UDC 517.948

ON AN ANTIPERIODIC TYPE BOUNDARY-VALUE PROBLEM FOR FIRST ORDER NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS OF NON-VOLTERRA'S TYPE

ПРО ГРАНИЧНІ ЗАДАЧІ АНТИПЕРІОДИЧНОГО ТИПУ ДЛЯ НЕЛІНІЙНИХ ФУНКЦІОНАЛЬНО-ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ НЕВОЛЬТЕРРІВСЬКОГО ТИПУ

R. Hakl

Math. Inst. Czech Acad. Sci.
Žižkova 22, 616 62 Brno, Czech Republic
e-mail: hakl@ipm.cz

A. Lomtatidze

Masaryk Univ.
Janáčkovo nám. 2a, 662 95 Brno, Czech Republic
e-mail: bacho@math.muni.cz

J. Šremr

Math. Inst. Czech Acad. Sci.
Žižkova 22, 616 62 Brno, Czech Republic
e-mail: sremr@ipm.cz

Nonimprovable, in a certain sense, sufficient conditions are established for the solvability and unique solvability of the boundary-value problem

$$u'(t) = F(u)(t), \qquad u(a) + \lambda u(b) = h(u),$$

where $F : C([a,b];R) \to L([a,b];R)$ *is a continuous operator satisfying the Carathèodory conditions,* $h : C([a,b];R) \to R$ *is a continuous functional, and* $\lambda \in R_+$.

Отримано неполіпшувані у певному сенсі достатні умови для існування розв'язків або єдиного розв'язку граничної задачі

$$u'(t) = F(u)(t), \qquad u(a) + \lambda u(b) = h(u),$$

де $F : C([a,b];R) \to L([a,b];R)$ *— неперервний оператор, що задовольняє умови Каратеодорі,* $h : C([a,b];R) \to R$ *— неперервний функціонал і* $\lambda \in R_+$.

Introduction. The following notation is used throughout.

R is the set of all real numbers, $R_+ = [0, +\infty[$.

$C([a,b];R)$ is the Banach space of continuous functions $u : [a,b] \to R$ with the norm $\|u\|_C = \max\{|u(t)| : a \le t \le b\}$.

$C([a,b];R_+) = \{u \in C([a,b];R) : u(t) \ge 0 \text{ for } t \in [a,b]\}$.

$\widetilde{C}([a,b];R)$ is the set of absolutely continuous functions $u : [a,b] \to R$.

$B^i_{\lambda c}([a,b];R) = \{u \in C([a,b];R) : (u(a)+\lambda u(b))\,\mathrm{sgn}((2-i)u(a)+(i-1)u(b)) \le c\}$, where $c \in R$, $i = 1, 2$.

$L([a,b];R)$ is the Banach space of Lebesgue integrable functions $p : [a,b] \to R$ with the norm $\|p\|_L = \int\limits_a^b |p(s)|ds$.

$L([a,b];R_+) = \{p \in L([a,b];R) : p(t) \ge 0$ for almost all $t \in [a,b]\}$.

$\mathcal{M}_{ab}$ is the set of measurable functions $\tau : [a,b] \to [a,b]$.

$\widetilde{\mathcal{L}}_{ab}$ is the set of linear operators $\ell : C([a,b];R) \to L([a,b];R)$ for which there is a function $\eta \in L([a,b];R_+)$ such that

$$|\ell(v)(t)| \le \eta(t)\|v\|_C \qquad \text{for } t \in [a,b], \quad v \in C([a,b];R).$$

$\mathcal{P}_{ab}$ is the set of linear operators $\ell \in \widetilde{\mathcal{L}}_{ab}$ transforming the set $C([a,b];R_+)$ into the set $L([a,b];R_+)$.

K_{ab} is the set of continuous operators $F : C([a,b];R) \to L([a,b];R)$ satisfying the Caratheòdory conditions, i.e., for every $r > 0$ there exists $q_r \in L([a,b];R_+)$ such that

$$|F(v)(t)| \le q_r(t) \qquad \text{for } t \in [a,b], \quad \|v\|_C \le r.$$

$K([a,b] \times A; B)$, where $A \subseteq R^2$, $B \subseteq R$, is the set of functions $f : [a,b] \times A \to B$ satisfying the Caratheòdory conditions, i.e., $f(\cdot, x) : [a,b] \to B$ is a measurable function for all $x \in A$, $f(t, \cdot) : A \to B$ is a continuous function for almost all $t \in [a,b]$, and for every $r > 0$ there exists $q_r \in L([a,b];R_+)$ such that

