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ON AVERAGING DIFFERENCE EQUATIONS ON AN INFINITE INTERVAL

ПРО УСЕРЕДНЕННЯ РІЗНИЦЕВИХ РІВНЯНЬ НА НЕСКІНЧЕННОМУ ПРОМІЖКУ

A. N. Stanzhytsky

Kyiv Nat. Taras Shevchenko Univ.
Volodymyrs'ka St., 64, 01033, Kyiv, Ukraine
e-mail: stom@mail.univ.kiev.ua

A. M. Ateiwi

Al-Hussein Bin Talal Univ.
P.O.Box(20), Ma'an, Jordan
e-mail: ateiwi@hotmail.com

The averaging method on the semi-axis is justified for systems of difference equations. A theorem on closeness of the corresponding solutions of the exact and averaged systems on the semi-axis is proved. This theorem is analogous to N. N. Bogoliubov's second theorem about an averaging method for systems of difference equations.

Наведено метод усереднення на півосі для систем різницевих рівнянь. Доведено теорему про близькість розв'язків точної та усередненої систем на півосі. Дана теорема є аналогом другої теореми М. М. Боголюбова методу усереднення для систем різницевих рівнянь.

We consider a system of difference equations of the form

$$x_{n+1} = x_n + \varepsilon f_n(x_n), \tag{1}$$

where $n = 0, 1, 2, \ldots, x \in \mathbb{R}^m, \varepsilon > 0$ is a small parameter.

Let us assume that there exists the mean value of the sequence f_n,

$$f(x) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N} f_n(x). \tag{2}$$

Then system (1) associates with the averaged system

$$y_{n+1} = y_n + \varepsilon f_n(y_n). \tag{3}$$

The questions of closeness of the corresponding solutions $x_n(x_0)$ $(x_0(x_0) = x_0)$ and $y_n(x_0)$ $(y_0(x_0) = x_0)$ of systems (1) and (3) for small values of the parameter ε are included in the results representing the essence of N. N. Bogoliubov's averaging method.

This method for systems of differential equations in a standard form was initially substantiated in the work [1] where the system

$$\frac{dx}{dt} = \varepsilon X(t, x)$$

is associated with the averaged system

$$\frac{dy}{dt} = \varepsilon X_0(y),$$

$$X_0(x) = \lim_{T \to \infty} \frac{1}{T} \int_0^T X(t, x) dt.$$

By employing a series of theorems related to the averaging method one can investigate the following two aspects:

1) establishing the closeness between the exact and averaged solutions on asymptotically finite time intervals $\left(\text{of order } \frac{1}{\varepsilon}\right)$;

2) establishing the correspondence between the exact and averaged solutions on the semi-axis.

Further, the averaging method was extended to systems of differential equations of a special form, systems of integro-differential equations, difference and partial differential equations, equations in a Banach space, stochastic equations. In this connection, the reader is referred to monographs [2, 3] which contain numerous references.

However, it is worth noticing that mainly the first N. N. Bogoliubov's theorem on averaging on a finite time interval was extended.

As for the averaging method on the semi-axis, N. N. Bogoliubov proved the corresponding theorem in the case where the averaged system has a quasistatic equilibrium position and closeness is established for this particular solution.

The works [4, 5] proved the Banfi – Filatov averaging theorem on the semi-axis in the case where the averaged system has an asymptotically stable solution different from the equilibrium position.

For systems of differential equations of the form (1), the averaging theorem on a finite interval was obtained in the communication [6] and on an infinite one, in the case of existence of an equilibrium position in the averaged system (3) in works [7, 8].

The purpose of this work is to obtain an analogue of the Banfi – Filatov theorem for systems of the form (1).

Consider system (1), where the function $f_n(x)$ is determined at $n = 0, 1, 2, \ldots, x \in D \subset \subset \mathbb{R}^m$, D being some domain.

The following theorem holds.

