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ASYMPTOTIC SOLUTIONS TO THE FIRST ORDER DIFFERENTIAL EQUATION WITH DEVIATED ARGUMENT AND SLOWLY VARYING COEFFICIENTS

АСИМПТОТИЧНІ РОЗВ'ЯЗКИ ДИФЕРЕНЦІАЛЬНОГО РІВНЯННЯ ПЕРШОГО ПОРЯДКУ З ВІДХИЛЕННЯМ АРГУМЕНТУ І ПОВІЛЬНО ЗМІННИМИ КОЕФІЦІЄНТАМИ

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The object of this paper is to study the problem of constructing an approximate solution to the first order weakly nonlinear ordinary differential equation with deviated argument and slowly varying coefficients. Based on asymptotic techniques in nonlinear mechanics an algorithm for asymptotic integration of the differential equation under consideration is given.

Розглядається задача про побудову наближених розв'язків слабконелінійного диференціального рівняння першого порядку з відхиленням аргументу і повільно змінними коефіцієнтами. На основі асимптотичних методів нелінійної механіки запропоновано алгоритм асимптотичного інтегрування диференціального рівняння вказаного вище класу.

1. Introduction. Studying oscillatory processes is known to be of a great importance for different fields of mechanics, physics, technology and a number of other areas of natural sciences. In many cases the mathematical model of the system under consideration is presented by differential equations with deviating argument and slowly varying coefficients. As a rule, such equations are nonlinear, but in many cases they are close to linear ones, since they contain some small parameter such that, if its value is zero, the differential equations become linear. For such equations a number of perturbation theory methods is developed, including asymptotic methods of nonlinear mechanics [1–6] that are of great efficacy.

Up to the present, many problems of the theory of asymptotic methods of nonlinear mechanics have been thoroughly elaborated for different classes of differential equations by a number of authors; unfortunately it is impossible to list all of them here. We will only recall some papers of N. M. Krylov and N. N. Bogolyubov, devoted to the development of asymptotic method and connected with the discussed problem. V. P. Rubanik [7 – 9] and V. I. Fodchuk [10] have considered a weakly nonlinear nonstationary differential equation with delayed almost constant arguments and, in general, have described an algorithm for asymptotic integration, found a formula for the first approximation and the first improved approximations. Yu. A. Mitropol'skiy and Le Suan Kan [11, 12] have studied multifrequency and one-requency oscillations in systems with slowly varying parameters and constant delay. Algorithms for asymptotic integration of different types of differential equations with constant delay using the techniques due to N. M. Krylov and N. N. Bogolyubov are given by Yu. A. Mitropol'skiy and D. I. Martynyuk [13].The problem of constructing asymptotic solutions to delay differential equations of the second order with slowly varying coefficients was studied in papers [14, 15].

In the present paper we consider the problem of constructing asymptotic approximations for the first order ordinary differential equation with deviated argument and slowly varying coefficients. Such a problem is of a certain practical importance, through its connection to a study of nonstationary processes in oscillatory systems.

2. Formulation of the problem. We study the problem of constructing asymptotic solutions for the first order weakly nonlinear ordinary differential equation with deviated argument and slowly varying coefficients of the following form:

$$\frac{dx(t)}{dt} + \beta(\tau)\frac{dx(t-\sigma(\tau))}{dt} + \omega_1^2(\tau)x(t) + \omega_2^2(\tau)x(t-\sigma(\tau)) =$$

$$= \varepsilon f\left(\tau, \theta, x(t), x(t-\sigma(\tau)), \frac{dx(t)}{dt}, \frac{dx(t-\sigma(\tau))}{dt}\right). \tag{2.1}$$

Here ε is a small parameter; $\tau = \varepsilon t$ is *slow time*; $d\theta/dt = \nu(\tau)$ is an instant frequency of an external periodical force; $\sigma(\tau) \geq \sigma_0 > 0$ is a delay; the function $f(\tau,\theta,x,y,u,v)$ is supposed to be 2π-periodic with respect to θ and have the from

$$f(\tau,\theta,x,y,u,v) = \sum_{|n|\leq N} e^{in\theta} f_n(\tau,x,y,u,v). \tag{2.2}$$

We also suppos that the coefficients $f_n(\tau,x,y,u,v), |n| \leq N$, are infinitely differentiable with respect to τ and are polynomials of the variables x,y,u,v.

The quantities $\beta = \beta(\tau), \omega_1 = \omega_1(\tau), \omega_2 = \omega_2(\tau), \nu = \nu(\tau), \sigma = \sigma(\tau)$ are supposed to be smooth enough as functions of τ. We also assume that the equation (2.1) has a unique solution of the initial problem for all $t > t_0$, where t_0 is an arbitrary (but fixed) moment of time.

