B}{t} + c(t)h +$$

$$+ \int\limits_{t_1}^{\infty} f(u, hg_1(u), \ldots, hg_m(u))\Delta u < h, \quad t \ge t_1,$$

and

$$(Sz)(t) = (Sz)(t_1) \le h, \quad t_0 \le t < t_1.$$

It is easy to see that $(Sz)(t) \ge d$ for $t \ge t_0$. Hence, $T\Omega \subset \Omega$. Define a series of sequences $\{z_k(t)\}$, $k \in \mathbf{N}$, by

$$z_0(t) = d, \quad z_k(t) = (Sz_{k-1})(t), \quad t \ge t_0, \quad k \in \mathbf{N_0}.$$

We can prove that

$$d \le z_k(t) \le z_{k+1}(t) \le h, \quad t \ge t_0, \quad k \in \mathbf{N_0}.$$

Then there exists $z(t) \in \Omega$ such that $\lim\limits_{k\to\infty} z_k(t) = z(t)$, $t \ge t_0$ and $d \le z(t) \le h$. Clearly, $z(t) = (Sz)(t)$ $(t \ge t_0)$, i.e.,

$$z(t) = \begin{cases} d + \dfrac{B}{t} + c(t)\dfrac{t-\tau}{t}\, z(t-\tau) + \\ + \dfrac{1}{t}\displaystyle\int_{t_1}^{t} \sigma(u) f(u, g_1(u)z(g_1(u)), \ldots, g_m(u)z(g_m(u)))\Delta u + \\ + \displaystyle\int\limits_{t}^{\infty} f(u, g_1(u)z(g_1(u)), \ldots, g_m(u)z(g_m(u)))\Delta u, \quad \text{if} \quad t \ge t_1, \\ z(t_1), \quad \text{if} \quad t_0 \le t < t_1. \end{cases}$$

Let $x(t) = tz(t)$, $t \ge t_0$. Then we have

$$x(t) = \begin{cases} dt + B + c(t)x(t-\tau) + \\ + \displaystyle\int_{t_1}^{t} \sigma(u) f(u, x(g_1(u)), \ldots, x(g_m(u)))\Delta u + \\ + \displaystyle\int_{t}^{\infty} t f(u, x(g_1(u)), \ldots, x(g_m(u)))\Delta u, \quad \text{if} \quad t \ge t_1, \\ x(t_1), \quad \text{if} \quad t_0 \le t < t_1. \end{cases} \tag{3.13}$$

Hence, $x(t)$ is a positive solution of (1.1). On the other hand, $x(t) \geq y(t) \geq dt + B$. Hence $\lim_{t\to\infty} x(t) = \infty$ and $\lim_{t\to\infty} y(t) = \infty$. From (3.13), we have

$$y^{\Delta}(t) = d + \int\limits_{t}^{\infty} f(u, x(g_1(u)), \ldots, x(g_m(u)))\Delta u =$$

$$= d + \int\limits_{t}^{\infty} f(u, g_1(u)z(g_1(u)), \ldots, g_m(u)z(g_m(u)))\Delta u \leq$$

$$\leq d + \int\limits_{t}^{\infty} f(u, hg_1(u), \ldots, hg_m(u))\Delta u.$$

Hence, $\lim_{t\to\infty} y^{\Delta}(t) = d$. Therefore, $x(t) \in E(\infty, \infty, d)$.

Theorem 3.4. *Assume that* $\lim_{t\to\infty} c(t) = c \in [0, 1)$. *Further, assume that*

$$\int\limits_{t_0}^{\infty} |f(u, hg_1(u), \ldots, hg_m(u))|\Delta u < \infty \quad \textit{for some} \quad h \neq 0 \tag{3.14}$$

and

$$\int\limits_{t_0}^{\infty} \sigma(u)|f(u, b_1, \ldots, b_1)|\Delta u = \infty \quad \textit{for some} \quad b_1 \neq 0, \tag{3.15}$$

where $b_1 h > 0$. *Then Eq. (1.1) has a nonoscillatory solution* $x(t) \in E(\infty, \infty, 0)$.

