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HOMOGENIZATION OF THE ROBIN PROBLEM IN A THICK MULTILEVEL JUNCTION

УСЕРЕДНЕННЯ ЗАДАЧІ РОБІНА В ГУСТОМУ БАГАТОРІВНЕВОМУ З'ЄДНАННІ

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In the paper we consider a mixed boundary-value problem for the Poisson equation in a plane two-level junction Ω_ε, which is the union of a domain Ω_0 and a large number $2N$ of thin rods with variable thickness of order $\varepsilon = \mathcal{O}(N^{-1})$. The thin rods are divided into two levels depending on their length. In addition, the thin rods from each level are ε-periodically alternated. We investigate the asymptotic behaviour of the solution as $\varepsilon \to 0$ under the Robin conditions on the boundaries of the thin rods. By using some special extension operators, the convergence theorem is proved.

Розглядається мішана крайова задача для рівняння Пуассона у плоскому дворівневому з'єднанні Ω_ε, яке є об'єднанням деякої області Ω_0 та великої кількості $2N$ тонких стержнів із змінною товщиною порядку $\varepsilon = \mathcal{O}(N^{-1})$. Тонкі стержні розділено на два рівні в залежності від їх довжини. Крім того, тонкі стержні з кожного рівня ε-періодично чергуються. Вивчено асимптотичну поведінку розв'язку, коли $\varepsilon \to 0$, при крайових умовах Робіна на межах тонких стержнів. Із використанням спеціальних операторів продовження доведено теорему збіжності.

Introduction. In this paper we consider a new type of thick junctions, namely, thick multilevel junctions. A thick multilevel junction is the union of some domain, which is called the junction's body, and a large number $N = \mathcal{O}(\varepsilon^{-1})$ of thin domains of thickness of order $\mathcal{O}(\varepsilon)$. Here ε is a small parameter. The thin domains are divided into a finite number of levels depending on their length. In addition, the thin domains from each levels are ε-periodically alternated along some manifold on the boundary of the junction's body. This manifold is called the joint zone.

The aim of researches is to develop rigorous asymptotic methods for boundary-value problems in thick multilevel junctions when the parameter ε goes to 0, i.e., when the number of the attached thin domains increases and their thickness decreases. The asymptotic methods, which were developed in [1 – 3] are used. A spectral problem in a plane thick multilevel junction with the flat boundaries of the thin rods was considered in [4]. Here we consider a mixed boundary-value problem for the Poisson equation in a plane thick two-level junction with variable thickness of the thin rods.

1. Statement of the problem. Let $a,\ d_1,\ d_2,\ b_1, b_2$ be positive real numbers and let $d_1 \leq d_2,\ 0 < b_1 < b_2 < 1$. Consider two positive piecewise smooth functions h_1 and h_2 on the the segments $[-d_1, 0]$ and $[-d_2, 0]$, respectively. Suppose the functions h_1 and h_2 satisfy the following conditions:

$$\exists\, \delta_0 \in (b_1, b_2)\ \forall\, x_2 \in [-d_1, 0]: \quad 0 < b_1 - h_1(x_2)/2, \qquad b_1 + h_1(x_2)/2 < \delta_0;$$

$$\forall\, x_2 \in [-d_2, 0]: \quad \delta_0 < b_2 - h_2(x_2)/2, \qquad b_2 + h_2(x_2)/2 < 1.$$

It follows from these assumptions that there exist positive constants $m_0,\ M_0$ such that

$$\begin{gathered} 0 < m_0 \leq h_1(x_2) < \delta_0 \quad \text{and} \quad |h_1'(x_2)| \leq M_0 \quad \text{a. e. in } [-d_1, 0], \\ 0 < m_0 \leq h_2(x_2) < 1 - \delta_0 \quad \text{and} \quad |h_2'(x_2)| \leq M_0 \quad \text{a. e. in } [-d_2, 0]. \end{gathered} \tag{1}$$

Let us divide segment $[0, a]$ into N equal segments $[\varepsilon j, \varepsilon(j+1)],\ j = 0, \ldots, N-1$. Here N is a large integer, therefore, the value $\varepsilon = a/N$ is a small discrete parameter.

A model plane thick two-level junction Ω_ε consists of the junction's body

$$\Omega_0 = \{x \in \mathbb{R}^2 : \ 0 < x_1 < a, \quad 0 < x_2 < \gamma(x_1)\,\},$$

where $\gamma \in C^1([0, a]),\ \gamma(0) = \gamma(a),\ \min_{[0,a]} \gamma > 0,$ and a large number of the thin rods

$$G_j^{(1)}(\varepsilon) = \{x \in \mathbb{R}^2 : \ |x_1 - \varepsilon\,(j + b_1)| < \ \varepsilon h_1(x_2)/2, \quad x_2 \in (-d_1, 0]\}, \quad j = 0, 1, \ldots, N-1,$$

$$G_j^{(2)}(\varepsilon) = \{x \in \mathbb{R}^2 : \ |x_1 - \varepsilon\,(j + b_2)| < \ \varepsilon h_2(x_2)/2, \quad x_2 \in (-d_2, 0]\}, \quad j = 0, 1, \ldots, N-1,$$

i.e.,

$$\Omega_\varepsilon = \Omega_0 \cup G^{(1)}(\varepsilon) \cup G^{(2)}(\varepsilon),$$

where $G^{(1)}(\varepsilon) = \cup_{j=0}^{N-1} G_j^{(1)}(\varepsilon),\ \ G^{(2)}(\varepsilon) = \cup_{j=0}^{N-1} G_j^{(2)}(\varepsilon)$.

We see that the number of the thin rods is equal to $2N$ and they are divided into two levels $G^{(1)}(\varepsilon)$ and $G^{(2)}(\varepsilon)$ depending on their length (we recall that $d_1 \leq d_2$). The small parameter ε characterizes the distance between the thin neighboring rods and their thickness. The thickness of the rods from the first level is equal to εh_1 and to εh_2 for the rods from the second level. These thin rods from each level are ε-periodically alternated along the segment $I_0 = \{x : x_1 \in \in [0, a],\ \ x_2 = 0\}$.

Denote by $\Upsilon_j^{(i,\pm)}(\varepsilon)$ the lateral surfaces of the thin rod $G_j^{(i)}(\varepsilon)$; the signs "$+$"or "$-$" indicate the right or left surface respectively. The base of $G_j^{(i)}(\varepsilon)$ will be denoted by $\Theta_j^{(i)}(\varepsilon)$. Also we introduce the following notations:

$$\Upsilon^{(i,\pm)}(\varepsilon) := \bigcup_{j=0}^{N-1} \Upsilon_j^{(i,\pm)}(\varepsilon), \quad \Theta^{(i)}(\varepsilon) := \bigcup_{j=0}^{N-1} \Theta_j^{(i)}(\varepsilon),$$

$$\Upsilon^{(i)}(\varepsilon) := \Upsilon^{(i,+)}(\varepsilon) \cup \Upsilon^{(i,-)}(\varepsilon) \cup \Theta^{(i)}(\varepsilon),$$

for $i = 1, 2$.

In Ω_ε we consider the following mixed boundary-value problem:

$$\begin{array}{rcll} -\Delta_x u_\varepsilon(x) & = & f_\varepsilon(x), & x \in \Omega_\varepsilon, \\ \partial_\nu u_\varepsilon(x) & = & -\varepsilon\, k_1\, u_\varepsilon(x), & x \in \Upsilon^{(1)}(\varepsilon), \\ \partial_\nu u_\varepsilon(x) & = & -\varepsilon\, k_2\, u_\varepsilon(x), & x \in \Upsilon^{(2)}(\varepsilon), \\ \partial^p_{x_1} u_\varepsilon(0, x_2) & = & \partial^p_{x_1} u_\varepsilon(a, x_2), & x_2 \in [0, \gamma(0)], \ \ p = 0, 1, \\ \partial_\nu u_\varepsilon(x) & = & 0, & x \in \Gamma_\varepsilon. \end{array} \qquad (2)$$

Here $\partial_\nu = \partial/\partial\nu$ is the outward normal derivative, $\partial_{x_1} = \partial/\partial x_1$, the constants k_1 and k_2 are positive. Thus, we have the Robin conditions on the boundaries of the thin rods, the periodic conditions on the vertical sides of Ω_0 and the Neumann condition on the other part Γ_ε of $\partial\Omega_\varepsilon$.

We can regard without loss of generality that the right-hand side f_ε belongs to $L^2(\Omega_2)$, where Ω_2 is the interior of $\overline{\Omega_0} \cup \overline{D_2}$, $D_2 = (0, a) \times (-d_2, 0)$ is a rectangle that is filled up by the thin rods from the second level in the limit passage as $\varepsilon \to 0$. Similarly, $D_1 = (0, a) \times (-d_1, 0)$ and Ω_1 is the interior of $\overline{\Omega_0} \cup \overline{D_1}$.

We assume that

$$f_\varepsilon \to f_0 \quad \text{in} \quad L^2(\Omega_2) \quad \text{as} \quad \varepsilon \to 0. \qquad (3)$$

The aim of our research is to study the asymptotic behaviour of the solution to problem (2) as $\varepsilon \to 0$, i.e., when the number of attached thin rods infinitely increases and their thickness tends to 0.