Assume that characteristic equation of system (2.1),

$$\lambda + \beta(\tau)\lambda e^{-\lambda\sigma(\tau)} + \omega_1^2(\tau) + \omega_2^2(\tau)e^{-\lambda\sigma(\tau)} = 0, \tag{2.3}$$

has solutions $\lambda = \pm i\omega(\tau)$ at every moment of the slow time τ. It means that the fundamental frequency $\omega = \omega(\tau)$ of the unperturbed system (2.1) (when $\varepsilon = 0$) for every τ satisfies the system of characteristic equations,

$$D_1(\omega) = \beta\omega \sin \omega\sigma + \omega_1^2 + \omega_2^2 \cos \omega\sigma = 0, \tag{2.4}$$

$$D_2(\omega) = \omega + \beta\omega \cos \omega\sigma - \omega_2^2 \sin \omega\sigma = 0, \tag{2.5}$$

and is smooth enough in the variable τ.

Thus the unperturbed system (2.1), for every value of the parameter τ, has a set of periodic solutions $x(t) = a \cos(\omega t + \varphi)$, where $a = a(\tau)$ and $\varphi = \varphi(\tau)$ are arbitrary values.

On the basis of ideas used in asymptotic techniques in nonlinear mechanics [1], developed by N. M. Krylov, N. N. Bogolyubov and Yu. A. Mitropol'skiy, we describe an algorithm that allows to find an approximate solution to the problem (2.1) as an asymptotic expansion in the small parameter ε, which is asymptotically close to the periodic solution $x(t) = a\cos(\omega t + \varphi)$, which would give a solution to the unperturbed system (2.1).

3. Asymptotic expansion. In the context of solving the problem of constructing approximate (asymptotic) solutions, one usually studies nonresonance and resonance cases, and solutions to the problem are found separately for each of these cases. A dependence of both the fundamental frequency $\omega(\tau)$ and the frequency of external force $\nu(\tau)$ on the slow time τ does not allow to apply such an approach to the problem under consideration. As is known [1], the problem is connected with the possibility for the system (2.1) from one, for example, nonresonance state, to another, resonance state, and vice versa, caused transform by varying the frequencies $\omega(\tau), \nu(\tau)$ with the slow time τ.

Thus, an asymptotic solution of the problem (2.1) we seek in the following form:

$$x(t) = a \cos \varphi + \sum_{k=1}^{\infty} \varepsilon^k U_k(\tau, a, \theta, \varphi), \;\; \varphi = \frac{p}{q}\theta + \psi, \tag{3.1}$$

where the numbers p, q are mutually distinct natural and depend on the relation between the frequencies $\omega(\tau), \nu(\tau)$.

The functions $a(t)$ and $\psi(t)$ satisfy the following differential equations:

$$\frac{da}{dt} = \sum_{k=1}^{\infty} \varepsilon^k A_k(\tau, a, \psi), \;\; \frac{d\psi}{dt} = \omega(\tau) - \frac{p}{q}\nu(\tau) + \sum_{k=1}^{\infty} \varepsilon^k B_k(\tau, a, \psi), \tag{3.2}$$

where the functions $A_k(\tau, a, \psi), B_k(\tau, a, \psi)$, for any $k \in \mathbf{N}$, are 2π-periodic in the variable ψ.

The functions $U_k(\tau, a, \theta, \varphi), k \in \mathbf{N}$, are supposed to be 2π-periodic in the variables θ, ψ, and not to have the first harmonics in their Fourier series expansion in φ, i.e.,

$$\int_0^{2\pi} U_k(\tau, a, \theta, \varphi) e^{\pm i\varphi} d\varphi = 0, \;\; k \in \mathbf{N}. \tag{3.3}$$

It is known that condition (3.3) gives a possibility to uniquely determine the functions $U_k(\tau, a, \theta, \varphi)$ for every $k \in \mathbf{N}$. On the other hand, these conditions allow to construct asymptotic solutions containing no secular terms.

As is well known, in order to find differential equations for the functions $A_k(\tau, a, \psi)$, $B_k(\tau, a, \psi)$, $U_k(\tau, a, \theta, \varphi)$, $k \in \mathbf{N}$, it becomes necessary to substitute the expansion (3.1) into equation (2.1), taking into account the differential equations (3.2), expand the obtained relation into the series in the small parameter ε, and finally equate the terms with the same degree of ε.