$$|f(t,x)| \le q_r(t) \qquad \text{for } t \in [a,b],\ x \in A,\ \|x\| \le r.$$

$$[x]_+ = \frac{1}{2}(|x|+x), \qquad [x]_- = \frac{1}{2}(|x|-x).$$

By a solution of the equation

$$u'(t) = F(u)(t), \tag{0.1}$$

where $F \in K_{ab}$, we understand a function $u \in \widetilde{C}([a,b];R)$ satisfying the equation (0.1) almost everywhere in $[a,b]$.

Consider the problem on the existence and uniqueness of a solution of (0.1) satisfying the boundary condition

$$u(a) + \lambda u(b) = h(u), \tag{0.2}$$

where $\lambda \in R_+$ and $h : C([a,b];R) \to R$ is a continuous functional.

The general boundary-value problems for functional differential equations have been studied very intensively. There are a lot of interesting general results (see, e.g., [1–27] and the references therein), but still only a few effective criteria for the solvability of special boundary-value problems for functional differential equations are known even in the linear case. In the present paper, we try to fill to some extent the existing gap. More precisely, in Section 1 there are established nonimprovable effective sufficient conditions for the solvability and unique solvability of the problem (0.1), (0.2). Sections 2, 3 and 4 are devoted respectively to the auxiliary propositions, the proofs of the main results and the examples verifying their optimality.

All results will be concretized for the differential equation with deviating arguments of the form

$$u'(t) = p(t)u(\tau(t)) - g(t)u(\mu(t)) + f(t, u(t), u(\nu(t))), \tag{0.3}$$

where $p, g \in L([a,b]; R_+)$, $\tau, \mu, \nu \in \mathcal{M}_{ab}$, and $f \in K([a,b] \times R^2; R)$.

The special case of the discussed boundary-value problem is the Cauchy problem (for $\lambda = 0$ and $h \equiv \text{Const}$). In this case, the below theorems coincide with the results obtained in [5]. The periodic type boundary-value problem (i.e. the case $\lambda < 0$) for the linear equation and for the nonlinear one is studied respectively in [14] and [15].

From the general theory of linear boudary-value problems for functional differential equations we need the following well-known result (see, e.g., [3, 19, 27]).

Theorem 0.1. *Let $\ell \in \widetilde{\mathcal{L}}_{ab}$. Then the problem*

$$u'(t) = \ell(u)(t) + q_0(t), \qquad u(a) + \lambda u(b) = c_0, \tag{0.4}$$

where $q_0 \in L([a,b]; R)$, $c_0 \in R$, is uniquely solvable if and only if the corresponding homogeneous problem

$$u'(t) = \ell(u)(t), \tag{0.1$_0$}$$

$$u(a) + \lambda u(b) = 0 \tag{0.2$_0$}$$

has only the trivial solution.

Remark 0.1. From the Riesz–Schauder theory it follows that if $\ell \in \widetilde{\mathcal{L}}_{ab}$ and the problem (0.1_0), (0.2_0) has a nontrivial solution, then there exist $q_0 \in L([a,b]; R)$ and $c_0 \in R$ such that the problem (0.4) has no solution.

1. Main results. Throughout the paper we assume that $q \in K([a,b] \times R_+; R_+)$ is nondecreasing in the second argument, and satisfies

$$\lim_{x \to +\infty} \frac{1}{x} \int_a^b q(s, x)\,ds = 0. \tag{1.1}$$

Theorem 1.1. *Let $\lambda \in \ ]0,1]$, $c \in R_+$,*

$$h(v) \operatorname{sgn} v(a) \le c \qquad \textit{for} \quad v \in C([a,b]; R), \tag{1.2}$$

and let there exist

$$\ell_0, \ell_1 \in \mathcal{P}_{ab} \tag{1.3}$$

such that on the set $B^1_{\lambda c}([a,b];R)$ *the inequality*

$$[F(v)(t) - \ell_0(v)(t) + \ell_1(v)(t)]\operatorname{sgn} v(t) \le q(t, \|v\|_C) \qquad \textit{for} \quad t \in [a,b] \tag{1.4}$$

holds. If, moreover,

$$\|\ell_0(1)\|_L < 1, \qquad \|\ell_1(1)\|_L < \alpha(\lambda), \tag{1.5}$$

where

$$\alpha(\lambda) = \begin{cases} -\lambda + 2\sqrt{1 - \|\ell_0(1)\|_L} & \textit{for} \quad \|\ell_0(1)\|_L < 1 - \lambda^2; \\ \dfrac{1}{\lambda}\,(1 - \|\ell_0(1)\|_L) & \textit{for} \quad \|\ell_0(1)\|_L \ge 1 - \lambda^2, \end{cases} \tag{1.6}$$

then the problem (0.1), (0.2) has at least one solution.

