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ON STABILITY OF THE HILL'S EQUATION WITH DAMPING

ПРО СТІЙКІСТЬ РІВНЯННЯ ХІЛЛА ЗІ ЗГАСАННЯМ

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We consider the Hill's equation with damping describing the parametric oscillations of the torsional pendulum excited by means of varying the moment of inertia of the rotating body. Using the method of a small parameter we have calculated analytically a fundamental system of solutions of this equation in the form of power series in the excitation amplitude ε with accuracy $O(\varepsilon^2)$ and verified the conditions of its stability. In the first-order approximation in ε, we have proved that the resonance domain exists only if the excitation frequency Ω is sufficiently close to the double natural frequency of the pendulum, and the corresponding equation of the stability boundary has been obtained.

Розглядається рівняння Хілла зі згасанням, що описує параметричні коливання крутильного маятника, які збуджуються зміною моменту інерції тіла, що обертається. За допомогою методу малого параметра аналітичним шляхом отримано фундаментальну систему розв'язків цього рівняння у вигляді степеневих рядів відносно амплітуди збудження ε з точністю до $O(\varepsilon^2)$ та перевірено виконання умов його стійкості. У першому наближенні по ε доведено, що область резонансу існує лише в області частот збудження Ω, близьких до подвійної власної частоти маятника, і отримано рівняння межі області стійкості.

Introduction. We consider the second order linear differential equation of the form

$$\frac{d^2\theta}{dt^2} + \beta(t,\varepsilon)\frac{d\theta}{dt} + \kappa(t,\varepsilon)\theta(t) = 0, \qquad (1)$$

where $\beta(t,\varepsilon)$ and $\kappa(t,\varepsilon)$ are continuous periodic functions of time t with a period T, i. e., $\beta(t+T,\varepsilon) = \beta(t,\varepsilon)$, $\kappa(t+T,\varepsilon) = \kappa(t,\varepsilon)$ for all t and ε is a small parameter. Equations of this type describe dynamical systems with intrinsic periodicity and appear in many branches of science and engineering. A physical example which is considered here is the torsional oscillations of the body mounted on an elastic shaft and excited by means of alternating its moment of inertia.

In the case where $\beta(t,\varepsilon) = 0$, equation (1) reduces to the Hill's equation which has been the subject of many papers (see, for example, [1–3]). The case $\beta(t,\varepsilon) = \mathrm{const}$ was analyzed by P. Pedersen [4]. It was shown that in both cases, depending on the parameters of the system, there are values of the excitation frequency $\Omega = 2\pi/T$ and amplitude ε such that the solution $\theta(t)$ increases unboundedly as $t \to \infty$ and the motion of the system becomes unstable. This phenomenon is known as a parametric resonance.

The parametric resonance in linear oscillating systems has been studied quite well and different methods were developed [5]. The most general method is the classic Floquet method [6] which is based on a calculation of the monodromy matrix and an analysis of the behaviour of its eigenvalues. It was used for studying equation (1) and some more general systems of differential equations in [7, 8]. But this method requires a large number of numerical integrations and this limits its possibilities, especially, if coefficients of the equations depend on some parameters. The main aim of the present paper is to study the stability of equation (1) in the case of parametric oscillations of the torsional pendulum with damping and to determine analytically boundaries of the domains of instability in the space of the parameters. It should be noted that the stability analysis of differential equations with periodic coefficients is rather cumbersome but it can be successfully done with a modern computer software such as, for example, the computer algebra system *Mathematica* [9].

Criteria of the system stability. According to the general theory of linear differential equations with periodic coefficients (see, for example, [1]), behaviour of the solutions of equation (1) is determined by its characteristic multipliers ρ which are just the eigenvalues of the monodromy matrix $X(T)$ and, hence, are given by the characteristic equation

$$\det(X(T) - \rho I_2) = 0, \tag{2}$$

where I_2 is an 2×2 identity matrix. Here $X(t)$ is the principal fundamental matrix for the equation (1) which is defined as

$$X(t) = \begin{pmatrix} \theta_1(t) & \theta_2(t) \\ \theta_1'(t) & \theta_2'(t) \end{pmatrix},$$

where $\theta_1(t)$ and $\theta_2(t)$ are two linearly independent solutions of equation (1) satisfying the following initial conditions

$$\begin{aligned} &\theta_1(0) = 1, \ \ \theta_1'(0) = 0, \\ &\theta_2(0) = 0, \ \ \theta_2'(0) = 1. \end{aligned} \tag{3}$$