Let us introduce the following notations:

$$Q_0 = \left(\omega(\tau) - \frac{p}{q}\nu(\tau)\right)\frac{\partial}{\partial\psi},\ Q_1 = \frac{\partial}{\partial\tau} + A_1(\tau, a, \psi)\frac{\partial}{\partial a} + B_1(\tau, a, \psi)\frac{\partial}{\partial\psi},$$

$$Q_k = A_k(\tau, a, \psi)\frac{\partial}{\partial a} + B_k(\tau, a, \psi)\frac{\partial}{\partial\psi},\ k \geq 2, \tag{3.4}$$

$$T_{nm} = \sum_{k_1\geq 0, k_2\geq 0, \ldots, k_n\geq 0}^{k_1+k_2+\ldots+k_n=m} Q_{k_1}Q_{k_2}\ldots Q_{k_n},\ \mathcal{P}_m = \sum_{n=0}^{\infty}\frac{(-\sigma)^{n+1}}{(n+1)!}T_{nm},$$

where $T_{0m} = \delta_{0m}$, δ_{0m} is the Kronecker symbol, $n \in \mathbf{N}, m = 0, 1, \ldots$.

In particular, $T_{00} = 0, T_{1m} = Q_m, T_{n0} = Q_0^n$ for $n, m \in \mathbf{N}$; the operators $Q_k, T_{nm}, \mathcal{P}_m$ act on the functions $A_k(\tau, a, \psi), B_k(\tau, a, \psi)$, where $k, n \in \mathbf{N}, m = 0, 1, \ldots$. We also note here that $\mathcal{P}_m = 0, m = 0, 1, \ldots$, if $\sigma = 0$, i.e., the delay is absent.

By direct calculations it is possible to find the following formulas:

$$\frac{d^n a}{dt^n} = \sum_{k=1}^{\infty}\varepsilon^k\left(\sum_{m=0}^{k-1}T_{n-1,m}A_{k-m}\right),\ n \geq 2,$$

$$\frac{d^n\psi}{dt^n} = \varepsilon^{n-1}\frac{d^{n-1}}{d\tau^{n-1}}\left(\omega(\tau) - \frac{p}{q}\nu(\tau)\right) + \sum_{k=1}^{\infty}\varepsilon^k\left(\sum_{m=0}^{k-1}T_{n-1,m}B_{k-m}\right),\ n \geq 2,$$

$$\frac{dx}{dt} = -a\omega\sin\varphi + \sum_{k=1}^{\infty}\varepsilon^k\left\{ - aB_k\sin\varphi + A_k\cos\varphi + \left(\nu\frac{\partial}{\partial\theta} + \omega\frac{\partial}{\partial\varphi}\right)U_k\right\} +$$

$$+ \sum_{k=1}^{\infty}\varepsilon^k g_k(\tau, a, \psi, \varphi, \theta), \tag{3.5}$$

$$\frac{d^2x}{dt^2} = -a\omega^2\cos\varphi + \sum_{k=1}^{\infty}\varepsilon^k\left\{ - \left(2\omega A_k + a\left(\omega - \frac{p}{q}\nu\right)\frac{\partial B_k}{\partial\psi}\right)\sin\varphi + \right.$$

$$\left. + \left(\left(\omega - \frac{p}{q}\nu\right)\frac{\partial A_k}{\partial\psi} - 2a\omega B_k\right)\cos\varphi\right\} +$$

$$+\sum_{k=1}^{\infty}\varepsilon^k\left\{\left(\nu^2\frac{\partial^2}{\partial\theta^2}+2\nu\omega\frac{\partial^2}{\partial\theta\partial\varphi}+\omega^2\frac{\partial^2}{\partial\varphi^2}\right)U_k+s_k(\tau,a,\psi,\varphi,\theta)\right\},$$

where $g_k(\tau,a,\psi,\varphi,\theta)$, $s_k(\tau,a,\psi,\varphi,\theta)$, $k\in\mathbf{N}$, are determined from the functions $A_m=A_m(\tau, a,\psi)$, $B_m=B_m(\tau,a,\psi)$, $U_m=U_m(\tau,a,\theta,\varphi)$, $m=\overline{1,k-1}$.