2. Auxiliary inequalities. First we recall that for every fixed value ε, in accordance with the main results of the theory of boundary-value problems, there exists a unique weak solution $u_\varepsilon \in \mathcal{H}_\varepsilon$ to problem (2) such that the integral identity

$$\int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla \varphi \, dx + \varepsilon\, k_1 \int\limits_{\Upsilon^{(1)}(\varepsilon)} u_\varepsilon\, \varphi \, dl_x + \varepsilon\, k_2 \int\limits_{\Upsilon^{(2)}(\varepsilon)} u_\varepsilon\, \varphi \, dl_x = \int\limits_{\Omega_\varepsilon} f_\varepsilon(x)\, \varphi(x)\, dx \qquad (4)$$

holds for any function $\varphi \in \mathcal{H}_\varepsilon$, where

$$\mathcal{H}_\varepsilon = \{u \in H^1(\Omega_\varepsilon): \ \ \partial^p_{x_1} u(0, x_2) = \partial^p_{x_1} u(a, x_2), \ \ x_2 \in [0, \gamma(0)], \ \ p = 0, 1\}.$$

In addition, the solution u_ε satisfies the inequality

$$\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)} \le c_1 \|f_\varepsilon\|_{L^2(\Omega_\varepsilon)}. \qquad (5)$$

Let us show that the constant c_1 in (5) is independent of the small parameter ε.

Lemma 1. *For ε small enough, the usual norm $\|\cdot\|_{H^1(\Omega_\varepsilon)}$ in the Sobolev space $H^1(\Omega_\varepsilon)$ and the norm*:

$$\|v\|_{\varepsilon,k_1,k_2} = \left(\int\limits_{\Omega_\varepsilon} |\nabla v|^2 \, dx + \varepsilon \, k_1 \int\limits_{\Upsilon^{(1)}(\varepsilon)} v^2 \, dl_x + \varepsilon \, k_2 \int\limits_{\Upsilon^{(2)}(\varepsilon)} v^2 \, dl_x \right)^{1/2}$$

are uniformly equivalent, i.e., there exist constants $C_1 > 0$, $C_2 > 0$ and ε_0 such that for all $\varepsilon \in (0, \varepsilon_0)$ and any function $v \in H^1(\Omega_\varepsilon)$ the inequalities

$$C_1 \|v\|_{H^1(\Omega_\varepsilon)} \le \|v\|_{\varepsilon,k_1,k_2} \le C_2 \|v\|_{H^1(\Omega_\varepsilon)} \tag{6}$$

are satisfied.

Proof. Let us defined the following function:

$$Y(t) = \begin{cases} -t + b_1, & t \in [0, \delta_0), \\ -t + b_2, & t \in [\delta_0, 1), \end{cases} \tag{7}$$

and then periodically extend it to $\mathbb{R}$.

Integrating by parts the integral $\varepsilon \int_{G^{(1)}(\varepsilon) \cup G^{(2)}(\varepsilon)} Y(x_1/\varepsilon) \, \partial_{x_1} v \, dx$ and taking into account that the outward normal to the lateral surfaces $\Upsilon_j^{(i,\pm)}(\varepsilon)$ of the thin rod $G_j^{(i)}(\varepsilon)$, except for some set of measure zero, has the form

$$\nu_{\pm}^{(i)}(\varepsilon) = \frac{1}{\sqrt{1 + \varepsilon^2 4^{-1} |h_i'(x_2)|^2}} \left(\pm 1 \, , \, -\varepsilon \, \frac{h_i'(x_2)}{2} \right), \qquad i = 1, 2, \quad j = 0, \ldots, N-1, \tag{8}$$

we get the identity

$$\varepsilon \sum_{i=1}^{2} \int\limits_{\Upsilon^{(i,\pm)}(\varepsilon)} \frac{h_i(x_2)}{2\sqrt{1 + \varepsilon^2 4^{-1} |h_i'(x_2)|^2}} \, v \, dl_x =$$

$$= \int\limits_{G^{(1)}(\varepsilon) \cup G^{(2)}(\varepsilon)} v \, dx - \varepsilon \int\limits_{G^{(1)}(\varepsilon) \cup G^{(2)}(\varepsilon)} Y\left(\frac{x_1}{\varepsilon}\right) \partial_{x_1} v \, dx \quad \forall \, v \in H^1(\Omega_\varepsilon). \tag{9}$$

Using the identity (9), the properties of the trace operator and taking into account that $\max_{\mathbb{R}} |Y| \le 1$, we obtain

$$\|v\|^2_{\varepsilon,k_1,k_2} = \int\limits_{\Omega_\varepsilon} |\nabla v|^2\,dx +$$

$$+\,\varepsilon\sum_{i=1}^{2} k_i \int\limits_{\Upsilon^{(i,\pm)}(\varepsilon)} \frac{2\sqrt{1+\varepsilon^2 4^{-1}|h_i'(x_2)|^2}}{h_i(x_2)}\,\frac{h_i(x_2)}{2\sqrt{1+\varepsilon^2 4^{-1}|h_i'(x_2)|^2}}\;v^2\,dl_x +$$

$$+\,\varepsilon\sum_{i=1}^{2} k_i \int\limits_{\Theta^{(i)}(\varepsilon)} v^2\,dx_1 \le$$

$$\le \int\limits_{\Omega_\varepsilon} |\nabla v|^2\,dx + c_1\sum_{i=1}^{2}\int\limits_{\Upsilon^{(i,\pm)}(\varepsilon)} \frac{\varepsilon h_i(x_2)}{2\sqrt{1+\varepsilon^2 4^{-1}|h_0'(x_2)|^2}} v^2\,dl_x + \varepsilon c_2\sum_{i=1}^{2}\|v\|^2_{H^1(G^{(i)}(\varepsilon))} \le$$

$$\le c_3\|v\|^2_{H^1(\Omega_\varepsilon)} + c_1\Bigg(\int\limits_{G^{(1)}(\varepsilon)\cup G^{(2)}(\varepsilon)} v^2\,dx - \varepsilon\int\limits_{G^{(1)}(\varepsilon)\cup G^{(2)}(\varepsilon)} Y\Big(\frac{x_1}{\varepsilon}\Big) 2v\,\partial_{x_1} v\,dx\Bigg) \le$$

$$\le c_3\|v\|^2_{H^1(\Omega_\varepsilon)} + c_1\Bigg(\int\limits_{G^{(1)}(\varepsilon)\cup G^{(2)}(\varepsilon)} v^2\,dx + \int\limits_{G^{(1)}(\varepsilon)\cup G^{(2)}(\varepsilon)} \varepsilon\Big((\partial_{x_1} v)^2 + v^2\Big)\,dx\Bigg) \le$$

$$\le C_2\|u\|^2_{H^1(\Omega_\varepsilon)}. \tag{10}$$

Similarly, we obtain

$$\|v\|^2_{H^1(\Omega_\varepsilon)} = \int\limits_{\Omega_\varepsilon} |\nabla v|^2\,dx + \int\limits_{\Omega_0} v^2\,dx + \int\limits_{G^{(1)}(\varepsilon)\cup G^{(2)}(\varepsilon)} v^2\,dx = \int\limits_{\Omega_\varepsilon} |\nabla v|^2\,dx + \int\limits_{\Omega_0} v^2\,dx +$$

$$+\,\varepsilon\sum_{i=1}^{2}\int\limits_{\Upsilon^{(i,\pm)}(\varepsilon)} \frac{h_i(x_2)}{2\sqrt{1+\varepsilon^2 4^{-1}|h_i'(x_2)|^2}}\;v^2\,dl_x + \varepsilon\int\limits_{G^{(1)}(\varepsilon)\cup G^{(2)}(\varepsilon)} Y\Big(\frac{x_1}{\varepsilon}\Big) 2v\,\partial_{x_1} v\,dx \le$$

$$\le C_3\|v\|^2_{\varepsilon,k_1,k_2} + \int\limits_{\Omega_0} v^2\,dx + \varepsilon\int\limits_{G^{(1)}(\varepsilon)\cup G^{(2)}(\varepsilon)} v^2\,dx,$$

whence

$$\|v\|^2_{H^1(\Omega_\varepsilon)} \le C_4\left(\|v\|^2_{\varepsilon,k_1,k_2} + \int\limits_{\Omega_0} v^2\,dx\right). \tag{11}$$

Now let us show that there exists a positive constant C_5 such that for ε small enough and for any $v \in H^1(\Omega_\varepsilon)$,

$$\int\limits_{\Omega_0} v^2\,dx \;\le\; C_5 \|v\|^2_{\varepsilon,k_1,k_2}. \tag{12}$$

We argue by contradiction. If not, then there exist sequences $\{\varepsilon_m : m \in \mathbb{N}\}$ and $\{v_m\} \subset \subset H^1(\Omega_{\varepsilon_m})$ such that $\lim_{m\to 0}\varepsilon_m = 0$,

$$\int\limits_{\Omega_0} v_m^2\,dx = 1, \tag{13}$$

$$\int\limits_{\Omega_{\varepsilon_m}} |\nabla v_m|^2\,dx + \varepsilon_m \sum_{i=1}^{2} k_i \int\limits_{\Upsilon^{(i)}(\varepsilon_m)} v_m^2\,dl_x < \frac{1}{m}. \tag{14}$$

Since the sequence $\{v_m\}$ is bounded in $H^1(\Omega_0)$, we may assume without loss of generality that it is a Cauchy sequence in $L^2(\Omega_0)$. From inequality (14) it follows that $\{v_m\}$ is a Cauchy sequence also in $H^1(\Omega_0)$,

$$\|v_m - v_n\|^2_{H^1(\Omega_0)} \le \|v_m - v_n\|^2_{L_2(\Omega_0)} + \frac{1}{m} + \frac{1}{n}.$$

Hence, $\{v_m\}$ converges to some element $v_0 \in H^1(\Omega_0)$. Obviously, $v_0 \equiv \mathrm{const}$ in $H^1(\Omega_0)$ and, due to (13), $v_0 = |\Omega_0|^{-1/2}$, where $|\Omega_0|$ denotes the measure of the domain Ω_0.