Hence, the characteristic equation (2) can be written in the form

$$\rho^2 - 2A\rho + B = 0, \tag{4}$$

where

$$A = \frac{1}{2}(\theta_1(T) + \theta_2'(T)),$$

$$B = \theta_1(T)\theta_2'(T) - \theta_1'(T)\theta_2(T).$$

Thus, the characteristic multipliers $\rho_{1,2}$ are functions of two parameters A and B and are given by

$$\rho_{1,2} = A \pm \sqrt{A^2 - B}. \tag{5}$$

In order to determine $\rho_{1,2}$ we should find two linearly independent solutions $\theta_1(t)$ and $\theta_2(t)$ of the equation (1) satisfying the initial conditions (3). Although these solutions are not found yet, we can characterize the properties of $\rho_{1,2}$ in terms of the parameters A and B.

a) If $0 \le A^2 < B$ then, according to (5), $\rho_{1,2}$ are a complex-conjugate pair of characteristic multipliers with absolute value $|\rho_{1,2}| = \sqrt{B}$ and can be represented as

$$\rho_{1,2} = \sqrt{B}\exp{(\pm i\,\frac{2\pi\sigma}{\Omega})},$$

where i is the imaginary unit and σ is a real number. The corresponding characteristic exponents $\mu_{1,2}$ are defined then as

$$\mu_{1,2} = \frac{1}{T}\ln\rho_{1,2} = \frac{\Omega}{4\pi}\,\ln B \pm i\,\sigma$$

and the general solution of equation (1) may be written in the form

$$\theta(t) = (C_1\mathrm{Re}(e^{(i\sigma t)}f(t)) + C_2\mathrm{Im}(e^{i\sigma t}f(t)))\;\exp\left(t\,\frac{\Omega}{4\pi}\,\ln B\right), \tag{6}$$

where $f(t)$ is a complex-valued periodic function with the period $T = \dfrac{2\pi}{\Omega}$ and C_1, C_2 are arbitrary constants. It is obvious now from (6) that in the case where $0 \le A^2 < B < 1$ the function $\theta(t) \to 0$ as $t \to \infty$ and the motion of the system is asymptotically stable. For $B = 1$ solution (6) is bounded and oscillatory, and the motion of the system is stable. Thus, the system becomes unstable only if $B > 1$.

b) In the case where $A^2 = B$ there is a single real characteristic multiplier $\rho_1 = A$. It can be regarded as the limit $\sigma \to 0$ in case a). Again, the system is unstable for $|A| > 1$ and asymptotically stable for $|A| < 1$. In the case where $A = 1$ and $A = -1$ there exists a periodic solution with periods T and $2T$, respectively. Besides, there may exist an additional solution growing linearly with $t \to \infty$ and the system will be unstable.

c) If $A^2 > B$ then, according to (5), the characteristic multipliers $\rho_{1,2}$ are different real numbers. They are both positive or negative if $B > 0$ and $A > 0$ or $A < 0$, respectively. In the case where $B < 0$, the characteristic multipliers $\rho_{1,2}$ have opposite signs. The general solution of equation (1) can be written in the form

$$\theta(t) = C_1 f_1(t)\exp\left(t\,\frac{\Omega}{2\pi}\,\ln|\rho_1|\right) + C_2 f_2(t)\exp\left(t\,\frac{\Omega}{2\pi}\,\ln|\rho_2|\right), \tag{7}$$

where $f_1(t)$, $f_2(t)$ are real-valued periodic functions with the periods $T = \dfrac{2\pi}{\Omega}$ or $T = \dfrac{4\pi}{\Omega}$ depending on the sign of the corresponding characteristic multiplier. Hence, the system will be unstable if at least one characteristic multiplier has the absolute value greater than 1.