Using the Rubanik's idea [10] of representing the function $g(t-\sigma)$ in ferms of the power series

$$g(t-\sigma)=g(t)+\sum_{k=1}^{\infty}\left(\frac{d^k}{dt^k}g(t)\frac{\sigma^k}{k!}\right),$$

which is possible, for instance, under the conditions

$$\left|\frac{d^k}{dt^k}g(t)\right|\le CL^k \quad \text{for all} \quad t\in\mathbf{R}, k\in\mathbf{N},$$

with some constancs C,L, the functions $a(t-\sigma(\tau)),\psi(t-\sigma(\tau))$ and $\theta(t-\sigma(\tau))$ can also be represented an asymptotic expansion in the small parameter ε with the coefficients being functions of $a(t),\psi(t),\theta(t)$ without delay as follows:

$$a(t-\sigma(\tau))=a(t)+\sum_{k=1}^{\infty}\varepsilon^k\left(\sum_{m=0}^{k-1}\mathcal{P}_mA_{k-m}(\tau,a(t),\psi(t))\right),$$

$$\psi(t-\sigma(\tau))=\psi(t)+\sum_{k=1}^{\infty}\varepsilon^k\left(\sum_{m=0}^{k-1}\mathcal{P}_mB_{k-m}(\tau,a(t),\psi(t))\right)+$$

$$+\sum_{k=0}^{\infty}\varepsilon^k\frac{(-\sigma)^{k+1}}{(k+1)!}\,\frac{d^k}{d\tau^k}\left(\omega(\tau)-\frac{p}{q}\nu(\tau)\right), \tag{3.6}$$

$$\theta(t-\sigma(\tau))=\theta(t)+\sum_{k=0}^{\infty}\varepsilon^k\frac{(-\sigma)^{k+1}}{(k+1)!}\,\frac{d^k\nu(\tau)}{d\tau^k}.$$

Further, as a consequence, we find

$$x(t-\sigma(\tau))=a\cos(\varphi-\omega\sigma)+\sum_{k=1}^{\infty}\varepsilon^k\big\{\mathcal{P}_0A_k(\tau,a,\psi)\cos(\varphi-\omega\sigma)-$$

$$-a\mathcal{P}_0B_k(\tau,a,\psi)\sin(\varphi-\omega\sigma)+U_k(\tau,a,\theta-\nu\sigma,\varphi-\omega\sigma)+q_k(\tau,a,\psi,\varphi,\theta)\big\}, \tag{3.7}$$

$$\frac{dx(t-\sigma(\tau))}{dt} = -a\omega\sin(\varphi-\omega\sigma)+$$

$$+\sum_{k=1}^{\infty}\varepsilon^k\big\{\,[\mathcal{R}_0 A_k(\tau,a,\psi) - a\omega\mathcal{P}_0 B_k(\tau,a,\psi)]\cos(\varphi-\omega\sigma)-$$

$$-\,[\omega\mathcal{P}_0 A_k(\tau,a,\psi) - a\mathcal{R}_0 B_k(\tau,a,\psi)]\sin(\varphi-\omega\sigma)\big\}+$$

$$+\sum_{k=1}^{\infty}\varepsilon^k\left\{\left(\nu\frac{\partial}{\partial\theta}+\omega\frac{\partial}{\partial\varphi}\right)U_k(\tau,a,\theta-\nu\sigma,\varphi-\omega\sigma)+r_k(\tau,a,\psi,\varphi,\theta)\right\},$$

where the explicit form of each function $q_k(\tau,a,\psi,\varphi,\theta)$, $r_k(\tau,a,\psi,\varphi,\theta)$, for any $k \in \mathbf{N}$, is determined by the functions $A_m(\tau,a,\psi)$, $B_m(\tau,a,\psi)$, $U_m(\tau,a,\theta,\varphi)$, $m = \overline{1,k-1}$.

In addition,

$$\mathcal{P}_0 = \sum_{n=0}^{\infty}\frac{(-\sigma)^{n+1}}{(n+1)!}\left(\left(\omega-\frac{p}{q}\nu\right)\frac{\partial}{\partial\psi}\right)^n,$$

$$\mathcal{R}_0 = \sum_{n=0}^{\infty}\frac{(-\sigma)^{n}}{n!}\left(\left(\omega-\frac{p}{q}\nu\right)\frac{\partial}{\partial\psi}\right)^n, \tag{3.8}$$

the values assumed as $a = a(t), \psi = \psi(t), \varphi = \varphi(t), \theta = \theta(t)$, i.e., containing no delay.

Note that there are many functions that can be represented in a way similar to relations (3.6), (3.7). Specifically, this takes place if the power series for such functions are convergent for all values of τ.