Then, the sequence of the traces of $\{v_m\}$ converges to v_0 in $L^2(\partial\Omega_0)$ as well and it is easy to verify that

$$\int\limits_{I_0(\varepsilon_m)} v_m^2(x_1,0)\,dx_1 = \sum_{i=1}^{2} \int\limits_{I_0} \chi_i(x_1/\varepsilon_m)\, v_m^2(x_1,0)\,dx_1 \;\to$$

$$\to \sum_{i=1}^{2} h_i(0) \int\limits_{I_0} v_0^2(x_1,0)\,dx_1 = \big(h_1(0)+h_2(0)\big)|\Omega_0|^{-1} a \ne 0, \quad m\to\infty, \tag{15}$$

where $I_0(\varepsilon) := I_0 \cap \Omega_\varepsilon$ and $\chi_i(\cdot)$ is a 1-periodic function such that

$$\chi_i(t) = \begin{cases} 1, & t \in \Big(b_i - \dfrac{h_i(0)}{2},\, b_i + \dfrac{h_i(0)}{2}\Big), \\ 0, & t \in [0,1] \setminus \Big(b_i - \dfrac{h_i(0)}{2},\, b_i + \dfrac{h_i(0)}{2}\Big), \end{cases} \tag{16}$$

for $i = 1, 2$. Obviously,

$$\chi_i(x_1/\varepsilon) \to \int\limits_0^1 \chi_i(t)\,dt = h_i(0) \quad \text{weakly in } L^2(0,a) \quad \text{as } \varepsilon \to 0.$$

On the other hand, from (9) and (14) it follows that

$$\int\limits_{G^{(1)}(\varepsilon_m)\cup G^{(2)}(\varepsilon_m)} \left(|\nabla v_m|^2 + v_m^2\right) dx \le \frac{C_6}{m}$$

and, therefore,

$$\int\limits_{I_0(\varepsilon_m)} v_m^2(x_1,0)\,dx_1 \le C_7 \int\limits_{G^{(1)}(\varepsilon_m)\cup G^{(2)}(\varepsilon_m))} \left(|\nabla v_m|^2 + v_m^2\right) dx \le \frac{C_8}{m},$$

where the constants C_6, C_7, C_8 are independent of m. This means that

$$\int\limits_{I_0(\varepsilon_m)} v_m^2(x_1,0)\,dx_1 \to 0 \quad \text{as } m \to \infty. \tag{17}$$

However (17) varies from (15). This contradiction establishes estimate (12).

Thus, by virtue of (11) and (12), we obtain the left inequality in (6).

The lemma is proved.

Remark 1. Hereafter all constants $\{c_i, C_i\}$ in asymptotic inequalities are independent of the parameter ε.

3. Extension operator. Due to the a-periodic condition in problem (2), we can assume that the function f_ε and the solution u_ε are a-periodic functions with respect to x_1.

Theorem 1. *Let condition (3) be satisfied and, in addition, there exist constants C_1 and ε_0 such that for all values $\varepsilon \in (0, \varepsilon_0)$*

$$\int\limits_{\Omega_\varepsilon} (F_\varepsilon(x))^2\,dx \le C_1, \tag{18}$$

where $F_\varepsilon(x) = \varepsilon^{-1}(f_\varepsilon(x+\varepsilon\bar{e}_1) - f_\varepsilon(x)) \quad (\bar{e}_1 = (1,0))$.

Then there exist extension operators

$$\mathbf{P}_\varepsilon^{(1)} : H^1(\Omega_0 \cup G^{(1)}(\varepsilon)) \mapsto H^1(\Omega_1) \ \text{ and } \ \mathbf{P}_\varepsilon^{(2)} : H^1(\Omega_0 \cup G^{(2)}(\varepsilon)) \mapsto H^1(\Omega_2)$$

such that, for the solution u_ε,

$$\|\, \mathbf{P}_\varepsilon^{(1)} u_\varepsilon \,\|_{H^1(\Omega_1)} + \|\, \mathbf{P}_\varepsilon^{(2)} u_\varepsilon \,\|_{H^1(\Omega_2)} \le C_2 \left(\|\mathbf{F}_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|f_\varepsilon\|_{L^2(\Omega_\varepsilon)}\right) \le C_3. \tag{19}$$

Proof. The first step in the proof is to show that scattering of the values of the solution u_ε on neighboring thin rods is small in some sense.

Here we assume for simplicity that $\gamma \equiv \mathrm{const}$. In general case similarly as in the proof of Theorem 4.1 [4], we should multiply the differential equation of problem (2) by a smooth cut-off function χ_0 such that $\chi_0(x_2) = 0$ for $x_2 \geq \gamma_0$, and $\chi_0(x_2) = 1$ for $x_2 \leq \gamma_0/2$, where $\gamma_0 = \min_{x_1 \in [0,a]} \gamma(x_1)$; and consider the function $v_\varepsilon = \chi_0 \, u_\varepsilon$ which is a solution to the corresponding boundary-value problem in a thick two-level junction whose junction's body is the rectangle $[0,a] \times [0,\gamma_0]$.

Thus, the problem (2) is invariant under the ε-shift along the axis x_1. This means that the function

$$\mathbf{U}_\varepsilon(x) = \varepsilon^{-1}(u_\varepsilon(x + \varepsilon \bar{e}_1) - u_\varepsilon(x)), \quad \bar{e}_1 = (1,0), \tag{20}$$

is a solution, a-periodic in x_1, to the following problem:

$$\begin{array}{rcll} -\Delta_x \mathbf{U}_\varepsilon(x) & = & \mathbf{F}_\varepsilon(x), & x \in \Omega_\varepsilon, \\ \partial_\nu \mathbf{U}_\varepsilon(x) & = & -\varepsilon \, k_1 \, \mathbf{U}_\varepsilon(x), & x \in \Upsilon^{(1)}(\varepsilon), \\ \partial_\nu \mathbf{U}_\varepsilon(x) & = & -\varepsilon \, k_2 \, \mathbf{U}_\varepsilon(x), & x \in \Upsilon^{(2)}(\varepsilon), \\ \partial^p_{x_1} \mathbf{U}_\varepsilon(0,x_2) & = & \partial^p_{x_1} \mathbf{U}_\varepsilon(a,x_2), & x_2 \in [0,\gamma(0)], \ \ p = 0,1, \\ \partial_\nu \mathbf{U}_\varepsilon(x) & = & 0, & x \in \Gamma_\varepsilon. \end{array} \tag{21}$$

By virtue of Lemma 1 and condition (18), we get the following estimate:

$$\|\mathbf{U}_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq C_2 \, \|\mathbf{F}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C_3. \tag{22}$$

At first we extend the solution u_ε the domain Ω_1 by using the "linear matching"

$$\widehat{P}^{(1)}_\varepsilon(u_\varepsilon)(x) = \left\{ \begin{array}{ll} u_\varepsilon, & x \in G^{(1)}(\varepsilon), \\ B^\varepsilon_j(x_2) + S^\varepsilon_j(x_2) \left(x_1 - \varepsilon \Big(j + b_1 + \dfrac{h_1(x_2)}{2} \Big) \right), & x \in \widetilde{Q}^{(1)}_j(\varepsilon), \end{array} \right. \tag{23}$$

in the domain $\Omega_0 \cup G^{(1)}(\varepsilon) \cup \widetilde{Q}^{(1)}(\varepsilon)$. Here

$$B^\varepsilon_j(x_2) = u_\varepsilon \left(\varepsilon \Big(j + b_1 + \frac{h_1(x_2)}{2} \Big), x_2 \right),$$

$$S^\varepsilon_j(x_2) = \frac{1}{\varepsilon(1 - h_1(x_2))} \left(u_\varepsilon \Big(\varepsilon \Big(j + 1 + b_1 - \frac{h_1(x_2)}{2} \Big), x_2 \Big) - B^\varepsilon_j(x_2) \right),$$

$$\widetilde{Q}^{(1)}(\varepsilon) = \bigcup_{j=-1}^{N} \widetilde{Q}^{(1)}_j(\varepsilon),$$

and the domain

$$\widetilde{Q}_j^{(1)}(\varepsilon) = \left\{ x : \; x_2 \in (-d_1, -\varepsilon), \; x_1 \in \left(\varepsilon\Big(j + b_1 + \frac{h_1(x_2)}{2}\Big), \varepsilon\Big(j + 1 + b_1 - \frac{h_1(x_2)}{2}\Big) \right) \right\}$$

is situated between two rods $G_j^{(1)}(\varepsilon)$ and $G_{j+1}^{(1)}(\varepsilon)$. In the case of the extreme rods, we perform the a-periodic extension of problem (2) with respect to the axis Ox_1.