As a result, we can conclude that the domain of asymptotic stability of the equation (1) is inside a triangle bounded by the lines $B = 1$, $B = -1 \pm 2A$ in the $A - B$ plane. The points that lie on the boundary of the triangle determine stable behaviour of its solutions, while the domain being outside the triangle is just the domain of instability.

The parameter B can be found without solving equation (1). Indeed, since the functions $\theta_1(t)$ and $\theta_2(t)$ are the solutions of equation (1), we can write

$$\theta_1''(t) + \beta(t,\varepsilon)\theta_1'(t) + \kappa(t,\varepsilon)\theta_1(t) = 0, \tag{8}$$

$$\theta_2''(t) + \beta(t,\varepsilon)\theta_2'(t) + \kappa(t,\varepsilon)\theta_2(t) = 0. \tag{9}$$

Multiplying equations (8), (9) by $(-\theta_2(t))$ and $\theta_1(t)$, respectively, and adding them we obtain the following relationship:

$$\theta_1(t)\theta_2''(t) - \theta_2(t)\theta_1''(t) = \frac{d}{dt}(\theta_1(t)\theta_2'(t) - \theta_2(t)\theta_1'(t)) =$$

$$= -\beta(t,\varepsilon)(\theta_1(t)\theta_2'(t) - \theta_2(t)\theta_1'(t)).$$

Hence, the function $y(t) = \theta_1(t)\theta_2'(t) - \theta_2(t)\theta_1'(t)$ satisfies the following differential equation:

$$y'(t) = -\beta(t,\varepsilon)\, y(t). \tag{10}$$

Using the initial conditions (3) we obtain the solution of equation (10) in the form

$$y(t) = \exp\left(-\int_0^t \beta(\tau,\varepsilon)d\tau\right).$$

Thus, the parameter B is determined only by the function $\beta(\tau,\varepsilon)$ and is given by

$$B = \exp\left(-\int_0^T \beta(\tau,\varepsilon)d\tau\right). \tag{11}$$

Now we can conclude that $B > 0$ for any $\beta(t,\varepsilon)$. Hence, we can formulate the following criteria for stability and instability of the system.

1. If the average value of the function $\beta(t,\varepsilon)$ is negative, i.e. $\int_0^T \beta(t,\varepsilon)dt < 0$, then $B > 1$ and the system is unstable.

2. If the average value of the function $\beta(t,\varepsilon)$ is equal to zero then $B = 1$ and the system is stable for $|A| \le 1$ and unstable for $|A| > 1$.

3. If the average value of the function $\beta(t,\varepsilon)$ is positive then $0 < B < 1$ and the system is asymptotically stable for $|A| < \frac{1}{2}(B+1)$, stable for $|A| = \frac{1}{2}(B+1)$, and unstable for $|A| > \frac{1}{2}(B+1)$.

Equation of motion of the system. Let us imagine that a disk having a moment of inertia J_0 is mounted on an elastic shaft and two point bodies of equal masses m are placed on its surface symmetrically with respect to the axis of the shaft and can move without friction along radius of the disk. The distance of each body from the axis of the shaft oscillates near the equilibrium value r_0 according to the law

$$r(t) = r_0 + \varepsilon\, \varphi(\Omega t),$$

where Ω and ε are the excitation frequency and amplitude respectively and $\varphi(t)$ is a continuous periodic function of time t with the period 2π. Hence, the moment of inertia of the system J varies as

$$J(t) = J_0 + 2m(r_0 + \varepsilon\, \varphi(\Omega t))^2. \tag{12}$$

Denoting the twisting angle of the disk by θ we can write the equation of motion of the system in the form

$$\frac{d}{dt}\left((t)\frac{d\theta}{dt}\right) = -\gamma\frac{d\theta}{dt} - c\theta, \tag{13}$$

where γ and c are the coefficient of viscous friction and the stiffness of the shaft respectively. Substituting (12) into equation (13) we see that it is just the equation (1) with the coefficients