Remark 1.1. Theorem 1.1 is nonimprovable in a certain sense. More precisely, the second inequality in (1.5) cannot be replaced by

$$\|\ell_1(1)\|_L < (1+\varepsilon)\alpha(\lambda)$$

no matter how small $\varepsilon > 0$ would be (see Examples 4.1 – 4.3).

Theorem 1.2. *Let* $\lambda \in\ ]0,1]$, $c \in R_+$,

$$h(v)\operatorname{sgn} v(b) \le c \qquad \textit{for} \quad v \in C([a,b];R), \tag{1.7}$$

and let there exist $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ *such that on the set* $B^2_{\lambda c}([a,b];R)$ *the inequality*

$$[F(v)(t) - \ell_0(v)(t) + \ell_1(v)(t)]\operatorname{sgn} v(t) \ge -q(t, \|v\|_C) \qquad \textit{for} \quad t \in [a,b] \tag{1.8}$$

holds. If, moreover,

$$\|\ell_0(1)\|_L + \lambda\|\ell_1(1)\|_L < \lambda, \tag{1.9}$$

then the problem (0.1), (0.2) has at least one solution.

Remark 1.2. Theorem 1.2 is nonimprovable in a certain sense. More precisely, the inequality (1.9) cannot be replaced by

$$\|\ell_0(1)\|_L + \lambda\|\ell_1(1)\|_L < \lambda + \varepsilon$$

no matter how small $\varepsilon > 0$ would be (see Examples 4.4 and 4.5).

Remark 1.3. Let $\lambda \in [1, +\infty[$. Define an operator $\psi : L([a,b];R) \to L([a,b];R)$ by

$$\psi(w)(t) \stackrel{\mathrm{df}}{=} w(a+b-t) \qquad \text{for } t \in [a,b].$$

Let φ be a restriction of ψ to the space $C([a,b];R)$. Put $\vartheta = \dfrac{1}{\lambda}$, and

$$\widehat{F}(w)(t) \overset{\mathrm{df}}{=} -\psi(F(\varphi(w)))(t), \qquad \widehat{h}(w) \overset{\mathrm{df}}{=} \vartheta h(\varphi(w)).$$

It is clear that if u is a solution of the problem (0.1), (0.2), then the function $v \overset{\mathrm{df}}{=} \varphi(u)$ is a solution of the problem

$$v'(t) = \widehat{F}(v)(t), \qquad v(a) + \vartheta v(b) = \widehat{h}(v), \tag{1.10}$$

and vice versa, if v is a solution of the problem (1.10), then the function $u \overset{\mathrm{df}}{=} \varphi(v)$ is a solution of the problem (0.1), (0.2).

Therefore, the following theorems immediately follow from Theorems 1.1 and 1.2.

Theorem 1.3. *Let $\lambda \in [1,+\infty[$, $c \in R_+$, the condition (1.7) be fulfilled, and let there exist $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ such that on the set $B^2_{\lambda c}([a,b];R)$ the inequality (1.8) holds. If, moreover,*

$$\|\ell_1(1)\|_L < 1, \qquad \|\ell_0(1)\|_L < \beta(\lambda), \tag{1.11}$$

where

$$\beta(\lambda) = \begin{cases} -\dfrac{1}{\lambda} + 2\sqrt{1 - \|\ell_1(1)\|_L} & \text{for} \quad \|\ell_1(1)\|_L < 1 - \dfrac{1}{\lambda^2}; \\ \lambda\,(1 - \|\ell_1(1)\|_L) & \text{for} \quad \|\ell_1(1)\|_L \geq 1 - \dfrac{1}{\lambda^2}, \end{cases} \tag{1.12}$$

then the problem (0.1), (0.2) has at least one solution.