Without the loss of generality, we can assume here that h_1 is smooth on $[-d_1, 0]$. It is easy to calculate that

$$\|\widehat{P}_\varepsilon^{(1)}(u_\varepsilon)\|^2_{H^1(\widetilde{Q}_j^{(1)}(\varepsilon))} =$$

$$= \int\limits_{\widetilde{Q}_j^{(1)}(\varepsilon)} \left| B_j^\varepsilon(x_2) + S_j^\varepsilon(x_2) \left[x_1 - \varepsilon \left(j + b_1 + \frac{h_1(x_2)}{2} \right) \right] \right|^2 dx +$$

$$+ \int\limits_{\widetilde{Q}_j^{(1)}(\varepsilon)} \Big| \big(B_j^\varepsilon(x_2)\big)' + S_j^\varepsilon(x_2) + \big(S_j^\varepsilon(x_2)\big)' \times$$

$$\times \left[x_1 - \varepsilon \left(j + b_1 + \frac{h_1(x_2)}{2} \right) \right] \Big|^2 dx. \qquad (24)$$

Further, we will not indicate that functions B_j^ε, S_j^ε and h_1 are depending on x_2 if it doesn't lead to a confusion. By using the inequality $(a+b)^2 \le 2a^2 + 2b^2$ and properties (1) for h_1, we get

$$\|\widehat{P}_\varepsilon^{(1)}(u_\varepsilon)\|^2_{H^1(\widetilde{Q}_j^{(1)}(\varepsilon))} \le 2 \int\limits_{\widetilde{Q}_j^{(1)}(\varepsilon)} \big(B_j^\varepsilon\big)^2 +$$

$$+ 2 \int\limits_{\widetilde{Q}_j^{(1)}(\varepsilon)} \big(S_j^\varepsilon\big)^2 \left[x_1 - \varepsilon \left(j + b_1 + \frac{h_1}{2} \right) \right]^2 dx +$$

$$+ 2 \int\limits_{\widetilde{Q}_j^{(1)}(\varepsilon)} \Big(\big(B_j^\varepsilon\big)' \Big)^2 dx + 4 \int\limits_{\widetilde{Q}_j^{(1)}(\varepsilon)} \big(S_j^\varepsilon\big)^2 dx +$$

$$+ 4 \int\limits_{\widetilde{Q}_j^{(1)}(\varepsilon)} \Big(\big(S_j^\varepsilon\big)' \Big)^2 \left[x_1 - \varepsilon \left(j + b_1 + \frac{h_1}{2} \right) \right]^2 dx.$$

Now, taking into account the geometry of the domain $\widetilde{Q}_j^{(1)}(\varepsilon)$, we deduce

$$\left\| \widehat{P}_\varepsilon^{(1)}(u_\varepsilon) \right\|^2_{H^1(\widetilde{Q}_j^{(1)}(\varepsilon))} \le 4\varepsilon\,(1-m_0) \int\limits_{-d_1}^{-\varepsilon} \left[\left(B_j^\varepsilon\right)^2 + \left(\left(B_j^\varepsilon\right)' \right)^2 + \left(S_j^\varepsilon\right)^2 \right] dx_2 + 4 \int\limits_{-d_1}^{-\varepsilon} \left[\left(S_j^\varepsilon\right)^2 + \right.$$

$$\left. + \left(\left(S_j^\varepsilon\right)' \right)^2 \right] \int\limits_{\varepsilon\left(j+b_1+\frac{h_1}{2}\right)}^{\varepsilon\left(j+1+b_1-\frac{h_1}{2}\right)} \left[x_1 - \varepsilon\left(j + b_1 + \frac{h_1}{2} \right) \right]^2 dx_1\, dx_2 \le$$

$$\le 4\varepsilon\,(1-m_0) \int\limits_{-d_1}^{-\varepsilon} \left[\left(B_j^\varepsilon\right)^2 + \left(\left(B_j^\varepsilon\right)' \right)^2 + \left(S_j^\varepsilon\right)^2 \right] dx_2 +$$

$$+ 4 \frac{\varepsilon^3\,(1-m_0)^3}{3} \int\limits_{-d_1}^{-\varepsilon} \left[\left(S_j^\varepsilon\right)^2 + \left(\left(S_j^\varepsilon\right)' \right)^2 \right] dx_2 \le$$

$$\le C_1 \left\{ \varepsilon \int\limits_{-d_1}^{-\varepsilon} \left[\left(B_j^\varepsilon\right)^2 + \left(\left(B_j^\varepsilon\right)' \right)^2 + \left(S_j^\varepsilon\right)^2 \right] dx_2 + \right.$$

$$\left. + \varepsilon^3 \int\limits_{-d_1}^{-\varepsilon} \left[\left(S_j^\varepsilon\right)^2 + \left(\left(S_j^\varepsilon\right)' \right)^2 \right] dx_2 \right\}. \tag{25}$$

Now, let us estimate each term in the right-hand side of (25) by using the following two inequalities:

$$u^2(0) \le 2\varepsilon^{-1} \int\limits_0^\varepsilon u^2(t)\,dt + 2\varepsilon \int\limits_0^\varepsilon \left(u'(t)\right)^2 dt, \tag{26}$$

$$\left(u(0) - u(\varepsilon)\right)^2 \le \varepsilon \int\limits_0^\varepsilon \left(u'\right)^2(t)\,dt \tag{27}$$

that hold for every $u \in H^1\left([0,\varepsilon]\right)$.

By adapting (26) to our case, we obtain

$$u_\varepsilon^2\left(\varepsilon\left(j+b_1+\frac{h_1}{2}\right),x_2\right) \le 2\varepsilon^{-1}\int\limits_{\varepsilon\left(j+b_1-\frac{h_1}{2}\right)}^{\varepsilon\left(j+b_1+\frac{h_1}{2}\right)} u_\varepsilon^2(x)\,dx_1, +2\varepsilon\int\limits_{\varepsilon\left(j+b_1-\frac{h_1}{2}\right)}^{\varepsilon\left(j+b_1+\frac{h_1}{2}\right)} \left(\partial_{x_1}u_\varepsilon(x)\right)^2 dx_1,$$

and integrating over $(-d_1,-\varepsilon)$, we have

$$\varepsilon\int\limits_{-d_1}^{-\varepsilon}\left(B_j^\varepsilon\right)^2 dx_2 \le 2\left\{\|u_\varepsilon\|^2_{L^2\left(\widetilde{G}_j^{(1)}(\varepsilon)\right)}+\varepsilon^2\|\partial_{x_1}u_\varepsilon\|^2_{L^2\left(\widetilde{G}_j^{(1)}(\varepsilon)\right)}\right\} \le c_1\|u_\varepsilon\|^2_{H^1\left(G_j^{(1)}(\varepsilon)\right)}, \tag{28}$$

where $\widetilde{G}_j^{(1)}(\varepsilon)=G_j^{(1)}(\varepsilon)\cap\{x: -d_1<x_2<-\varepsilon\}$.

Moreover,

$$\int\limits_{-d_1}^{-\varepsilon}\left(S_j^\varepsilon\right)^2 dx_2 =$$

$$=\int\limits_{-d_1}^{-\varepsilon}\left\{\frac{1}{\varepsilon(1-h_1)}\left[u_\varepsilon\left(\varepsilon\left(j+1+b_1-\frac{h_1}{2}\right),x_2\right)-u_\varepsilon\left(\varepsilon\left(j+b_1+\frac{h_1}{2}\right),x_2\right)\right]\right\}^2 dx_2 \le$$

$$\le c_2\varepsilon^{-2}\int\limits_{-d_1}^{-\varepsilon}\left[u_\varepsilon\left(\varepsilon\left(j+1+b_1-\frac{h_1}{2}\right),x_2\right)-u_\varepsilon\left(\varepsilon\left(j+b_1-\frac{h_1}{2}\right),x_2\right)+\right.$$

$$\left.+\,u_\varepsilon\left(\varepsilon\left(j+b_1-\frac{h_1}{2}\right),x_2\right)-u_\varepsilon\left(\varepsilon\left(j+b_1+\frac{h_1}{2}\right),x_2\right)\right]^2 dx_2 =$$

$$= c_2\varepsilon^{-2}\int\limits_{-d_1}^{-\varepsilon}\left[\varepsilon\mathbf{U}_\varepsilon\left(\varepsilon\left(j+b_1-\frac{h_1}{2}\right),x_2\right)+\right.$$

$$\left.+\,u_\varepsilon\left(\varepsilon\left(j+b_1-\frac{h_1}{2}\right),x_2\right)-u_\varepsilon\left(\varepsilon\left(j+b_1+\frac{h_1}{2}\right),x_2\right)\right]^2 dx_2 \le$$

$$\le 2c_2\int\limits_{-d_1}^{-\varepsilon}\left[\mathbf{U}_\varepsilon\left(\varepsilon\left(j+b_1-\frac{h_1}{2}\right),x_2\right)\right]^2 dx_2+$$

$$+\,2c_2\varepsilon^{-2}\int\limits_{-d_1}^{-\varepsilon}\left[u_\varepsilon\left(\varepsilon\left(j+b_1+\frac{h_1}{2}\right),x_2\right)-u_\varepsilon\left(\varepsilon\left(j+b_1-\frac{h_1}{2}\right),x_2\right)\right]^2 dx_2.$$