$$\beta(t,\varepsilon) = \frac{\gamma + 4\varepsilon m\Omega(r_0 + \varepsilon\, \varphi(\Omega t))\varphi'(\Omega t)}{J_0 + 2m(r_0 + \varepsilon\, \varphi(\Omega t))^2}, \qquad \kappa(t,\varepsilon) = \frac{c}{J_0 + 2m(r_0 + \varepsilon\, \varphi(\Omega t))^2}, \tag{14}$$

that are periodic functions of the period $T = \dfrac{2\pi}{\Omega}$. Substituting $\beta(t,\varepsilon)$ in (11) we can represent the parameter B in the form

$$B = \exp\left(-\int_0^T \frac{\gamma + 4\varepsilon m\Omega(r_0 + \varepsilon\, \varphi(\Omega t))\varphi'(\Omega t)}{J_0 + 2m(r_0 + \varepsilon\, \varphi(\Omega t))^2} dt\right). \tag{15}$$

Since the function $\varphi(t)$ is supposed to be periodic with the period 2π, we have

$$\varphi(\Omega\, T) = \varphi\left(\Omega\, \frac{2\pi}{\Omega}\right) = \varphi(2\pi) = \varphi(0)$$

and

$$\int_0^T \frac{4\varepsilon m\Omega(r_0 + \varepsilon\, \varphi(\Omega t))\varphi'(\Omega t)}{J_0 + 2m(r_0 + \varepsilon\, \varphi(\Omega t))^2} dt = \int_0^T d(\ln(J_0 + 2m(r_0 + \varepsilon\, \varphi(\Omega t))^2)) = 0.$$

Thus, the relationship (15) can be rewritten as

$$B = \exp\left(-\int_0^T \frac{\gamma}{J_0 + 2m(r_0 + \varepsilon\, \varphi(\Omega t))^2}\, dt\right). \tag{16}$$

The denominator in the expression under the integral sign in (16) is a positive function of t. Hence, if $\gamma < 0$ then $B > 1$ and the system will be unstable for any values of other parameters. So, we'll suppose further that $\gamma \geq 0$ and, hence, $0 < B \leq 1$. This means that the system can be unstable only if $|A| > \frac{1}{2}(B+1)$. And the lines

$$B = -1 - 2A, \quad -1 \leq A < 0, \tag{17}$$

and

$$B = -1 + 2A, \quad 0 \leq A \leq 1, \tag{18}$$

are stability boundaries in the $A - B$ plane.

Calculation of the parameters A and B with the method of a small parameter. The coefficients $\beta(t,\varepsilon)$ and $k(t,\varepsilon)$ defined in (14) can be represented as series expansions in powers of ε

$$\beta(t,\varepsilon) = 2\beta_0 + \sum_{j=1}^{\infty} \beta_j(t)\varepsilon^j, \; \kappa(t,\varepsilon) = \omega_0^2 + \sum_{j=1}^{\infty} \kappa_j(t)\varepsilon^j, \tag{19}$$

where

$$\beta_0 = \frac{\gamma}{2p}, \; \omega_0^2 = \frac{c}{p}, \; p = J_0 + 2mr_0^2, \; \beta_1 = \frac{4mr_0}{p}(-2\beta_0\varphi(\Omega t) + \Omega\varphi'(\Omega t)),$$

$$\beta_2 = \frac{4m\varphi(\Omega t)}{p^2}(-\beta_0\varphi(\Omega t)(p - 8mr_0^2) + \Omega(p - 4mr_0^2)\varphi'(\Omega t)),$$

$$\kappa_1 = -\frac{4}{p}mr_0\omega_0^2\varphi(\Omega t), \qquad \kappa_2 = \frac{2m}{p^2}(-p + 8mr_0^2)\omega_0^2\varphi(\Omega t)^2, \ldots .$$

The series (19) converge for any t and sufficiently small ε and $\beta_j(t)$, $\kappa_j(t)$ are continuous functions. So, according to Poincare – Liapunov theorem [10 – 12], a general solution of equation (1) can be also represented as a power series

$$\theta(t) = \sum_{j=0}^{\infty} \theta_j(t)\varepsilon^j \tag{20}$$

that converges for any t and sufficiently small ε with $\theta_j(t)$ being continuous functions.