Theorem 1.4. *Let $\lambda \in [1,+\infty[$, $c \in R_+$, the condition (1.2) be fulfilled, and let there exist $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ such that on the set $B^1_{\lambda c}([a,b];R)$ the inequality (1.4) holds. If, moreover,*

$$\lambda\|\ell_1(1)\|_L + \|\ell_0(1)\|_L < 1, \tag{1.13}$$

then the problem (0.1), (0.2) has at least one solution.

Remark 1.4. On account of Remarks 1.1 – 1.3, it is clear that Theorems 1.3 and 1.4 are also nonimprovable.

Next we establish theorems on the unique solvability of the problem (0.1), (0.2).

Theorem 1.5. *Let $\lambda \in\ ]0,1]$,*

$$[h(v) - h(w)]\operatorname{sgn}(v(a) - w(a)) \leq 0 \qquad \text{for} \quad v, w \in C([a,b];R), \tag{1.14}$$

and let there exist $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ such that on the set $B^1_{\lambda c}([a,b];R)$, where $c = |h(0)|$, the inequality

$$[F(v)(t) - F(w)(t) - \ell_0(v-w)(t) + \ell_1(v-w)(t)]\operatorname{sgn}(v(t) - w(t)) \leq 0 \tag{1.15}$$

holds. Let, moreover, (1.5) be fulfilled, where $\alpha(\lambda)$ is defined by (1.6). Then the problem (0.1), (0.2) is uniquely solvable.

Theorem 1.6. *Let* $\lambda \in \,]0,1]$,

$$[h(v) - h(w)] \operatorname{sgn}(v(b) - w(b)) \le 0 \qquad \textit{for} \quad v, w \in C([a,b]; R), \tag{1.16}$$

and let there exist $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ *such that on the set* $B^2_{\lambda c}([a,b]; R)$, *where* $c = |h(0)|$, *the inequality*

$$[F(v)(t) - F(w)(t) - \ell_0(v-w)(t) + \ell_1(v-w)(t)] \operatorname{sgn}(v(t) - w(t)) \ \ge 0 \tag{1.17}$$

holds. Let, moreover, (1.9) be fulfilled. Then the problem (0.1), (0.2) is uniquely solvable.

According to Remark 1.3, Theorems 1.5 and 1.6 imply the following results.

Theorem 1.7. *Let* $\lambda \in [1, +\infty[$, *the condition (1.16) be satisfied, and let there exist* $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ *such that on the set* $B^2_{\lambda c}([a,b]; R)$, *where* $c = |h(0)|$, *the inequality (1.17) holds. Let, moreover, (1.11) be fulfilled, where* $\beta(\lambda)$ *is defined by (1.12). Then the problem (0.1), (0.2) is uniquely solvable.*

Theorem 1.8. *Let* $\lambda \in [1, +\infty[$, *the condition (1.14) be satisfied, and let there exist* $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ *such that on the set* $B^1_{\lambda c}([a,b]; R)$, *where* $c = |h(0)|$, *the inequality (1.15) holds. Let, moreover, (1.13) be fulfilled. Then the problem (0.1), (0.2) is uniquely solvable.*

Remark 1.5. Theorems 1.5 – 1.8 are nonimprovable in a certain sense (see Examples 4.1 – 4.5).

For the equation of the type (0.3), from Theorems 1.1 – 1.8 we get the following assertions.

Corollary 1.1. *Let* $\lambda \in \,]0,1]$, $c \in R_+$, *the condition (1.2) be fulfilled, and let*

$$f(t,x,y) \operatorname{sgn} x \le q(t) \qquad \textit{for} \quad t \in [a,b],\ x,y \in R, \tag{1.18}$$

where $q \in L([a,b]; R_+)$. *If, moreover,*

$$\int_a^b p(s)ds < 1, \qquad \int_a^b g(s)ds < \gamma(\lambda), \tag{1.19}$$

where

$$\gamma(\lambda) = \begin{cases} -\lambda + 2\sqrt{1 - \displaystyle\int_a^b p(s)ds} & \textit{for} \quad \displaystyle\int_a^b p(s)ds < 1 - \lambda^2; \\ \dfrac{1}{\lambda}\left(1 - \displaystyle\int_a^b p(s)ds\right) & \textit{for} \quad \displaystyle\int_a^b p(s)ds \ge 1 - \lambda^2, \end{cases} \tag{1.20}$$

then the problem (0.3), (0.2) has at least one solution.