By (26) and (27), we have

$$\int\limits_{-d_1}^{-\varepsilon} \left(S_j^\varepsilon\right)^2 dx_2 \le 4c_2\varepsilon^{-1} \|\mathbf{U}_\varepsilon\|^2_{L^2\left(\widetilde{G}_j^{(1)}(\varepsilon)\right)} + 4c_2\varepsilon \|\partial_{x_1}\mathbf{U}_\varepsilon\|^2_{L^2\left(\widetilde{G}_j^{(1)}(\varepsilon)\right)} +$$

$$+ 2c_2\varepsilon^{-1}M_0 \int\limits_{-d_1}^{-\varepsilon} \int\limits_{\varepsilon\left(j+b_1-\frac{h_1}{2}\right)}^{\varepsilon\left(j+b_1+\frac{h_1}{2}\right)} \left(\partial_{x_1}u_\varepsilon(x)\right)^2 dx_1 dx_2 \le$$

$$\le c_4\left(\varepsilon^{-1}\|\mathbf{U}_\varepsilon\|^2_{L^2\left(\widetilde{G}_j^{(1)}(\varepsilon)\right)} + \varepsilon\|\partial_{x_1}\mathbf{U}_\varepsilon\|^2_{L^2\left(\widetilde{G}_j^{(1)}(\varepsilon)\right)} + \varepsilon^{-1}\|\partial_{x_1}u_\varepsilon\|^2_{L^2\left(\widetilde{G}_j^{(1)}(\varepsilon)\right)}\right). \tag{29}$$

Thus

$$\varepsilon^3 \int\limits_{-d_1}^{-\varepsilon} \left(S_j^\varepsilon\right)^2 dx_2 \le \varepsilon \int\limits_{-d_1}^{-\varepsilon} \left(S_j^\varepsilon\right)^2 dx_2 \le$$

$$\le c_4\left(\|\mathbf{U}_\varepsilon\|^2_{L^2\left(\widetilde{G}_j^{(1)}(\varepsilon)\right)} + \varepsilon^2\|\partial_{x_1}\mathbf{U}_\varepsilon\|^2_{L^2\left(\widetilde{G}_j^{(1)}(\varepsilon)\right)} + \|\partial_{x_1}u_\varepsilon\|^2_{L^2\left(\widetilde{G}_j^{(1)}(\varepsilon)\right)}\right) \le$$

$$\le c_5\left(\|\mathbf{U}_\varepsilon\|^2_{H^1\left(G_j^{(1)}(\varepsilon)\right)} + \|u_\varepsilon\|^2_{H^1\left(G_j^{(1)}(\varepsilon)\right)}\right). \tag{30}$$

Now we are going to estimate the other terms in (25),

$$\int\limits_{-d_1}^{-\varepsilon} \left(\left(B_j^\varepsilon(x_2)\right)'\right)^2 dx_2 =$$

$$= \int\limits_{-d_1}^{-\varepsilon} \left[\varepsilon\frac{h_1'(x_2)}{2}\,\partial_{x_1}u_\varepsilon\Big(\varepsilon\Big(j+b_1+\frac{h_1}{2}\Big),x_2\Big) + \partial_{x_2}u_\varepsilon\Big(\varepsilon\Big(j+b_1+\frac{h_1}{2}\Big),x_2\Big)\right]^2 dx_2 \le$$

$$\le \frac{M_0^2}{2}\,\varepsilon^2 \int\limits_{-d_1}^{-\varepsilon} \left[\partial_{x_1}u_\varepsilon\Big(\varepsilon\Big(j+b_1+\frac{h_1}{2}\Big),x_2\Big)\right]^2 dx_2 +$$

$$+ 2\int\limits_{-d_1}^{-\varepsilon} \left(\partial_{x_2}u_\varepsilon\Big(\varepsilon\Big(j+b_1+\frac{h_1}{2}\Big),x_2\Big)\right)^2 dx_2. \tag{31}$$

If we apply to the last two integrals in (31) the same calculation of (28) we obtain

$$\varepsilon \int\limits_{-d_1}^{-\varepsilon} \Big(\big(B_j^{\varepsilon}(x_2)\big)' \Big)^2 dx_2 \leq c_7 \bigg(\varepsilon^2 \left\| \partial_{x_1} u_\varepsilon \right\|^2_{L^2\left(\widetilde{G}_j^{(1)}(\varepsilon)\right)} + \varepsilon^4 \left\| \partial^2_{x_1} u_\varepsilon \right\|^2_{L^2\left(\widetilde{G}_j^{(1)}(\varepsilon)\right)} +$$

$$+ \left\| \partial_{x_2} u_\varepsilon \right\|^2_{L^2\left(\widetilde{G}_j^{(1)}(\varepsilon)\right)} + \varepsilon^2 \left\| \partial^2_{x_1,x_2} u_\varepsilon \right\|^2_{L^2\left(\widetilde{G}_j^{(1)}(\varepsilon)\right)} \bigg) \leq$$

$$\leq c_7 \bigg(\left\| \nabla u_\varepsilon \right\|^2_{L^2\left(\widetilde{G}_j^{(1)}(\varepsilon)\right)} + \varepsilon^2 \left\| u_\varepsilon \right\|^2_{H^2\left(\widetilde{G}_j^{(1)}(\varepsilon)\right)} \bigg).$$

To estimate the norm $\| u_\varepsilon \|^2_{H^2\left(\widetilde{G}_j^{(1)}(\varepsilon)\right)}$, we use the second energy inequality [5] with the smooth cut-off function

$$\chi_\varepsilon(x_2) = \begin{cases} 0, & x_2 \geq -\dfrac{\varepsilon}{2}, \\ 1, & x_2 \leq -\varepsilon. \end{cases}$$

We obtain that $\varepsilon^2 \| u_\varepsilon \|^2_{H^2\left(\widetilde{G}_j^{(1)}(\varepsilon)\right)} \leq c \left(\| u_\varepsilon \|^2_{H^1\left(G_j^{(1)}(\varepsilon)\right)} + \| f_\varepsilon \|^2_{L^2\left(G_j^{(1)}(\varepsilon)\right)} \right)$. So,

$$\varepsilon \int\limits_{-d_1}^{-\varepsilon} \Big(\big(B_j^{\varepsilon}(x_2)\big)' \Big)^2 dx_2 \leq c_8 \left(\| u_\varepsilon \|^2_{H^1\left(G_j^{(1)}(\varepsilon)\right)} + \| f_\varepsilon \|^2_{L^2\left(G_j^{(1)}(\varepsilon)\right)} \right). \tag{32}$$

Now, let us estimate the integral of square of the derivative,

$$\big(S_j^\varepsilon(x_2)\big)' = \frac{h_1'(x_2)}{1-h_1(x_2)}\, S_j^\varepsilon(x_2) + \frac{1}{\varepsilon\,(1-h_1(x_2))} \bigg\{ \varepsilon\, \partial_{x_2} \mathbf{U}_\varepsilon\Big(\varepsilon\Big(j + b_1 - \frac{h_1}{2}\Big), x_2\Big) +$$

$$+ \bigg[\partial_{x_2} u_\varepsilon\Big(\varepsilon\Big(j + b_1 - \frac{h_1}{2}\Big), x_2\Big) - \partial_{x_2} u_\varepsilon\Big(\varepsilon\Big(j + b_1 + \frac{h_1}{2}\Big), x_2\Big) \bigg] \bigg\} -$$

$$- \frac{h_1'(x_2)}{2\,(1-h_1(x_2))} \bigg[\partial_{x_1} u_\varepsilon\Big(\varepsilon\Big(j + 1 + b_1 - \frac{h_1}{2}\Big), x_2\Big) + \partial_{x_1} u_\varepsilon\Big(\varepsilon\Big(j + b_1 + \frac{h_1}{2}\Big), x_2\Big) \bigg].$$