Proof. Without loss of generality, assume that $h > 0$ and $b_1 > 0$. From (3.14) there exists a sufficiently large t_1 so that for $t \geq t_1$ we have $t - \tau \geq t_0$ and $g_i(\sigma(t)) \geq t_0$, $i = 1, 2, \ldots, m$, and

$$\frac{b_1}{th} + c(t) + \frac{1}{h}\int\limits_{t_1}^{\infty} f(u, hg_1(u), \ldots, hg_m(u))\Delta u < 1. \tag{3.16}$$

Define a set Ω by

$$\Omega = \{z(t) \in X | 0 \leq z(t) \leq h, t \geq t_0\}$$

and an operator S on Ω by

$$(Sz)(t) = \begin{cases} \dfrac{b_1}{t} + c(t)\dfrac{t-\tau}{t}\, z(t-\tau) + \\ +\dfrac{1}{t}\displaystyle\int_{t_1}^{t} \sigma(u) f(u, g_1(u)z(g_1(u)), \ldots, g_m(u)z(g_m(u)))\Delta u + \\ +\displaystyle\int_{t}^{\infty} f(u, g_1(u)z(g_1(u)), \ldots, g_m(u)z(g_m(u)))\Delta u, \quad \text{if} \quad t \geq t_1, \\ (Sz)(t), \quad \text{if} \quad t_0 \leq t < t_1. \end{cases}$$

Clearly, for $z(t) \in \Omega$,

$$(Sz)(t) \le \frac{b_1}{t} + c(t)h + \frac{1}{t}\int\limits_{t_1}^{t} \sigma(u) f(u, hg_1(u), \ldots, hg_m(u))\Delta u +$$

$$+ \int\limits_{t}^{\infty} f(u, hg_1(u), \ldots, hg_m(u))\Delta u \le \frac{b_1}{t} + c(t)h +$$

$$+ \int\limits_{t_1}^{\infty} f(u, hg_1(u), \ldots, hg_m(u))\Delta u \le h, \quad t \ge t_1,$$

and $(Sz)(t) = (Sz)(t_1) \le h$, $t_0 \le t < t_1$, i.e., $S\Omega \subset \Omega$.

Define a series of sequences $\{z_k(t)\}$, $k \in \mathbf{N}$, by

$$z_0(t) = 0, \quad z_k(t) = (Sz_{k-1})(t), \quad t \ge t_0, \quad k \in \mathbf{N_0}. \tag{3.17}$$

By induction, we can prove that

$$0 \le z_k(t) \le z_{k+1}(t) \le h, \quad t \ge t_0, \quad k \in \mathbf{N}.$$

Then there exists $z(t) \in \Omega$ such that $\lim_{k\to\infty} z_k(t) = z(t)$, $t \ge t_0$.

Clearly, $z(t) = (Sz)(t)$, $t \ge t_0$, i.e.,

$$z(t) = \begin{cases} \dfrac{b_1}{t} + c(t)\dfrac{t-\tau}{t} z(t-\tau) + \\ + \dfrac{1}{t}\displaystyle\int_{t_1}^{t} \sigma(u) f(u, g_1(u)z(g_1(u)), \ldots, g_m(u)z(g_m(u)))\Delta u + \\ + \displaystyle\int_{t}^{\infty} f(u, g_1(u)z(g_1(u)), \ldots, g_m(u)z(g_m(u)))\Delta u, \quad \text{if} \quad t \ge t_1, \\ z(t_1), \quad \text{if} \quad t_0 \le t < t_1. \end{cases}$$

Let $x(t) = tz(t)$, $t \ge t_0$. Then we have

$$x(t) = \begin{cases} b_1 + c(t)x(t-\tau) + \\ + \displaystyle\int_{t_1}^{t} \sigma(u) f(u, x(g_1(u)), \ldots, x(g_m(u)))\Delta u + \\ + \displaystyle\int_{t}^{\infty} t f(u, x(g_1(u)), \ldots, x(g_m(u)))\Delta u, \quad \text{if} \quad t \ge t_1, \\ x(t_1), \quad \text{if} \quad t_0 \le t < t_1. \end{cases} \tag{3.18}$$