Corollary 1.2. *Let* $\lambda \in \,]0,1]$, $c \in R_+$, *the condition (1.7) be fulfilled, and let*

$$f(t,x,y) \operatorname{sgn} x \ge -q(t) \qquad \textit{for} \quad t \in [a,b],\ x,y \in R, \tag{1.21}$$

where $q \in L([a,b];R_+)$. If, moreover,

$$\lambda \int_a^b g(s)ds + \int_a^b p(s)ds < \lambda, \tag{1.22}$$

then the problem (0.3), (0.2) has at least one solution.

Corollary 1.3. *Let $\lambda \in [1,+\infty[$, $c \in R_+$, the conditions (1.7) and (1.21) be fulfilled, and let*

$$\int_a^b g(s)ds < 1, \qquad \int_a^b p(s)ds < \delta(\lambda), \tag{1.23}$$

where

$$\delta(\lambda) = \begin{cases} -\dfrac{1}{\lambda} + 2\sqrt{1 - \displaystyle\int_a^b g(s)ds} & \text{for} \quad \displaystyle\int_a^b g(s)ds < 1 - \dfrac{1}{\lambda^2}; \\ \lambda\left(1 - \displaystyle\int_a^b g(s)ds\right) & \text{for} \quad \displaystyle\int_a^b g(s)ds \geq 1 - \dfrac{1}{\lambda^2}. \end{cases} \tag{1.24}$$

Then the problem (0.3), (0.2) has at least one solution.

Corollary 1.4. *Let $\lambda \in [1,+\infty[$, $c \in R_+$, the conditions (1.2) and (1.18) be fulfilled, and let*

$$\lambda \int_a^b g(s)ds + \int_a^b p(s)ds < 1. \tag{1.25}$$

Then the problem (0.3), (0.2) has at least one solution.

Corollary 1.5. *Let $\lambda \in \,]0,1]$, the condition (1.14) be fulfilled and let*

$$[f(t,x_1,y_1) - f(t,x_2,y_2)]\operatorname{sgn}(x_1 - x_2) \leq 0$$

$$\text{for } \ t \in [a,b], \quad x_1,x_2,y_1,y_2 \in R. \tag{1.26}$$

If, moreover, (1.19) holds, where $\gamma(\lambda)$ is defined by (1.20), then the problem (0.3), (0.2) is uniquely solvable.

Corollary 1.6. *Let $\lambda \in \,]0,1]$, the conditions (1.16),*

$$[f(t,x_1,y_1) - f(t,x_2,y_2)]\operatorname{sgn}(x_1 - x_2) \geq 0$$

$$\text{for } \ t \in [a,b], \quad x_1,x_2,y_1,y_2 \in R, \tag{1.27}$$

and (1.22) hold. Then the problem (0.3), (0.2) is uniquely solvable.

Corollary 1.7. *Let $\lambda \in [1, +\infty[$, and the conditions (1.16) and (1.27) be fulfilled. Let, moreover, (1.23) hold, where $\delta(\lambda)$ is defined by (1.24). Then the problem (0.3), (0.2) is uniquely solvable.*

Corollary 1.8. *Let $\lambda \in [1, +\infty[$, and the conditions (1.14), (1.25), and (1.26) hold. Then the problem (0.3), (0.2) is uniquely solvable.*

2. Auxiliary propositions. First we formulate the result from [22] (Theorem 1) in a suitable for us form.

Lemma 2.1. *Let there exist a positive number ρ and an operator $\ell \in \widetilde{\mathcal{L}}_{ab}$ such that the homogeneous problem (0.1_0), (0.2_0) has only the trivial solution, and let for every $\delta \in\ ]0, 1[$ and for an arbitrary function $u \in \widetilde{C}([a, b]; R)$ satisfying*

$$u'(t) = \ell(u)(t) + \delta[F(u)(t) - \ell(u)(t)], \qquad u(a) + \lambda u(b) = \delta h(u), \tag{2.1}$$

the estimate

$$\|u\|_C \le \rho \tag{2.2}$$

hold. Then the problem (0.1), (0.2) has at least one solution.