Taking into account the properties of h_1, its derivative (see (1)), estimate (29) and applying (26) and (27), we deduce

$$\int_{-d_1}^{-\varepsilon} \left((S_j^\varepsilon)' \right)^2 dx_2 \le$$

$$\le c_9 \Big[c_4 \left(\varepsilon^{-1} \|\mathbf{U}_\varepsilon\|^2_{L^2\left(\widetilde{G}_j^{(1)}(\varepsilon)\right)} + \varepsilon \|\partial_{x_1} \mathbf{U}_\varepsilon\|^2_{L^2\left(\widetilde{G}_j^{(1)}(\varepsilon)\right)} + \varepsilon^{-1} \|\partial_{x_1} u_\varepsilon\|^2_{L^2\left(\widetilde{G}_j^{(1)}(\varepsilon)\right)} \right) +$$

$$+ c_{10} \left(\varepsilon^{-1} \|\partial_{x_2} \mathbf{U}_\varepsilon\|^2_{L^2\left(\widetilde{G}_j^{(1)}(\varepsilon)\right)} + \varepsilon \left\|\partial^2_{x_1,x_2} \mathbf{U}_\varepsilon\right\|^2_{L^2\left(\widetilde{G}_j^{(1)}(\varepsilon)\right)} \right) +$$

$$+ c_{11} \varepsilon^{-1} \left\|\partial^2_{x_1,x_2} u_\varepsilon\right\|^2_{L^2\left(\widetilde{G}_j^{(1)}(\varepsilon)\right)} + c_{12} \left(\varepsilon^{-1} \|\partial_{x_1} u_\varepsilon\|^2_{L^2\left(\widetilde{G}_{j+1}^{(1)}(\varepsilon)\right)} + \varepsilon \left\|\partial^2_{x_1} u_\varepsilon\right\|^2_{L^2\left(\widetilde{G}_{j+1}^{(1)}(\varepsilon)\right)} \right) +$$

$$+ c_{13} \left(\varepsilon^{-1} \|\partial_{x_1} u_\varepsilon\|^2_{L^2\left(\widetilde{G}_j^{(1)}(\varepsilon)\right)} + \varepsilon \left\|\partial^2_{x_1} u_\varepsilon\right\|^2_{L^2\left(\widetilde{G}_j^{(1)}(\varepsilon)\right)} \right) \Big] \le$$

$$\le c_{14} \left(\varepsilon^{-1} \|\mathbf{U}_\varepsilon\|^2_{H^1\left(\widetilde{G}_j^{(1)}(\varepsilon)\right)} + \varepsilon \|\nabla \mathbf{U}_\varepsilon\|^2_{L^2\left(\widetilde{G}_j^{(1)}(\varepsilon)\right)} + \varepsilon \|\mathbf{U}_\varepsilon\|^2_{H^2\left(\widetilde{G}_j^{(1)}(\varepsilon)\right)} + \right.$$

$$\left. + \varepsilon^{-1} \|u_\varepsilon\|^2_{H^2\left(\widetilde{G}_j^{(1)}(\varepsilon) \cup \widetilde{G}_{j+1}^{(1)}(\varepsilon)\right)} + \varepsilon^{-1} \|\nabla u_\varepsilon\|^2_{L^2\left(\widetilde{G}_j^{(1)}(\varepsilon) \cup \widetilde{G}_{j+1}^{(1)}(\varepsilon)\right)} \right).$$

Again applying the second energy inequality as above, we have

$$\varepsilon^3 \int_{-d_1}^{-\varepsilon} \left((S_j^\varepsilon)' \right)^2 \le c_{15} \left(\varepsilon^2 \|\mathbf{U}_\varepsilon\|^2_{H^1\left(G_j^{(1)}(\varepsilon)\right)} + \varepsilon^4 \|\mathbf{U}_\varepsilon\|^2_{H^1\left(\widetilde{G}_j^{(1)}(\varepsilon)\right)} + \varepsilon^2 \|\mathbf{F}_\varepsilon\|^2_{L^2\left(G_j^{(1)}(\varepsilon)\right)} + \right.$$

$$\left. + \|u_\varepsilon\|^2_{H^1\left(G_j^{(1)}(\varepsilon) \cup G_{j+1}^{(1)}(\varepsilon)\right)} + \|f_\varepsilon\|^2_{L^2\left(G_j^{(1)}(\varepsilon) \cup G_{j+1}^{(1)}(\varepsilon)\right)} \right). \tag{33}$$

Thus, by (28), (30), (32) and (33), the right-hand side of (25) is estimated in the following way:

$$\|\widehat{P}_\varepsilon^{(1)}(u_\varepsilon)\|^2_{H^1\left(\widetilde{Q}_j^{(1)}(\varepsilon)\right)} \le c_{16} \left(\|u_\varepsilon\|^2_{H^1\left(G_j^{(1)}(\varepsilon)\right)} + \|\mathbf{U}_\varepsilon\|^2_{H^1\left(G_j^{(1)}(\varepsilon)\right)} + \|f_\varepsilon\|^2_{L^2\left(G_j^{(1)}(\varepsilon)\right)} + \right.$$

$$\left. + \varepsilon^2 \|\mathbf{F}_\varepsilon\|^2_{L^2\left(G_j^{(1)}(\varepsilon)\right)} + \|u_\varepsilon\|^2_{H^1\left(G_j^{(1)}(\varepsilon) \cup G_{j+1}^{(1)}(\varepsilon)\right)} + \|f_\varepsilon\|^2_{L^2\left(G_j^{(1)}(\varepsilon) \cup G_{j+1}^{(1)}(\varepsilon)\right)} \right). \tag{34}$$

Summing (33) over j from -1 to N, using Lemma 1 and (3), we get

$$\|\widehat{P}_\varepsilon^{(1)}(u_\varepsilon)\|^2_{H^1\left(G^{(1)}(\varepsilon) \cup \widetilde{Q}^{(1)}(\varepsilon)\right)} \le c_{17} \left(\|f_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + \|\mathbf{F}_\varepsilon\|^2_{L_2(\Omega_\varepsilon)} \right). \tag{35}$$

Now it remains to extend $\widehat{P}_\varepsilon^{(1)}(u_\varepsilon)$ to

$$T_j^{(1)}(\varepsilon) = \Big\{x: \ \ x_2 \in (-\varepsilon, 0)\,, \ \ x_1 \in \big(\varepsilon(j + b_1 + 2^{-1}h_1(x_2)), \varepsilon(j + 1 + b_1 - 2^{-1}h_1(x_2))\big)\Big\},$$

$$j = -1, 0, 1, \ldots, N.$$

Since the domains $T_j^{(1)}(\varepsilon)$, $j = -1, 0, 1, .., N$, are equal (each of this domain can be obtained from $T_0^{(1)}(\varepsilon)$ by a parallel shift along the axis Ox_1), we use results about extension operators in perforated domains [6]. It follows from these results that there exist an extension operator $\mathfrak{P}_\varepsilon^{(1)} : H^1\big(G^{(1)}(\varepsilon) \cup \widetilde{Q}^{(1)}(\varepsilon)\big) \mapsto H^1(\Omega_1)$, uniformly bounded in ε.

Thus, the extension operator $\mathbf{P}_\varepsilon^{(1)} := \mathfrak{P}_\varepsilon^{(1)} \circ \widehat{P}_\varepsilon^{(1)}$ is constructed and it satisfies the uniform estimate (19). Similarly we can construct the operator $\mathbf{P}_\varepsilon^{(2)} : H^1(\Omega_0 \cup G^{(2)}(\varepsilon)) \mapsto H^1(\Omega_2)$ which also satisfies (19).

The theorem is proved.

4. Convergence theorem. To prove this theorem we should pass to the limit in the integral identity (4) as $\varepsilon \to 0$. For this we will use identity (9), the extension operators constructed in Section 3, and the following characteristic functions

$$\chi_\varepsilon^{(i)}(x) := \chi^{(i)}\left(\frac{x_1}{\varepsilon}, x_2\right) = \begin{cases} 0, & x \in \Omega_0, \\ 1, & x \in G^{(i)}(\varepsilon), \\ 0, & x \in D_i \setminus G^{(i)}(\varepsilon), \end{cases} \qquad i = 1, 2.$$

We can assume that these functions are ε-periodic with respect to x_1. Similarly as in Section 4 [1], we can prove that $\chi_\varepsilon^{(i)} \to h_i$ weakly in $L_2(D_i)$ as $\varepsilon \to 0$.