In order to obtain differential equations determining functions $\theta_j(t)$ let us substitute expansions (19), (20) into equation (1). Then, equating coefficients of ε^j, $j = 0, 1, 2, \ldots$, in the left- and the right-hand sides of the equation we obtain the following system of differential equations:

$$\begin{gathered} \theta_0''(t) + 2\beta_0\theta_0'(t) + \omega_0^2\theta_0(t) = 0, \\ \theta_j''(t) + 2\beta_0\theta_j'(t) + \omega_0^2\theta_j(t) = f_j(t), \; j = 1, 2, \ldots , \end{gathered} \tag{21}$$

where

$$f_j(t) = -\sum_{n=1}^{j} \left(\beta_n(t)\theta'_{j-n}(t) + \kappa_n(t)\theta_{j-n}(t)\right). \tag{22}$$

Two linearly independent solutions $\theta_0(t)$ of the first equation in (21) must satisfy the initial conditions (3). The corresponding functions are easily found and are given by

$$\theta_0(t) = e^{-\beta_0 t}\left(\cos(\omega t) + \frac{\beta_0}{\omega}\sin(\omega t)\right), \tag{23}$$

$$\theta_0(t) = \frac{1}{\omega}\, e^{-\beta_0 t}\, \sin(\omega t), \tag{24}$$

where $\omega = \sqrt{\omega_0^2 - \beta_0^2}$. Initial conditions for the functions $\theta_j,\ j = 1, 2, \ldots,$ can be written then as

$$\theta_j(0) = \theta'_j(0) = 0. \tag{25}$$

Solving the second equation in (21) with initial conditions (25) we obtain the following expression for the functions $\theta_j\ j = 1, 2, \ldots,$

$$\theta_j(t) = -\frac{1}{2i\,\omega}\, e^{-(\beta_0 + i\,\omega)t}\int_0^t j(\tau)\, e^{(\beta_0 + i\,\omega)\tau}\, d\tau + \\ + \frac{1}{2i\,\omega}\, e^{-(\beta_0 - i\,\omega)t}\int_0^t f_j(\tau)\, e^{(\beta_0 - i\,\omega)\tau}\, d\tau, \tag{26}$$

where the functions $f_j(t)$ are defined in (22). Using the recurrence relation (26) we can successively calculate the coefficients θ_j in the expansion (20). But, as j is growing the calculations become more and more cumbersome. So this method can be reasonably realized only with a computer software.

Using the system *Mathematica* we have done the above calculations in the case of $\varphi(t) = \cos t$ for the initial functions θ_0 given in (23), (24) with accuracy of ε^2. As a result we have found the parameter A as a power series in ε,

$$A = \exp\Big(-\frac{2\pi\beta_0}{\Omega}\Big)\left(\cos\Big(\frac{2\pi\omega}{\Omega}\Big) + \varepsilon^2\, \frac{\pi m}{p^2\omega\Omega(\Omega^2 - 4\omega^2)}\Big(-p(4\omega^2 - \Omega^2)\, \times \right. \\ \times \Big((\omega^2 - \beta_0^2)\sin\Big(\frac{2\pi\omega}{\Omega}\Big) + 2\omega\beta_0\cos\Big(\frac{2\pi\omega}{\Omega}\Big)\Big) + 8mr_0^2\Big((\omega^2(3\omega^2 - \Omega^2) - \\ \left. - \beta_0^2(6\omega^2 - \Omega^2) - \beta_0^4)\sin\Big(\frac{2\pi\omega}{\Omega}\Big) + 2\omega\beta_0(4\omega^2 - \Omega^2)\cos\Big(\frac{2\pi\omega}{\Omega}\Big)\Big)\Big)\right), \tag{27}$$

where the error term is $O(\varepsilon^3)$. Substituting the expansion (19) into (16) we obtain the parameter B in the form

$$B = \exp\Big(-\frac{4\pi\beta_0}{\Omega}\Big)\Big(1 + \varepsilon^2\,\frac{4\pi m\beta_0}{p^2\Omega}(p - 8mr_0^2)\Big). \tag{28}$$

It should be noticed that the series (27), (28) converge for any Ω and sufficiently small ε. And calculating the parameters A and B we can easily find the characteristic multipliers $\rho_{1,2}$ according to (5).