Hence, $x(t)$ is a positive solution of (1.1). On the other hand, from (3.18), we have $x(t) \ge b_1$ and that

$$x(t) \ge y(t) = x(t) - c(t)x(t-\tau) \ge \int\limits_{t_1}^{t} \sigma(u) f(u, b_1, \ldots, b_1)\Delta u$$

which together with (3.15) imply $\lim_{t\to\infty} x(t) = \infty$ and $\lim_{t\to\infty} y(t) = \infty$. By (3.18), we get

$$y^{\Delta}(t) = \int_{t}^{\infty} f(u, x(g_1(u)), \ldots, x(g_m(u)))\Delta u =$$

$$= \int_{t}^{\infty} f(u, g_1(u)z(g_1(u)), \ldots, g_m(u)z(g_m(u)))\Delta u \leq$$

$$\leq \int_{t}^{\infty} f(u, hg_1(u), \ldots, hg_m(u))\Delta u.$$

Hence

$$0 \leq \lim_{t\to\infty} y^{\Delta}(t) \leq \lim_{t\to\infty} \int_{t}^{\infty} f(u, hg_1(u), \ldots, hg_m(u))\Delta u = 0,$$

i.e., $\lim_{t\to\infty} y^{\Delta}(t) = 0$. Therefore, $x(t) \in E(\infty, \infty, 0)$.

Theorem 3.5. *Assume that* $\lim_{t\to\infty} c(t) = c \in [0, 1)$. *Further assume that there exists* $d > 0$ *such that*

$$\int_{t_0}^{\infty} f(u, d_1, \ldots, d_1)\Delta u = \infty \quad \textit{for any} \quad d_1 \in (0, d]. \tag{3.19}$$

Then every solution $x(t)$ *of Eq. (1.1) either oscillates or* $\{x(t)\} \in E(0, 0, 0)$.

Proof. Let $x(t)$ be an eventually positive solution of (1.1). By Lemma 3.1, if $\lim_{t\to\infty} x(t) = 0$, then $\lim_{t\to\infty} y(t) = 0$ and so $\lim_{t\to\infty} y^{\Delta}(t) = 0$. Hence, $x(t) \in E(0, 0, 0)$. If $\lim_{t\to\infty} x(t) = 0$ fails, then $y(t) > 0$ eventually. Since $y^{\Delta\Delta}(t) < 0$, we have $y^{\Delta}(t) > 0$, eventually. Therefore, there exists $\overline{d} \in (0, d]$ such that $x(t) \geq y(t) \geq \overline{d}$. From (1.1) and (1.2), we have

$$y^{\Delta\Delta}(t) = -f(t, x(g_1(t)), \ldots, x(g_m(t))).$$

Integrating it from t_0 to t, we obtain

$$y^{\Delta}(t) - y^{\Delta}(t_0) = -\int_{t_0}^{t} f(u, x(g_1(u)), \ldots, x(g_m(u)))\Delta u \leq -\int_{t_0}^{t} f(u, \overline{d}, \ldots, \overline{d})\Delta u.$$

Let $t \to \infty$. Then we get $\int_{t_0}^{\infty} f(u, \overline{d}, \ldots, \overline{d})\Delta u < \infty$ which contradicts (3.19) and completes the proof.

The above results can be extended to the second order neutral equation

$$[x(t) - c(t)x(t-\tau)]^{\Delta\Delta} = f(t, x(g_1(t)), \ldots, x(g_m(t))). \tag{3.20}$$

With respect to equation (3.20), we assume that conditions (H_1), (H_2) and (H_3) hold. By the same argument, we have the following theorems.

Theorem 3.6. *Assume that* $\lim_{t\to\infty} c(t) = c \in [0,1)$. *Let* $x(t)$ *be a nonoscillatory solution of (3.20). Let* S *denote the set of all nonoscillatory solution of (3.20), and define*

$$E(0,0,0) = \{x(t) \in E : \lim_{t\to\infty} x(t) = 0, \lim_{t\to\infty} y(t) = 0, \lim_{t\to\infty} y^{\Delta}(t) = 0\},$$

$$E(b,a,0) = \left\{x(t) \in E : \lim_{t\to\infty} x(t) = b = \frac{a}{1-c}, \lim_{t\to\infty} y(t) = a, \lim_{t\to\infty} y^{\Delta}(t) = 0\right\},$$

$$E(\infty,\infty,d) = \{x(t) \in E : \lim_{t\to\infty} x(t) = \infty, \lim_{t\to\infty} y(t) = \infty, \lim_{t\to\infty} y^{\Delta}(t) = d \neq 0\},$$

$$E(\infty,\infty,\infty) = \{x(t) \in E : \lim_{t\to\infty} x(t) = \infty, \lim_{t\to\infty} y(t) = \infty, \lim_{t\to\infty} y^{\Delta}(t) = \infty\}.$$