Theorem 1. *Let, in the domain D, the following conditions be fulfilled*:

1) $f_n(x)$ satisfies the Lipschits condition with respect to x,

$$|f_n(x) - f_n(x')| \le |x - x'| \qquad \forall x, x' \in D \tag{4}$$

($|x|$ *being the Euclidean norm*);

2) $f_n(x)$ is uniform with respect to $x \in D$ and $n = 0, 1, 2, \ldots$, that is, there exists the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=k}^{k+N} f_n(x) = f(x) \tag{5}$$

and the function $f(x)$ is bounded;

3) the solution $y_n(x_0)$ $(y_0(x_0) = x_0(x_0))$ of the averaged system is determined for all $n \in \mathbb{N}$ and lies in the domain D along with some δ-neighborhood;

4) $y_n(x_0)$ is uniformly asymptotically stable.

Then for any $0 < \eta < g$ there exists ε_0 such that for $\varepsilon < \varepsilon_0$, $n \in \mathbb{N}$, the inequality

$$(x_n(x_0) - y_n(x_0)) < \eta, \tag{6}$$

holds, where $x_n(x_0)$ is a solution of system (1) such that $x_0(x_0) = x_0$.

Proof. Note that it follows from the conditions of the theorem that $f(x)$ is a Lipschits function. Indeed, let x and x' be arbitrary points from the domain D. Then for $\mu > 0$ we can find N' and N'' such that for $N > \max\{N', N''\}$, the following inequalities hold:

$$\left| \frac{1}{N} \sum_{n=0}^{N} f_n(x) - f(x) \right| < \frac{\mu}{2},$$

$$\left| \frac{1}{N} \sum_{n=0}^{N} f_n(x') - f(x') \right| < \frac{\mu}{2}.$$

Then

$$|f(x) - f(x')| = \left| f(x) - \frac{1}{N} \sum_{n=0}^{N} f_n(x) + \frac{1}{N} \sum_{n=0}^{N} f_n(x) - \right.$$

$$\left. - \frac{1}{N} \sum_{n=0}^{N} f_n(x') + \frac{1}{N} \sum_{n=0}^{N} f_n(x') - f(x') \right| \le$$

$$\le \frac{\mu}{2} + \frac{\mu}{2} + \frac{1}{N} \sum_{n=0}^{N} |f_n(x') - f_n(x)| \le \mu + L|x - x'|$$

whence, by virtue of μ being arbitrary, we obtain

$$|f(x) - f(x')| \le L|x - x'|.$$

Thus, the conditions of the averaging theorem on a finite interval [6] are fulfilled. By this theorem for any $\rho > 0$, $L > 0$ and $\bar{n}_0 \in \mathbb{N}$ there exists ε_0 so that for $\varepsilon \subset \varepsilon_0$ solutions x_n and y_n of the exact and averaged systems such that $x_{\bar{n}_0} = y_{\bar{n}_0}$ satisfy the following inequality:

$$|x_n - y_n| < \rho/2, \tag{7}$$

with $n \in \mathbb{N}$ satisfying the condition $\bar{n}_0 \le n \le \bar{n}_0 + \dfrac{L}{\varepsilon}$.

Since the solution $y_n(x_0)$ is uniformly asymptotically stable, for any $\eta > 0$ and $\bar{n} \in \mathbb{N}$, we can find $\rho > 0$ (independent of $\bar{n}$ and ε) such that for any solution $\bar{y}_n$ of the averaged system satisfying, at the moment $\bar{n}$, the inequality

$$|y_{\bar{n}} - \bar{y}_{\bar{n}}| < \rho$$

at $n > \bar{n}$ the inequality

$$|y_n - \bar{y}_n| < \frac{\eta}{2} \tag{8}$$

holds for $n > \bar{n}$.

By selecting η sufficiently small we can get solutions of the averaged system satisfying inequality (8) lying in a δ-neighborhood of the solution $y_n(x_0)$ for $n \ge \bar{n}$ and belonging to the domain D together with some of their neighborhoods.