Determination of the domains of instability. It follows from (27), (28) that in the case of $\varepsilon = 0$, the parameters A and B take the form

$$A = \exp\Big(-\frac{2\pi\beta_0}{\Omega}\Big)\cos\Big(\frac{2\pi\omega}{\Omega}\Big), \qquad B = \exp\Big(-\frac{4\pi\beta_0}{\Omega}\Big).$$

Hence,

$$B + 1 \pm 2A = 1 + \exp\Big(-\frac{4\pi\beta_0}{\Omega}\Big) \pm 2\exp\Big(-\frac{2\pi\beta_0}{\Omega}\Big)\cos\Big(\frac{2\pi\omega}{\Omega}\Big) \geq \Big(1 - \exp\Big(-\frac{2\pi\beta_0}{\Omega}\Big)\Big)^2 > 0$$

for any Ω and $\beta_0 > 0$. This means that the corresponding point (A, B) in the $A - B$ plane belongs to the domain of asymptotic stability of the system. This has been expected because for $\varepsilon = 0$ equation (1) reduces to the damped oscillator equation whose general solution is well-known. Since the parameters A and B are continuous functions of ε, the point (A, B) will belong to the asymptotic stability domain for sufficiently small $\varepsilon > 0$ and $\beta_0 = \mathrm{const} > 0$ as well. Only in the case of $\beta_0 = 0$, $\varepsilon = 0$, the point (A, B) belongs to the stability boundaries (17), (18) if

$$\Omega = \Omega_0 = \frac{2\omega_0}{n}, \qquad n = 1, 2, 3, \ldots\,. \tag{29}$$

Hence, the domains of instability in the space of the parameters $(\Omega, \beta_0, \varepsilon)$ can exist only in a vicinity of the points (29). The boundary of every such domain is some surface which degenerates into a point as $\varepsilon \to 0$. Thus, considering these domains we can represent $\Omega = \Omega(\varepsilon)$, $\beta_0 = \beta_0(\varepsilon)$ for sufficiently small ε as the power series

$$\begin{aligned} \Omega &= \Omega_0 + \Omega_1\varepsilon + \Omega_2\varepsilon^2 + \ldots\,, \\ \beta_0 &= \beta_{01}\varepsilon + \beta_{02}\varepsilon^2 + \ldots\,. \end{aligned} \tag{30}$$

Substituting (30) into (27), (28) and expanding their right-hand sides in powers of ε we obtain the parameters A and B in the form

$$
\begin{aligned}
A = {} & \cos\Big(\frac{2\pi\omega_0}{\Omega_0}\Big) - \frac{2\pi\varepsilon}{\Omega_0^2}\Big(\beta_{01}\Omega_0\cos\Big(\frac{2\pi\omega_0}{\Omega_0}\Big) - \omega_0\Omega_1\sin\Big(\frac{2\pi\omega_0}{\Omega_0}\Big)\Big) + \\
& + \frac{\pi\varepsilon^2}{\omega_0\Omega_0^4}\Big(\beta_{01}^2\Omega_0^2\Big(2\pi\omega_0\cos\Big(\frac{2\pi\omega_0}{\Omega_0}\Big) + \Omega_0\sin\Big(\frac{2\pi\omega_0}{\Omega_0}\Big)\Big) - \\
& - 2\beta_{01}\omega_0\Omega_0\Omega_1\Big(2\pi\omega_0\sin\Big(\frac{2\pi\omega_0}{\Omega_0}\Big) - \Omega_0\cos\Big(\frac{2\pi\omega_0}{\Omega_0}\Big)\Big) - \\
& - 2\omega_0(\beta_{02}\Omega_0^3 + \pi\omega_0^2\Omega_1^2)\cos\Big(\frac{2\pi\omega_0}{\Omega_0}\Big)\Big) + \\
& + \frac{\pi\omega_0\varepsilon^2}{p^2\Omega_0^3(\Omega_0^2 - 4\omega_0^2)}\Big(4\omega_0^2(m\Omega_0^2(-p + 6mr_0^2) + 2p^2\Omega_1^2 - 2p^2\Omega_0\Omega_2) + \\
& + \Omega_0^2(m\Omega_0^2(p - 8mr_0^2) - 2p^2\Omega_1^2 + 2p^2\Omega_0\Omega_2)\Big)\sin\Big(\frac{2\pi\omega_0}{\Omega_0}\Big), \\
B = {} & 1 - \frac{4\pi\beta_{01}\varepsilon}{\Omega_0} + \frac{4\pi\varepsilon^2}{\Omega_0^2}(2\pi\beta_{01}^2 - \beta_{02}\Omega_0 + \beta_{01}\Omega_1).
\end{aligned}
\tag{31}
$$