Definition 2.1. *We say that the operator $\ell \in \widetilde{\mathcal{L}}_{ab}$ belongs to the set $U_i(\lambda)$, $i \in \{1, 2\}$, if there exists a positive number r such that for any $q^* \in L([a, b]; R_+)$ and $c \in R_+$, every function $u \in \widetilde{C}([a, b]; R)$, satisfying the inequalities*

$$[u(a) + \lambda u(b)] \operatorname{sgn}((2 - i)u(a) + (i - 1)u(b)) \le c, \tag{2.3}$$

$$(-1)^{i+1}[u'(t) - \ell(u)(t)] \operatorname{sgn} u(t) \le q^*(t) \qquad \textit{for } t \in [a, b], \tag{2.4}$$

admits the estimate

$$\|u\|_C \le r\,(c + \|q^*\|_L)\,. \tag{2.5}$$

Lemma 2.2. *Let $i \in \{1, 2\}$, $c \in R_+$,*

$$h(v) \operatorname{sgn}((2 - i)v(a) + (i - 1)v(b)) \le c \qquad \textit{for}\ \ v \in C([a, b]; R), \tag{2.6}$$

and let there exist $\ell \in U_i(\lambda)$ such that on the set $B^i_{\lambda c}([a, b]; R)$ the inequality

$$(-1)^{i+1}[F(v)(t) - \ell(v)(t)] \operatorname{sgn} v(t) \le q(t, \|v\|_C) \qquad \textit{for}\ \ t \in [a, b] \tag{2.7}$$

is fulfilled. Then the problem (0.1), (0.2) has at least one solution.

Proof. First note that due to the condition $\ell \in U_i(\lambda)$, the homogeneous problem (0.1_0), (0.2_0) has only the trivial solution.

Let r be the number appearing in Definition 2.1. According to (1.1) there exists $\rho > 2rc$ such that

$$\frac{1}{x}\int_a^b q(s, x)ds < \frac{1}{2r} \qquad \text{for} \quad x > \rho.$$

Now assume that a function $u \in \widetilde{C}([a,b];R)$ satisfies (2.1) for some $\delta \in\ ]0,1[$. Then, according to (2.6), u satisfies the inequality (2.3), i.e., $u \in B^i_{\lambda c}([a,b];R)$. By (2.1) and (2.7) we obtain that the inequality (2.4) is fulfilled for $q^*(t) = q(t, \|u\|_C)$. Hence, by the condition $\ell \in U_i(\lambda)$ and the definition of the number ρ, we get the estimate (2.2).

Since ρ depends neither on u nor on δ, from Lemma 2.1 it follows that the problem (0.1), (0.2) has at least one solution.

The lemma is proved.

Lemma 2.3. *Let $i \in \{1,2\}$,*

$$[h(u_1) - h(u_2)]\operatorname{sgn}((2-i)(u_1(a) - u_2(a)) + (i-1)(u_1(b) - u_2(a))) \le 0$$
$$\text{for} \quad u_1, u_2 \in C([a,b];R), \tag{2.8}$$

and let there exist $\ell \in U_i(\lambda)$ such that on the set $B^i_{\lambda c}([a,b];R)$, where $c = |h(0)|$, the inequality

$$(-1)^{i+1}[F(u_1)(t) - F(u_2)(t) - \ell(u_1 - u_2)(t)]\operatorname{sgn}(u_1(t) - u_2(t)) \le 0 \tag{2.9}$$

holds. Then the problem (0.1), (0.2) is uniquely solvable.

Proof. From (2.8) it follows that the condition (2.6) is fulfilled, where $c = |h(0)|$. By (2.9), on the set $B^i_{\lambda c}([a,b];R)$ the inequality (2.7) holds, where $q \equiv |F(0)|$. Consequently, all the assumptions of Lemma 2.2 are fulfilled and this guarantees that the problem (0.1), (0.2) has at least one solution. It remains to show that the problem (0.1), (0.2) has at most one solution.

Let u_1, u_2 be arbitrary solutions of the problem (0.1), (0.2). Put $u(t) = u_1(t) - u_2(t)$ for $t \in [a,b]$. Then by (2.8) and (2.9) we get

$$[u(a) + \lambda u(b)]\operatorname{sgn}((2-i)u(a) + (i-1)u(b)) \le 0,$$

$$(-1)^{i+1}[u'(t) - \ell(u)(t)]\operatorname{sgn} u(t) \le 0 \qquad \text{for } t \in [a,b].$$

This, together with the condition $\ell \in U_i(\lambda)$, results in $u \equiv 0$. Consequently, $u_1 \equiv u_2$.

The lemma is proved.