Substituting formulas (3.5) – (3.7) into the right-hand side of equation (2.1) and expanding the result in the small parameter ε yields

$$\varepsilon f\left(\tau,\theta,x(t),x(t-\sigma(\tau)),\frac{dx(t)}{dt},\frac{dx(t-\sigma(\tau))}{dt}\right) = \sum_{k=0}^{\infty}\varepsilon^{k+1}f_k(\tau,a,\theta,\varphi).$$

In a standard way, we substitute the obtained expressions into equation (2.1), equate the coefficients at the same powers of the small parameter ε and find

$$\left[\nu\frac{\partial}{\partial\theta}+\omega\frac{\partial}{\partial\varphi}+\omega_1^2\right]U_k(\tau,a,\theta,\varphi)+$$

$$+\left[\beta\left(\nu\frac{\partial}{\partial\theta}+\omega\frac{\partial}{\partial\varphi}\right)+\omega_2^2\right]U_k(\tau,a,\theta-\omega\sigma,\varphi-\omega\sigma) =$$

$$= F_k(\tau,a,\varphi,\theta) + (\mathcal{L}A_k + a\mathcal{E}B_k)\cos\varphi + (\mathcal{E}A_k - a\mathcal{L}B_k)\sin\varphi, \tag{3.9}$$

where the differential operators $\mathcal{L}, \mathcal{E}$ are as follows:

$$\mathcal{L} = \left(\omega - \frac{p}{q}\nu\right)\frac{\partial}{\partial\psi} + 1 + \beta\mathcal{R}_0\cos\omega\sigma + \beta\omega\mathcal{P}_0\sin\omega\sigma + \omega_2^2\mathcal{P}_0\cos\omega\sigma,$$
$$\mathcal{E} = -2\omega + \beta\mathcal{R}_0\sin\omega\sigma - \beta\omega\mathcal{P}_0\cos\omega\sigma + \omega_2^2\mathcal{P}_0\sin\omega\sigma. \qquad (3.10)$$

An explicit form of the functions $F_k = f_{k-1} - (\alpha g_k + \beta r_k + \omega_2^2 q_k + s_k)$, $f_{k-1}, g_k, q_k, r_k, s_k$, for every $k \in \mathbf{N}$, can be found after a consequtive calculation of the functions $A_m(\tau, a, \psi)$, $B_m(\tau, a, \psi)$, $U_m(\tau, a, \theta, \varphi)$, $m = \overline{1, k-1}$. To implement this, we have to use the 2π-periodicity of the functions $A_k(\tau, a, \psi), B_k(\tau, a, \psi), U_k(\tau, a, \theta, \varphi), F_k(\tau, a, \varphi, \theta), k \in \mathbf{N}$, with respect to the variables ψ, φ, θ, correspondingly, and then represent these functions in terms of their Fourier series as follows:

$$U_k(\tau, a, \theta, \varphi) = \sum_{m,n=-\infty}^{+\infty} U_{kmn}(\tau, a)e^{i(m\theta+n\varphi)},$$
$$F_k(\tau, a, \varphi, \theta) = \sum_{m,n=-\infty}^{+\infty} F_{kmn}(\tau, a)e^{i(m\theta+n\varphi)},$$
$$A_k(\tau, a, \psi) = \sum_{n=-\infty}^{+\infty} A_{kn}(\tau, a)e^{in\psi}, \qquad (3.11)$$
$$B_k(\tau, a, \psi) = \sum_{n=-\infty}^{+\infty} B_{kn}(\tau, a)e^{in\psi},$$

where

$$U_{kmn}(\tau, a) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} U_k(\tau, a, \theta, \varphi)e^{-i(m\theta+n\varphi)}d\theta d\varphi,$$
$$F_{kmn}(\tau, a) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} F_k(\tau, a, \varphi, \theta)e^{-i(m\theta+n\varphi)}d\theta d\varphi,$$
$$A_{kn}(\tau, a) = \frac{1}{2\pi}\int_0^{2\pi} A_k(\tau, a, \psi)e^{-im\psi}d\psi,$$
$$B_{kn}(\tau, a) = \frac{1}{2\pi}\int_0^{2\pi} B_k(\tau, a, \psi)e^{-im\psi}d\psi.$$

Taking into consideration the relation $\varphi - \dfrac{p}{q}\theta = \psi$, we separate the resonance and the nonresonance terms in the Fourier expansion (3.11) of the functions $F_k(\tau, a, \psi, \theta, \varphi)$, $k \in \mathbf{N}$, in the following way:

$$F_k(\tau, a, \varphi, \theta) = \sum_{mq+(n\pm1)p\neq 0} F_{kmn}(\tau, a) e^{i(m\theta + n\varphi)} +$$

$$+ \sum_{mq+(n\pm1)p=0} F_{kmn}(\tau, a) e^{i(m\theta + n\varphi)}, \;\; m, n \in \mathbf{Z},$$

where the last term can be wriften as

$$\sum_{mq+(n\pm1)p=0} F_{kmn}(\tau, a) e^{i(m\theta + n\varphi)} =$$

$$= \sum_{n=-\infty}^{+\infty} \left[F_{k,-pn,qn-1}(\tau, a) e^{i\varphi} + F_{k,-pn,qn+1}(\tau, a) e^{-i\varphi} \right] e^{iqn\psi}.$$