Now one can easily see that, in fact, the relations (17), (18) can be fulfilled only in a vicinity of the points (29). Moreover, only one of them can be fulfilled in every point (29). Substituting (31) into (17), (18) we obtain, successively for $n = 1, 2, 3, 4, \ldots$, the following equations:

$$B + 1 + 2A = \frac{1}{4}\pi^2\varepsilon^2\Big(\frac{4\beta_{01}^2 + \Omega_1^2}{\omega_0^2} - \frac{4m^2r_0^2}{p^2}\Big) = 0, \tag{32}$$

$$B + 1 - 2A = \frac{4\pi^2\varepsilon^2}{\omega_0^2}(\beta_{01}^2 + \Omega_1^2) = 0, \tag{33}$$

$$B + 1 + 2A = \frac{9\pi^2\varepsilon^2}{4\omega_0^2}(4\beta_{01}^2 + 9\Omega_1^2) = 0, \tag{34}$$

$$B + 1 - 2A = \frac{16\pi^2\varepsilon^2}{\omega_0^2}(\beta_{01}^2 + 4\Omega_1^2) = 0, \tag{35}$$

. . . .

Equations (33) – (35) have the only solution $\beta_{01} = \Omega_1 = 0$. This means that the domains of instability in a vicinity of the points (29) for $n = 2, 3, 4, \ldots$ can be found only if we take into account the third and higher order terms in the expansion (20). Equation (32) shows that the

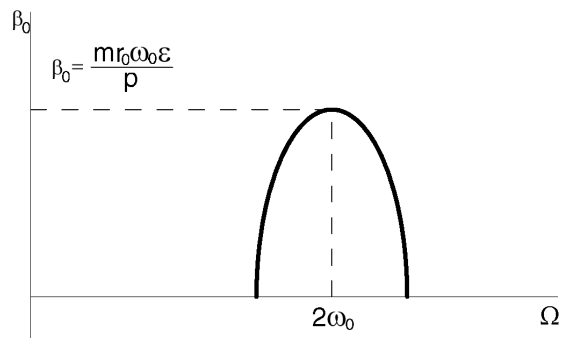

Fig. 1. The cross-section of the boundary surface by the plane $\beta_0 = \mathrm{const}$.

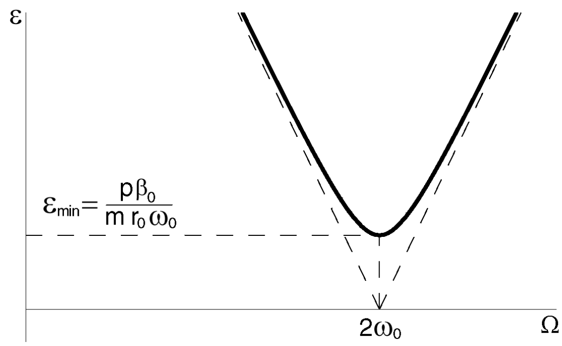

Fig. 2. The cross section of the boundary surface by the plane $\beta_0 = \mathrm{const}$.

domain of instability, where the inequality $B+1+2A < 0$ is fulfilled, exists in a vicinity on the point $\Omega_0 = 2\omega_0$. The boundary of this domain in the space of the parameters $(\Omega, \beta_0, \varepsilon)$ is given by the equation

$$4\beta_0^2 + (\Omega - 2\omega_0)^2 = \frac{4m^2 r_0^2 \omega_0^2}{p^2}\varepsilon^2, \tag{36}$$

where we have taken into account the relations $\beta_0 = \varepsilon\beta_{01}, \Omega = 2\omega_0 + \varepsilon\Omega_1$. Thus, we have found the stability boundary (36) in linear approximation in ε.