Theorem 2. *Let u_ε be a weak solution to problem (2). Then*

$$(u_\varepsilon)|_{\Omega_0} \to v_0^+, \qquad \left(\mathbf{P}_\varepsilon^{(1)} u_\varepsilon\right)\Big|_{D_1} \to v_0^{1,-}, \qquad \left(\mathbf{P}_\varepsilon^{(2)} u_\varepsilon\right)\Big|_{D_2} \to v_0^{2,-}$$

weakly in $H^1(\Omega_0)$, $H^1(D_1)$, $H^1(D_2)$ respectively as $\varepsilon \to 0$, where the vector-valued function

$$\mathbf{v}_0(x) = \begin{cases} v_0^+(x), & x \in \Omega_0, \\ v_0^{1,-}(x), & x \in D_1, \\ v_0^{2,-}(x), & x \in D_2, \end{cases}$$

is the unique weak solution to the following problem:

$$
\begin{array}{rcll}
-\Delta_x\, v_0^+(x) & = & f_0(x), & x \in \Omega_0, \\
\partial_{x_1}^p v_0^+(0,x_2) & = & \partial_{x_1}^p v_0^+(a,x_2), \ \ p=0,1, & x_2 \in [0,\gamma(0)], \\
\partial_\nu v_0^+(x) & = & 0, & x \in \Gamma_\gamma, \\
-\partial_{x_2}\big(h_1(x_2)\,\partial_{x_2} v_0^{1,-}(x)\big) + 2k_1\, v_0^{1,-}(x) & = & h_1(x_2) f_0(x), & x \in D_1, \\
\partial_{x_2} v_0^{1,-}(x_1,-d_1) & = & 0, & x_1 \in I_0, \\
-\partial_{x_2}\big(h_2(x_2)\,\partial_{x_2} v_0^{2,-}(x)\big) + 2k_2\, v_0^{2,-}(x) & = & h_2(x_2) f_0(x), & x \in D_2, \\
\partial_{x_2} v_0^{2,-}(x_1,-d_2) & = & 0, & x_1 \in I_0, \\
v_0^+(x_1,0) \ = \ v_0^{1,-}(x_1,0) & = & v_0^{2,-}(x_1,0), & x_1 \in I_0, \\
\partial_{x_2} v_0^+(x_1,0) & = & h_1(0)\partial_{x_2} v_0^{1,-}(x_1,0) + & \\
 & + & h_2(0)\partial_{x_2} v_0^{2,-}(x_1,0), & x_1 \in I_0,
\end{array}
\tag{36}
$$

where $\Gamma_\gamma = \{x : \ x_2 = \gamma(x_1),\ x_1 \in I_0\}$.

Proof. With the help of (9), the extension operators $\mathbf{P}_\varepsilon^{(i)}$ and the functions $\chi_\varepsilon^{(i)},\ i=1,2$, we rewrite identity (4) in the following way:

$$
\int\limits_{\Omega_0} \nabla u_\varepsilon \cdot \nabla\varphi\, dx + \sum_{i=1}^{2}\Bigg(\int\limits_{D_i} \chi_\varepsilon^{(i)}(x)\,\nabla\Big(\mathbf{P}_\varepsilon^{(i)} u_\varepsilon\Big)\cdot\nabla(\varphi)\, dx + \underbrace{\varepsilon k_i \int\limits_0^a \Big(\chi_\varepsilon^{(i)}\,\mathbf{P}_\varepsilon^{(i)} u_\varepsilon\,\varphi\Big)\Big|_{x_2=-d_i} dx_1}_{A_1} +
$$

$$
+\, 2k_i \int\limits_{D_i} \frac{\sqrt{1+\varepsilon^2 4^{-1}|h_i'(x_2)|^2}}{h_i(x_2)}\,\chi_\varepsilon^{(i)}(x)\,\Big(\mathbf{P}_\varepsilon^{(i)} u_\varepsilon\Big)(x)\,\varphi(x)\, dx\Bigg) =
$$

$$
= \underbrace{2\varepsilon \sum_{i=1}^{2} k_i \int\limits_{G^{(i)}(\varepsilon)} Y\Big(\frac{x_1}{\varepsilon}\Big)\frac{\sqrt{1+\varepsilon^2 4^{-1}|h_i'(x_2)|^2}}{h_i(x_2)}\,\partial_{x_1}\big(u_\varepsilon\,\varphi\big)\, dx}_{A_2} +
$$

$$
+ \int\limits_{\Omega_0} f_\varepsilon(x)\varphi(x)\, dx + \sum_{i=1}^{2}\int\limits_{D_i} \chi_\varepsilon^{(i)}(x) f_\varepsilon(x)\varphi(x)\, dx \quad \forall\, \varphi \in H^1_{\shortparallel,x_1}(\Omega_2),
\tag{37}
$$

where $H^1_{\shortparallel,x_1}(\Omega_2) = \{\varphi \in H^1(\Omega_2) : \varphi(0,x_2) = \varphi(a,x_2) \text{ for } x_2 \in (0,\gamma(0))\}$.

Because of (3), (5), (19), the sequences

$$
\Big\{\chi_\varepsilon^{(i)}\,\partial_{x_j}\big(\mathbf{P}_\varepsilon^{(i)} u_\varepsilon\big)\Big\}_{\varepsilon>0}, \quad j = 1,\, 2,
\tag{38}
$$

are bounded in $L_2(D_i),\ i = 1,\, 2$. Therefore, we can choose a subsequence of $\{\varepsilon\}$ (still denoted by $\{\varepsilon\}$) such that $\chi_\varepsilon^{(i)}\,\partial_{x_j}\big(\mathbf{P}_\varepsilon^{(i)}u_\varepsilon\big) \to \sigma_j^{(i)}$ weakly in $L_2(D_i),\ j = 1,\, 2,\ i = 1,\, 2,$ and

$$(u_\varepsilon)|_{\Omega_0} \to v_0^+, \qquad \Big(\mathbf{P}_\varepsilon^{(1)}u_\varepsilon\Big)\Big|_{D_1} \to v_0^{1,-}, \qquad \Big(\mathbf{P}_\varepsilon^{(2)}u_\varepsilon\Big)\Big|_{D_2} \to v_0^{2,-}$$

weakly in $H^1(\Omega_0),\ H^1(D_1),\ H^1(D_2)$ and strongly in $L^2(\Omega_0),\ L^2(D_1),\ L^2(D_2)$ respectively as $\varepsilon \to 0$. Since $(u_\varepsilon)|_{I_0} = (\mathbf{P}_\varepsilon^{(1)}u_\varepsilon)|_{I_0} = (\mathbf{P}_\varepsilon^{(2)}u_\varepsilon)|_{I_0}$, the traces of limit functions are equal as well, i.e., $v_0^+(x_1,0) = v_0^{1,-}(x_1,0) = v_0^{2,-}(x_1,0),\ \ x_1 \in I_0$.

Obviously, the summands A_1 and A_2 in (37) vanishe as $\varepsilon \to 0$. Now, passing to the limit in (37) and taking (3), (38) into account, we obtain

$$\int\limits_{\Omega_0} \nabla v_0^+ \cdot \nabla\varphi\, dx + \sum_{i=1}^{2}\left(\int\limits_{D_i}\sum_{j=1}^{2}\sigma_j^{(i)}(x)\,\partial_{x_j}\varphi(x)\,dx \;+\; 2k_i\int\limits_{D_i} v_0^{i,-}\,\varphi\,dx\right) =$$

$$= \int\limits_{\Omega_0} f_0(x)\varphi(x)\,dx \;+\; \sum_{i=1}^{2}\int\limits_{D_i} h_i(x_2)\,f_0(x)\varphi(x)\,dx, \qquad \varphi \in H^1_{\shortparallel,x_1}(\Omega_2). \tag{39}$$

Next we should find $\sigma_j^{(i)},\ j = 1,2,\ i = 1,2.$

In order to determine $\sigma_1^{(i)},\ i = 1,2,$ we consider the integral identity (4) with the following test functions :

$$\psi_1(x) = \varepsilon\begin{cases} 0, & x \in \Omega_0,\\ Y(x_1/\varepsilon)\,\phi_1(x), & x \in G^{(1)}(\varepsilon),\\ 0, & x \in G^{(2)}(\varepsilon),\end{cases} \qquad \psi_2(x) = \varepsilon\begin{cases} 0, & x \in \Omega_0,\\ 0, & x \in G^{(1)}(\varepsilon),\\ Y(x_1/\varepsilon)\,\phi_2(x), & x \in G^{(2)}(\varepsilon),\end{cases}$$

where ϕ_1 and ϕ_2 are arbitrary functions from $C_0^\infty(D_1)$ and $C_0^\infty(D_2)$ respectively. It is obvious that ψ_1 and ψ_2 belong to $H^1_{\shortparallel,x_1}(\Omega_\varepsilon)$. As a result, we get

$$\int\limits_{D_1}\chi_\varepsilon^{(1)}(x)\,\partial_{x_1}\mathbf{P}_\varepsilon^{(1)}(u_\varepsilon)\,\phi_1\,dx = \mathcal{O}(\varepsilon), \quad \int\limits_{D_2}\chi_\varepsilon^{(2)}(x)\,\partial_{x_1}\mathbf{P}_\varepsilon^{(2)}(u_\varepsilon)\,\phi_2\,dx = \mathcal{O}(\varepsilon), \quad \varepsilon \to 0,$$

whence $\sigma_1^{(1)} \equiv 0$ and $\sigma_1^{(2)} \equiv 0$.