Using both condition (3.3) and relation (3.9) we find the following differential equations for the functions $A_k(\tau, a, \psi)$, $B_k(\tau, a, \psi)$, $U_k(\tau, a, \theta, \varphi)$, $k \in \mathbf{N}$:

$$\left[\nu \frac{\partial}{\partial \theta} + \omega \frac{\partial}{\partial \varphi} + \omega_1^2 \right] U_k(\tau, a, \theta, \varphi) +$$

$$+ \left[\beta \left(\nu \frac{\partial}{\partial \theta} + \omega \frac{\partial}{\partial \varphi} \right) + \omega_2^2 \right] U_k(\tau, a, \theta - \omega\sigma, \varphi - \omega\sigma) =$$

$$= \sum_{m,n=-\infty}^{+\infty} F_{kmn}(\tau, a) e^{i(m\theta + n\varphi)}, \;\; mq + (n \pm 1)p \neq 0, \;\; k \in \mathbf{N}, \tag{3.12}$$

$$\begin{aligned} \mathcal{L}A_k + a\mathcal{E}B_k &= \sum_{n=-\infty}^{+\infty} G_{kn}(\tau, a) e^{in\psi}, \; k \in \mathbf{N}, \\ \mathcal{E}A_k - a\mathcal{L}B_k &= \sum_{n=-\infty}^{+\infty} H_{kn}(\tau, a) e^{in\psi}, \;\; k \in \mathbf{N}, \end{aligned} \tag{3.13}$$

where the differential operators $\mathcal{L}, \mathcal{E}$ are given by formula (3.10) and

$$G_{kn}(\tau,a) = -\frac{1}{2\pi}\int\limits_0^{2\pi}\sum_{m=-\infty}^{+\infty}[F_{k,-pm,qm-1}(\tau,a)+F_{k,-pm,qm+1}(\tau,a)]\times$$

$$\times\, e^{i(mq-n)\psi}\,d\psi = -\frac{1}{4\pi^3}\int\limits_0^{2\pi}\sum_{m=\infty}^{+\infty}\int\limits_0^{2\pi}\int\limits_0^{2\pi}F_k(\tau,a,\varphi,\theta)\times$$

$$\times\, e^{ipm\theta}e^{-imq\varphi}e^{i(mq-n)\psi}\cos\varphi\, d\theta d\varphi d\psi,$$

$$H_{kn}(\tau,a) = -\frac{1}{2\pi}\int\limits_0^{2\pi}\sum_{m=-\infty}^{+\infty}\Big[F_{k,-pm,qm+1}(\tau,a)-F_{k,-pm,qm-1}(\tau,a)\Big]\times$$

$$\times\, e^{i(mq-n)\psi}\,d\psi = \frac{1}{4\pi^3}\int\limits_0^{2\pi}\sum_{m=\infty}^{+\infty}\int\limits_0^{2\pi}\int\limits_0^{2\pi}F_k(\tau,a,\varphi,\theta)\times$$

$$\times\, e^{ipm\theta}e^{-imq\varphi}e^{i(mq-n)\psi}\sin\varphi\, d\theta d\varphi d\psi.$$

To solve equations (3.12), (3.13) we use the Fourier representations (3.11) and easily find systems of linear algebraic equations for the Fourier coefficients of the functions $U_k(\tau,a,\theta,\varphi)$, $k\notin\mathbf{N}$, as follows:

$$\Big[i(m\nu+n\omega)+\omega_1^2+\big(i\beta(m\nu+n\omega)+\omega_2^2\big)\,e^{-i(m\nu+n\omega)\sigma}\Big]\times$$

$$\times\, U_{kmn}(\tau,a) = \begin{cases} F_{kmn}(\tau,a,\psi), \text{ if } mq+(n\pm1)p\neq 0,\\ 0, \text{ if } mq+(n\pm1)p = 0,\end{cases} \tag{3.14}$$

where $m,n\in\mathbf{Z}$, and, respectively, the functions $A_k(\tau,a,\psi), B_k(\tau,a,\psi), k\in\mathbf{N}$, are given as follows:

$$\mathcal{L}_nA_{kn}+a\mathcal{E}_nB_{kn} = G_{kn}(\tau,a),\ \mathcal{E}_nA_{kn}-a\mathcal{L}_nB_{kn} = H_{kn}(\tau,a),\ n\in\mathbf{Z}, \tag{3.15}$$

where

$$\mathcal{L}_n = inl+1+\beta e^{-inl\sigma}\cos\omega\sigma+(\beta\omega\sin\omega\sigma+\omega_2^2\cos\omega\sigma)\rho(nl),$$

$$\mathcal{E}_n = -2\omega+\beta e^{-inl\sigma}\sin\omega\sigma+(-\beta\omega\cos\omega\sigma+\omega_2^2\sin\omega\sigma)\rho(nl),$$

$$l = \omega-\frac{p}{q}\nu,\qquad \rho=\rho(nl) = \begin{cases}(e^{-inl\sigma}-1)/(inl), \text{ if } nl\neq 0,\\ -\sigma, \text{ if } nl = 0.\end{cases}$$

Evidently, the system of algebraic equations (3.14) has a unique solution,

$$U_{kmn}(\tau, a) = \frac{F_{kmn}(\tau, a)}{\mathcal{M}_{mn} + \mathcal{N}_{mn} e^{-i(m\nu + n\omega)\sigma}},$$

if $mq + (n \pm 1)p \neq 0$ and $U_{kmn}(\tau, a) = 0$ if $mq + (n \pm 1)p = 0$, where

$$\mathcal{M}_{mn} = i(m\nu + n\omega) + \omega_1^2, \quad \mathcal{N}_{mn} = i\beta(m\nu + n\omega) + \omega_2^2,$$

if and only if the system of characteristic equations (2.4), (2.5) has no solution $\lambda = m\nu + n\omega, m, n \in \mathbf{Z}$, such that $mq + (n \pm 1)p \neq 0$.

Similarly, the system of algebraic equations (3.15) has a unique solution,

$$A_{kn}(\tau, a) = \frac{\mathcal{L}_n G_{kn}(\tau, a) + \mathcal{E}_n H_{kn}(\tau, a)}{\mathcal{L}_n^2 + \mathcal{E}_n^2},$$

$$B_{kn}(\tau, a) = \frac{\mathcal{E}_n G_{kn}(\tau, a) - \mathcal{L}_n H_{kn}(\tau, a)}{a(\mathcal{L}_n^2 + \mathcal{E}_n^2)},$$

if and only if the following condition takes place:

$$\pm 2\omega - nl + i + i\beta e^{i(\pm\omega - nl)\sigma} + (i\omega_2^2 \pm \beta\omega)\rho(nl)e^{\pm i\omega\sigma} \neq 0 \ \forall n \in \mathbf{Z}. \tag{3.16}$$

The last condition, if $n = 0$, is equivalent to the assumption that the numbers $\pm\omega$ are simple roots of the characteristic equations (2.4) and (2.5). In the opposite case, if $n \neq 0$, relationship (3.16) yields

$$1 + \beta \cos nl\sigma \cos \omega\sigma - (\beta\omega \sin \omega\sigma + \omega_2^2 \cos \omega\sigma)\frac{\sin nl\sigma}{nl} \neq 0,$$

$$nl - \beta \sin nl\sigma \cos \omega\sigma + (\beta\omega \sin \omega\sigma + \omega_2^2 \cos \omega\sigma)\frac{1 - \cos nl\sigma}{nl} \neq 0,$$

$$-2\omega + \beta \cos nl\sigma \sin \omega\sigma + (\beta\omega \cos \omega\sigma - \omega_2^2 \sin \omega\sigma)\frac{\sin nl\sigma}{nl} \neq 0,$$

$$-\beta \sin nl\sigma \sin \omega\sigma + (-\beta\omega \cos \omega\sigma + \omega_2^2 \sin \omega\sigma)\frac{1 - \cos nl\sigma}{nl} \neq 0.$$