Lemma 2.4. *Let $\lambda \in\ ]0,1]$, the operator ℓ admit the representation $\ell = \ell_0 - \ell_1$ with ℓ_0 and ℓ_1 satisfying the conditions (1.3) and (1.5), where α is defined by (1.6). Then ℓ belongs to the set $U_1(\lambda)$.*

Proof. Suppose $q^* \in L([a,b];R_+)$, $c \in R_+$ and $u \in \widetilde{C}([a,b];R)$ satisfies (2.3) and (2.4) for $i = 1$. We show that (2.5) holds, where

$$r = \begin{cases} \dfrac{\|\ell_1(1)\|_L + 1 + \lambda}{1 - \|\ell_0(1)\|_L - \dfrac{1}{4}(\|\ell_1(1)\|_L + \lambda)^2} & \text{if} \quad \|\ell_0(1)\|_L < 1 - \lambda^2; \\[2ex] \dfrac{\|\ell_1(1)\|_L + 1 + \lambda}{1 - \|\ell_0(1)\|_L - \lambda\|\ell_1(1)\|_L} & \text{if} \quad \|\ell_0(1)\|_L \ge 1 - \lambda^2. \end{cases} \tag{2.10}$$

It is clear that

$$u'(t) = \ell_0(u)(t) - \ell_1(u)(t) + \widetilde{q}(t), \tag{2.11}$$

where

$$\widetilde{q}(t) = u'(t) - \ell(u)(t) \qquad \text{for} \quad t \in [a,b]. \tag{2.12}$$

Obviously,

$$\widetilde{q}(t) \operatorname{sgn} u(t) \le q^*(t) \qquad \text{for} \quad t \in [a,b], \tag{2.13}$$

and

$$[u(a) + \lambda u(b)] \operatorname{sgn} u(a) \le c. \tag{2.14}$$

First suppose that u does not change its sign. According to (2.14) and the assumption $\lambda \in\ ]0,1]$, we obtain

$$|u(a)| \le c. \tag{2.15}$$

Choose $t_0 \in [a,b]$ such that

$$|u(t_0)| = \|u\|_C. \tag{2.16}$$

Due to (1.3) and (2.13), (2.11) implies

$$|u(t)|' \le \|u\|_C\ \ell_0(1)(t) + q^*(t) \qquad \text{for} \quad t \in [a,b]. \tag{2.17}$$

The integration of (2.17) from a to t_0, on account of (1.3), (2.15) and (2.16), results in

$$\|u\|_C - c \le \|u\|_C - |u(a)| \le \|u\|_C \int\limits_a^{t_0} \ell_0(1)(s)ds + \int\limits_a^{t_0} q^*(s)ds \le \|u\|_C \|\ell_0(1)\|_L + \|q^*\|_L.$$

Thus

$$\|u\|_C \left(1 - \|\ell_0(1)\|_L\right) \le c + \|q^*\|_L$$

and, consequently, the estimate (2.5) holds.

Now suppose that u changes its sign. Put

$$M = \max\{u(t) : t \in [a,b]\}, \quad m = -\min\{u(t) : t \in [a,b]\} \tag{2.18}$$

and choose $t_M, t_m \in [a,b]$ such that

$$u(t_M) = M, \quad u(t_m) = -m. \tag{2.19}$$

Obviously, $M > 0$, $m > 0$, and either

$$t_m < t_M, \tag{2.20}$$

or

$$t_m > t_M. \tag{2.21}$$

First suppose that (2.20) is fulfilled. It is clear that there exists $\alpha_2 \in \,]t_m, t_M[$ such that

$$u(t) > 0 \quad \text{for } \alpha_2 < t \le t_M, \qquad u(\alpha_2) = 0. \tag{2.22}$$

Let

$$\alpha_1 = \inf\{t \in [a, t_m] : u(s) < 0 \text{ for } t \le s \le t_m\}.$$

Obviously,

$$u(t) < 0 \quad \text{for } \alpha_1 < t \le t_m \quad \text{and} \quad u(\alpha_1) = 0 \quad \text{if } \alpha_1 > a. \tag{2.23}$$

From (2.14), (2.23) and the assumption $\lambda \in \,]0,1]$ it follows that

$$u(\alpha_1) \ge -\lambda[u(b)]_+ - c \ge -\lambda M - c. \tag{2.24}$$

The integration of (2.11) from α_1 to t_m and from α_2 to t_M, in view of (1.3), (2.13), (2.18), (2.19), (2.22), (2.23) and (2.24), yields

$$m - \lambda M - c \le m + u(\alpha_1) \le M \int\limits_{\alpha_1}^{t_m} \ell_1(1)(s)ds + m \int\limits_{\alpha_1}^{t_m} \ell_0(1)(s)ds + \int\limits_{\alpha_1}^{t_m} q^*(s)ds,$$

$$M \le M \int\limits_{\alpha_2}^{t_M} \ell_0(1)(s)ds + m \int\limits_{\alpha_2}^{t_M} \ell_1(1)(s)ds + \int\limits_{\alpha_2}^{t_M} q^*(s)ds.$$