Moreover, by virtue of uniform asymptotic stability of the solution $y_n(x_0)$, we can find $\bar{L} > 0$ independent of $\bar{n}$ and ε (by virtue of the uniform asymptotic stability) such that if $n \ge \bar{n} + [\bar{L}]$ ($[\,\cdot\,]$ is the integer part of the number) the inequality

$$|y_n - \bar{y}_n| < \frac{\rho}{2} \tag{9}$$

will be fulfilled.

Using the values of η, ρ and $\bar{L}$ we select ε_0 such that if $\varepsilon < \varepsilon_0$ on the intervals of length $\left[\dfrac{\bar{L}}{\varepsilon}\right]$, the corresponding solutions of the exact and averaged system will satisfy inequality (7).

Due to the fact that ρ can be taken smaller than η, we can affirm that inequality (6), required in the theorem, holds for $n \le \left[\dfrac{\bar{L}}{\varepsilon}\right]$ $(\varepsilon \le \varepsilon_0)$.

We now subdivide the semi-axis with points of the form $k[\bar{L}]$, $k = 1, 2, \ldots$. Let $k_0[\bar{L}]$ be the largest point of the integral points which belongs to the segment $[0, [\bar{L}/\varepsilon]]$.

Let us consider the solution $\bar{y}_n$ of the averaged system such that $\bar{y}_{k_0[\bar{L}]} = x_{k_0[\bar{L}]}$. From the last equality and inequalities (7), (8) and (9), we have that, for $n \in [k_0[\bar{L}], (k_0+1)[\bar{L}]]$, the following estimates hold:

$$|x_n^{(x_0)} - \bar{y}_n| < \frac{\rho}{2}, \tag{10}$$

$$|y_n^{(x_0)} - \bar{y}_n| < \frac{\eta}{2}, \tag{11}$$

$$|y_{(k_0+1)[\bar{L}]}^{(x_0)} - \bar{y}_{(k_0+1)[\bar{L}]}| < \frac{\rho}{2}, \tag{12}$$

$k_0[\bar{L}]$ is taken as $\bar{n}$.

Hence, for $n \in (k_0[\bar{L}], (k_0+1)[\bar{L}])$, we have

$$|x_n - y_n| \leq |x_n - \bar{y}_n| + |y_n - \bar{y}_n| < \frac{\rho}{2} + \frac{\eta}{2} < \eta,$$

i.e., the required inequality (6) is holds. Moreover, if $n = (k_0+1)[\bar{L}]$, we have the inequality

$$|x_n(x_0) - y_n(x_0)| \leq |x_n(x_0) - \bar{y}_n| + |\bar{y}_n - y_n(x_0)| < \frac{\rho}{2} + \frac{\rho}{2} = \rho,$$

which means that the point $x_{(k_0+1)[\bar{L}]}(x_0)$ falls into the zone of asymptotic stability of the solution $y_n(x_0)$ of the averaged equation.

In what follows we consider y_n as a solution of the averaged equation such that $y_{(k_0+1)[\bar{L}]} = x_{(k_0+1)[\bar{L}]}(x_0)$. After some reasoning analogous to the preceding, we obtain that on the interval $((k_0+1)[\bar{L}], (k_0+2)[\bar{L}])$ the inequality

$$|x_n(x_0) - y_n(x_0)| < \eta$$

holds and, if $n = (k_0+2)[\bar{L}]$, the inequality

$$|x_n(x_0) - y_n(x_0)| < \rho$$

is true as well.

This means that the required inequality (6) holds for $[(k_0+1)[\bar{L}], (k_0+2)[\bar{L}]]$, the latter means that the point $x_{(k_0+2)[\bar{L}]}(x_0)$ lies in the zone of asymptotic stability of the solution $y_n(x_0)$.

Keeping on the above process we obtain that for any $m \in \mathbb{N}$ on the interval $[(k_0+m)[\bar{L}], (k_0+m+1)[\bar{L}]]$, inequality (6) hold and, if $n = (k_0+m+1)[\bar{L}]$, the inequality

$$|x_n(x_0) - y_n(x_0)| < \rho$$

holds too.

Since $m \in \mathbb{N}$ is arbitrary, the theorem is proved.

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