Equation (36) determines a cone in three-dimensional space $(\Omega, \beta_0, \varepsilon)$ and the system will be unstable if the point determined by these parameters lies inside the cone. The cross sections of the cone by the planes $\varepsilon = \mathrm{const}$ and $\beta_0 = \mathrm{const}$ are shown in Fig. 1, 2 respectively. The first graph shows that for any value of the excitation frequency Ω from the interval

$$|\Omega - 2\omega_0| \le \frac{2mr_0\omega_0\varepsilon}{p}$$

and amplitude ε there exists a maximal value of the damping coefficient β_0 for which the parametric resonance can still occur. On the other hand, if the coefficient β_0 is small enough and fixed then the parametric resonance can occur only if the excitation amplitude ε is greater then some threshold value. Recall that the equation for the instability boundary (36) has been obtained in the first approximation in the excitation amplitude ε. Taking into account the higher approximations we can notice that it is deformed with ε growth. Besides, the domains of instability can arise in a vicinity of other points (29).

Conclusion. In the present paper we have studied the parametric oscillations of the torsional pendulum with damping which are described by the second order differential equation with periodic coefficients. The excitation of the pendulum is realized by means of varying the moment of inertia of the rotating body. We have calculated analytically two linearly independent solutions of the equation of motion in the form of power series in the excitation amplitude ε with accuracy $O(\varepsilon^2)$ and verified the conditions of its stability. It has been shown that the domains of a parametric resonance can exist only in a vicinity of the points $\Omega = \dfrac{2\omega_0}{n}$ $n = 1, 2, 3, \ldots$, where Ω and ω_0 are the excitation frequency and natural frequency of the pendulum, respectively. In the first approximation in the excitation amplitude ε it has been proved that the resonance domain exists only in a vicinity of the point $\Omega = 2\omega_0$ and the corresponding equation of the stability boundary was obtained.

1. *Yakubovich V. A., Starzhinskii V. M.* Linear differential equations with periodic coefficients. — Moscow: Nauka, 1972. — 720 p. (in Russian).
2. *Grebenikov E. A., Prokopenya A. N.* Determination of the boundaries between the domains of stability and instability for the Hill's equation // Nonlinear Oscillations. — 2003. — **6**, № 1. — P. 42 – 51.
3. *Nayfeh A. H., Mook D. T.* Nonlinear oscillations. — New York: John Wiley, 1979. — 704 p.
4. *Pedersen P.* On stability diagrams for damped Hill equations // Quart. Appl. Math. — 1985. — **42**. — P. 177 – 195.
5. *Yakubovich V. A., Starzhinskii V. M.* Parametric resonance in linear systems. – Moscow: Nauka, 1987. — 274 p. (in Russian).
6. *Cesari L.* Asymptotic behaviour and stability problems in ordinary differential equations. — Second ed. — New York: Springer, 1962. — 271 p.
7. *Seyranian A. P., Solem F., and Pedersen P.* Multi-parameter linear periodic systems: sensitivity, analysis and applications // J. Sound and Vibr. — 2000. – **229**. — P. 89 – 111.
8. *Mailybaev A. A., Seyranian A. P.* Parametric resonance in systems with small dissipation // J. Appl. Math. and Mech. — 2001. — **65**, No. 5. — P. 755 – 767.
9. *Wolfram S.* The Mathematica book. – Cambridge: Univ. Press, 1999. — 1470 p.
10. *Erouguine N. P.* Linear systems of differential equations. — Minsk: Byelorus. Acad. Sci., 1963. — 272 p. (in Russian).
11. *Bellomo N., Preziosi L., and Romano A.* Mechanics and dynamical systems with Mathematica. — Boston: Birkhauser, 2000. — 417 p.
12. *Grimshaw R.* Nonlinear ordinary differential equations. – CRC Press, 2000. — 328 p.

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