Thus the functions $U_k(\tau, a, \theta, \varphi)$, $A_k(\tau, a, \psi)$, $B_k(\tau, a, \psi)$, $k \in \mathbf{N}$, can be represented as follows:

$$U_k(\tau, a, \theta, \varphi) = \frac{1}{4\pi^2} \sum_{mq + (n \pm 1) \neq 0} \Big(\mathcal{M}_{mn} + \mathcal{N}_{mn} e^{-i(m\nu + n\omega)\sigma}\Big)^{-1} \times$$

$$\times e^{i(m\theta + n\varphi)} \int_0^{2\pi}\int_0^{2\pi} F_k(\tau, a, \varphi, \theta) e^{-i(m\theta + n\varphi)}\, d\theta d\varphi,$$

$$A_k(\tau, a, \psi) = \frac{1}{4\pi^3} \sum_{n=-\infty}^{+\infty} \left(\mathcal{L}_n^2 + \mathcal{E}_n^2\right)^{-1} e^{in\psi} \int_0^{2\pi} \sum_{m=-\infty}^{+\infty} \int_0^{2\pi} \int_0^{2\pi} F_k(\tau, a, \varphi, \theta) \times$$

$$\times\, e^{ipm\theta} e^{-iqm\varphi} (-\mathcal{L}_n \cos\varphi + \mathcal{E}_n \sin\varphi) e^{i(mq-n)\psi}\, d\theta d\varphi d\psi,$$

$$B_k(\tau, a, \psi) = \frac{1}{4a\pi^3} \sum_{n=-\infty}^{+\infty} \left(\mathcal{L}_n^2 + \mathcal{E}_n^2\right)^{-1} e^{in\psi} \int_0^{2\pi} \sum_{m=-\infty}^{+\infty} \int_0^{2\pi} \int_0^{2\pi} F_k(\tau, a, \varphi, \theta) \times$$

$$\times\, e^{ipm\theta} e^{-iqm\varphi} (-\mathcal{L}_n \sin\varphi - \mathcal{E}_n \cos\varphi) e^{i(mq-n)\psi}\, d\theta d\varphi d\psi.$$

Many properties of weakly-nonlinear oscillatory systems, such as the stationary mode, stability, dependence on the parameters of stationary oscillations, often appear in the first approximation and can be studied by the first approximation and the first improved approximation. The first improved approximate solution to the problem (2.1) has the following form:

$$x(t) = a \cos\left(\frac{p}{q}\theta + \psi\right) + \varepsilon U_1\left(\tau, a, \theta, \frac{p}{q}\theta + \psi\right),$$

where

$$U_1\left(\tau, a, \theta, \frac{p}{q}\theta + \psi\right) = \frac{1}{4\pi^2} \sum_{mq+(n\pm 1)\neq 0} \left(\mathcal{M}_{mn} + \mathcal{N}_{mn} e^{-i(m\nu + n\omega)\sigma}\right)^{-1} \times$$

$$\times\, e^{i(m\theta+n\varphi)} \int_0^{2\pi} \int_0^{2\pi} f_0(\tau, a, \theta, \varphi) e^{-i(m\theta+n\varphi)}\, d\theta d\varphi, \;\; \varphi = \frac{p}{q}\theta + \psi.$$

The functions $a = a(t)$ and $\psi = \psi(t)$ satisfy the differential equations for the first approximation,

$$\frac{da}{dt} = \varepsilon A_1(\tau, a\psi) = \frac{\varepsilon}{4\pi^3} \sum_{n=-\infty}^{+\infty} \left(\mathcal{L}_n^2 + \mathcal{E}_n^2\right)^{-1} e^{in\psi} \times$$

$$\times \int_0^{2\pi} \sum_{m=-\infty}^{+\infty} \int_0^{2\pi} \int_0^{2\pi} f_0(\tau, a, \theta, \varphi) e^{ipm\theta} e^{-iqm\varphi} \times$$

$$\times\, (-\mathcal{L}_n \cos\varphi + \mathcal{E}_n \sin\varphi) e^{i(mq-n)\psi} d\theta d\varphi d\psi,$$

$$\frac{d\psi}{dt} = \omega - \frac{p}{q}\nu + \varepsilon B_1(\tau, a\psi) =$$

$$= \omega - \frac{p}{q}\nu - \frac{\varepsilon}{4a\pi^3}\sum_{n=-\infty}^{+\infty}\left(\mathcal{L}_n^2 + \mathcal{E}_n^2\right)^{-1} e^{in\psi}\times$$

$$\times\int_0^{2\pi}\sum_{m=-\infty}^{+\infty}\int_0^{2\pi}\int_0^{2\pi} f_0(\tau, a, \theta, \varphi)e^{ipm\theta}e^{-iqm\varphi}\times$$

$$\times\,(\mathcal{L}_n\sin\varphi + \mathcal{E}_n\cos\varphi)e^{i(mq-n)\psi}d\theta d\varphi d\psi.$$

4. Conclusion. On the basis of asymptotic methods in nonlinear mechanics, an algorithm for constructing asymptotic approximations for the first order ordinary differential equation with deviated argument and slowly varying coefficients is given.

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