Next let us define $\sigma_2^{(1)}$. Take any function $\phi \in C_0^\infty(D_1)$ and perform the following calculations:

$$\int\limits_{D_1} \chi_\varepsilon^{(1)}(x)\partial_{x_2}\big(\mathbf{P}_\varepsilon^{(1)}u_\varepsilon(x)\big)\,\phi(x)\,dx = \sum_{j=0}^{N-1}\int\limits_{G_j^{(1)}(\varepsilon)} \partial_{x_2}\big(u_\varepsilon(x)\big)\phi(x)\,dx =$$

$$= \sum_{j=0}^{N-1}\Bigg(\int\limits_{\Upsilon_j^{(1,\pm)}(\varepsilon)} u_\varepsilon\,\phi\,\alpha_2^{(1)}(x_2,\varepsilon)\,dl_x - \int\limits_{G_j^{(1)}(\varepsilon)} u_\varepsilon\,\partial_{x_1}\phi\,dx\Bigg) =$$

$$= -2^{-1}\varepsilon\int\limits_{-d_1}^{0} h_1'(x_2)\sum_{j=0}^{N-1}\big(u_\varepsilon\phi\big)\big|_{x_1=\varepsilon(j+b_1\pm h_1(x_2)/2)}\,dx_2 -$$

$$- \int\limits_{D_1}\chi_\varepsilon^{(1)}(x)\big(\mathbf{P}_\varepsilon^{(1)}u_\varepsilon\big)\partial_{x_2}\phi\,dx =: B_1(\varepsilon) + B_2(\varepsilon). \tag{40}$$

Here $\alpha_2^{(1)}(x_2,\varepsilon) = -\varepsilon h_1'(x_2)\Big(2\sqrt{1+\varepsilon^2 4^{-1}(h_1'(x_2))^2}\Big)^{-1}$ is the second coordinate of the outward unit normal $\nu_\pm^{(1)}$ (see (8) to the lateral surfaces $\Upsilon_j^{(1,\pm)}(\varepsilon)$ of the thin rod $G_j^{(1)}(\varepsilon)$. It is easy to verify that

$$\lim_{\varepsilon\to 0} B_2(\varepsilon) = -\int\limits_{D_1} h_1(x_2) v_0^{1,-}(x)\partial_{x_2}\phi(x)\,dx. \tag{41}$$

To find the limit of $B_1(\varepsilon)$ we rewrite this value in the following way:

$$B_1(\varepsilon) = -2^{-1}\varepsilon\int\limits_{-d_1}^{0} h_1'(x_2)\left(\sum_{j=0}^{N-1}\int\limits_{\varepsilon(j+b_1-h_0(x_2)/2)}^{\varepsilon(j+b_1+h_0(x_2)/2)} \partial_{x_1}\big(u_\varepsilon\phi\big)\,dx_1\right)dx_2 -$$

$$- \varepsilon\int\limits_{-d_1}^{0} h_1'(x_2)\left(\sum_{j=0}^{N-1}\big((u_\varepsilon - v_0^{1,-})\,\phi\big)|_{x_1=\varepsilon(j+b_1-h_0(x_2)/2)}\right)dx_2 -$$

$$- \int\limits_{-d_1}^{0} h_1'(x_2)\left(\sum_{j=0}^{N-1}\big(v_0^{1,-}\,\phi\big)|_{x_1=\varepsilon(j+b_1-h_0(x_2)/2)}\,\big(\varepsilon(j+1)-\varepsilon j\big)\right)dx_2. \tag{42}$$

The first term in (42) is bounded by $\varepsilon\|u_\varepsilon\|_{H^1(G^{(1)}(\varepsilon))}\|\phi\|_{H^1(D_1)}$. Due to estimate (26), the second term in (42) is estimated by the value

$$c_1\,\Big(\big\|\mathbf{P}_\varepsilon^{(1)}u_\varepsilon - v_0^{1,-}\big\|_{L^2(G^{(1)}(\varepsilon))} + \varepsilon^2\big\|\partial_{x_1}(\mathbf{P}_\varepsilon^{(1)}u_\varepsilon - v_0^{1,-})\big\|_{L^2(G^{(1)}(\varepsilon))}\Big)\,\|\phi\|_{H^1(D_1)}. \tag{43}$$

Since for almost all points $x_2 \in (-d_1, 0)$ the function $v_0^{1,-} \in H^1(0,a)$, the inner sum in the third term in (42) is the Riemann sum for the integral $\int_0^a v_0^{1,-}\,\phi\,dx_1$. Then in view of Lebesgue's Theorem and Fubini's Theorem, the limit of the third term is equal to

$$-\int\limits_{D_1} h_1'(x_2)\,v_0^{1,-}(x)\phi(x)\,dx. \qquad (44)$$

Passing to the limit in (40) and taking (41) – (44) into account, we get

$$\sigma_2^{(1)}(x) = h_1(x_2)\,\partial_{x_2} v_0^{1,-}(x), \qquad x \in D_1.$$

Similarly we deduce that $\sigma_2^{(2)}(x) = h_2(x_2)\,\partial_{x_2} v_0^{2,-}(x),\ \ x \in D_2$.

Thus, we obtain that the vector-valued function $\mathbf{v}_0$ satisfies the following identity:

$$\int\limits_{\Omega_0} \nabla_x v_0^+ \cdot \nabla_x \varphi\,dx + \sum_{i=1}^{2} \int\limits_{D_i} \big(h_i(x_2)\,\partial_{x_2} v_0^{i,-}(x)\,\partial_{x_2}\varphi(x) \,+\, 2k_i\,v_0^{i,-}(x)\,\varphi(x)\big)\,dx =$$

$$= \int\limits_{\Omega_0} f_0(x)\varphi(x)\,dx \,+\, \sum_{i=1}^{2} \int\limits_{D_i} h_i(x_2)\,f_0(x)\varphi(x)\,dx \quad \forall\,\varphi \in H^1_{\shortparallel,x_1}(\Omega_2). \quad (45)$$

Identity (45) is the corresponding integral identity for problem (36) in the following anizotropic Sobolev vector-space:

$$\mathcal{H}_0 = \big\{\mathbf{u} = (u_0, u_1, u_2) \in \mathcal{V}_0 := L^2(\Omega_0) \times L^2(D_1) \times L^2(D_2) \;|$$

$$u_0 \in H^1(\Omega_0), \quad u_0(0, x_2) = u_0(a, x_2) \text{ for } x_2 \in (0, \gamma(0));$$

$$\exists\,\partial_{x_2} u_1 \in L^2(D_1); \quad \exists\,\partial_{x_2} u_2 \in L^2(D_2);$$

$$u_0(x_1, 0) = u_1(x_1, 0) = u_2(x_1, 0),\ \ x_1 \in I_0\big\}$$

with the scalar product

$$(\mathbf{u}, \mathbf{v})_{\mathcal{H}_0} = \int\limits_{\Omega_0} \nabla u_0 \cdot \nabla v_0\,dx \,+\, \sum_{i=1}^{2} \int\limits_{D_i} \Big(h_i(x_2)\partial_{x_2} u_i\,\partial_{x_2} v_i \,+\, 2k_i u_i v_i\Big)dx.$$

Obviously, the space $\mathcal{H}_0$ continuously embeds in $\mathcal{V}_0$.

By using standard Hilbert space methods, we can state that there exists a unique weak solution $v_0 \in \mathcal{H}$ to problem (36), which is called the limit problem for problem (2). It should be noted that in the rectangles D_1 and D_2 we have ordinary differential equations with respect to x_2 and there are no boundary conditions on the vertical sides of $D_i,\ i = 1, 2$.

Due to the uniqueness of the solution to problem (36), the above reasoning holds for any subsequence of $\{\varepsilon\}$ chosen at the beginning of the proof. Therefore, the theorem is proved.

Example. In the case $h_1 \equiv \text{const}$ and $h_2 \equiv \text{const}$, we can reduce problem (36) to some boundary-value problem in the junction's body. By solving these ordinary differential equations with regard to the Neumann conditions and the first transmission condition in I_0, we find

$$v_0^{i,-}(x) = -\frac{1}{\rho_i}\int\limits_{-d_i}^{x_2} \sinh\big(\rho_i(x_2-t)\big)\, f_0(x_1,t)\,dt +$$

$$+\frac{\cosh\big(\rho_i(x_2+d_i)\big)}{\cosh(\rho_i d_i)}\left(v_0^+(x_1,0) - \frac{1}{\rho_i}\int\limits_{-d_i}^{0}\sinh(\rho_i t)\, f_0(x_1,t)\,dt\right), \quad x\in D_i, \tag{46}$$

where $\rho_i = \sqrt{2k_i h_i^{-1}}$, $i=1,2$. Putting these functions in the second transmission condition, we obtain the following problem:

$$\begin{array}{rcll} -\Delta_x\, v_0^+(x) &=& f_0(x), & x\in\Omega_0,\\ \partial_{x_1}^p v_0^+(0,x_2) &=& \partial_{x_1}^p v_0^+(a,x_2),\ \ p=0,1, & x_2\in[0,\gamma(0)],\\ \partial_\nu v_0^+(x) &=& 0, & x\in\Gamma_\gamma,\\ \partial_{x_2} v_0^+(x_1,0) &=& \Big(\sum_{i=1}^2 h_i\rho_i\tanh(\rho_i d_i)\Big) v_0^+(x_1,0), + \widehat{f}_0(x_1), & x_1\in I_0, \end{array} \tag{47}$$

where

$$\widehat{f}_0(x_1) = -\sum_{i=1}^{2} h_i \int\limits_{-d_i}^{0}\big(\cosh(\rho_i t) + \tanh(\rho_i d_i)\sinh(\rho_i t)\big)\, f_0(x_1,t)\,dt.$$

Problem (47) is a classical boundary-value problem with the Robin condition on I_0. Obviously, it has a unique weak solution from $H^1(\Omega_0)$.

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