Then

$$E = E(0,0,0) \cup E(b,a,0) \cup E(\infty,\infty,d) \cup E(\infty,\infty,\infty).$$

Theorem 3.7. *Assume that* $\lim_{t\to\infty} c(t) = c \in [0,1)$. *Further, assume that*

$$\int_{t_0}^{\infty} \sigma(u)|f(u, b_1, \ldots, b_1)|\Delta u < \infty \quad \textit{for some} \quad b_1 \neq 0.$$

Then Eq. (3.20) has a nonoscillatory solution $x(t) \in E(b,a,0) (b \neq 0, a \neq 0)$.

Theorem 3.8. *Assume that* $\lim_{t\to\infty} c(t) = c \in [0,1)$. *Then the following statements are true.*
(i) If Eq. (3.20) has a nonoscillatory solution $x(t) \in E(\infty,\infty,d)$, $d \neq 0$, *then*

$$\int_{t_0}^{\infty} |f(u, hg_1(u), \ldots, hg_m(u))|\Delta u < \infty, \quad \textit{for some} \quad h \neq 0.$$

(ii) If

$$\int_{t_0}^{\infty} \sigma(u)|f(u, hg_1(u), \ldots, hg_m(u))|\Delta u < \infty, \quad \textit{for some} \quad h \neq 0,$$

then Eq. (3.20) has a nonoscillatory solution $x(t) \in E(\infty,\infty,d)$, $d \neq 0$.

Theorem 3.9. *Assume that* $\lim_{t\to\infty} c(t) = c \in [0,1)$. *Further assume that there exists* $d > 0$ *such that*

$$\int_{t_0}^{\infty} f(u, d_1, \ldots, d_1)\Delta u = \infty \quad \textit{for any} \quad d_1 \in (0,d].$$

Then every solution $x(t)$ of Eq. (1.1) either oscillates or $x(t) \in E(0,0,0)$, or $x(t) \in E(\infty,\infty,\infty)$.

In the following, we shall give some examples.

Example 3.1. In the case where $T = \mathbf{R}$, we consider the second order differential equation

$$\left(x(t) - \frac{1}{2}x(t-\tau)\right)'' + \frac{2(t-1)^3 - t^3}{(t-1)^6}\, x^3(t) = 0. \tag{3.21}$$

(3.4) becomes

$$\int_{t_0}^{\infty} u|f(u, b_1, \ldots, b_1)|\, du < \infty \quad \text{for some} \quad b_1 \neq 0. \tag{3.22}$$

It is easy to see that (3.22) holds. Therefore, (3.21) has a nonoscillatory solution $x(t) \in E(b, a, 0)$, $b \neq 0$, $a \neq 0$. In fact, $x(t) = 1 - \dfrac{1}{t}$ is such a solution, where $a = \dfrac{1}{2}$, $b = 1$.

Example 3.2. In the case where $T = \mathbf{N}$, we consider the second order delay difference equation

$$\Delta^2\left(x_n - \frac{1}{4}x_{n-1}\right) + \frac{2^{-n-3}}{(n-1-2^{-n+1})^5}x_{n-1}^5 = 0, \quad n \geq 2, \tag{3.23}$$

for which (3.8) becomes

$$\sum_{j=n_0}^{\infty} |f(j, hg_1(j), \ldots, hg_m(j))| < \infty. \tag{3.24}$$

It is easy to see that (3.24) is satisfied. In fact, the sequence $x_n = \left\{n - \dfrac{1}{2^n}\right\}$ is a nonoscillatory solution of (3.23) which belongs to the class $S\left(\infty, \infty, \dfrac{3}{4}\right)$.

Example 3.3. In the case where $T = h\mathbf{Z} = \{hk | k \in \mathbf{Z}\}$ for $h > 1$, we consider the second order delay dynamic equation

$$\left[x(t) - \frac{1}{2}x(t-h)\right]^{\Delta\Delta} + \frac{(\frac{1}{2} - 2^{-h})(2^{-h} - 1)^2}{h^2}\, x(t-h) = 0 \tag{3.25}$$

for which condition (3.19) of Theorem 3.5 is satisfied. In fact, $x(t) = \dfrac{1}{2^t}$ is a nonoscillatory solution of (3.25) which belongs to the class $S(0,0,0)$.

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