From the last two inequalities we obtain

$$m(1 - C_1) \le M(A_1 + \lambda) + \|q^*\|_L + c, \qquad M(1 - D_1) \le mB_1 + \|q^*\|_L, \tag{2.25}$$

where

$$A_1 = \int_{\alpha_1}^{t_m} \ell_1(1)(s)ds, \qquad B_1 = \int_{\alpha_2}^{t_M} \ell_1(1)(s)ds,$$

$$C_1 = \int_{\alpha_1}^{t_m} \ell_0(1)(s)ds, \qquad D_1 = \int_{\alpha_2}^{t_M} \ell_0(1)(s)ds.$$

Due to the first inequality in (1.5), $C_1 < 1$, $D_1 < 1$. Consequently, (2.25) implies

$$\begin{aligned} 0 < m(1-C_1)(1-D_1) &\le (A_1+\lambda)(mB_1 + \|q^*\|_L) + \|q^*\|_L + c \le \\ &\le m(A_1+\lambda)B_1 + (\|q^*\|_L + c)(\|\ell_1(1)\|_L + 1 + \lambda), \\ 0 < M(1-C_1)(1-D_1) &\le B_1(M(A_1+\lambda) + \|q^*\|_L + c) + \|q^*\|_L \le \\ &\le M(A_1+\lambda)B_1 + (\|q^*\|_L + c)(\|\ell_1(1)\|_L + 1 + \lambda). \end{aligned} \tag{2.26}$$

Obviously,

$$(1-C_1)(1-D_1) \ge 1 - (C_1 + D_1) \ge 1 - \|\ell_0(1)\|_L > 0. \tag{2.27}$$

If $\|\ell_0(1)\|_L \ge 1 - \lambda^2$, then, according to (1.6) and the second inequality in (1.5), we obtain $\|\ell_1(1)\|_L < \lambda$. Hence, $B_1 < \lambda$ and

$$(A_1+\lambda)B_1 = A_1B_1 + \lambda B_1 \le \lambda(A_1 + B_1) \le \lambda\|\ell_1(1)\|_L.$$

By the last inequality and (2.27), from (2.26) we get

$$\begin{aligned} m &\le r_0(\|\ell_1(1)\|_L + 1 + \lambda)(c + \|q^*\|_L), \\ M &\le r_0(\|\ell_1(1)\|_L + 1 + \lambda)(c + \|q^*\|_L), \end{aligned} \tag{2.28}$$

where

$$r_0 = (1 - \|\ell_0(1)\|_L - \lambda\|\ell_1(1)\|_L)^{-1}. \tag{2.29}$$

Therefore, the estimate (2.5) holds.

If $\|\ell_0(1)\|_L < 1 - \lambda^2$, then by the inequalities

$$4(A_1+\lambda)B_1 \le (A_1 + B_1 + \lambda)^2 \le (\|\ell_1(1)\|_L + \lambda)^2$$

and (2.27), (2.26) implies

$$\begin{aligned} m &\le r_1(\|\ell_1(1)\|_L + 1 + \lambda)(c + \|q^*\|_L), \\ M &\le r_1(\|\ell_1(1)\|_L + 1 + \lambda)(c + \|q^*\|_L), \end{aligned} \tag{2.30}$$

where

$$r_1 = \left[1 - \|\ell_0(1)\|_L - \frac{1}{4}(\|\ell_1(1)\|_L + \lambda)^2\right]^{-1}. \tag{2.31}$$

Therefore, the estimate (2.5) is valid.

Now suppose that (2.21) is satisfied. Obviously there exists $\alpha_4 \in \,]t_m, t_M[$ such that

$$u(t) < 0 \quad \text{for } \alpha_4 < t \leq t_m, \qquad u(\alpha_4) = 0. \tag{2.32}$$

Let

$$\alpha_3 = \inf\{t \in [a, t_M] : u(s) > 0 \text{ for } t \leq s \leq t_M\}.$$

Obviously,

$$u(t) > 0 \quad \text{for } \alpha_3 < t \leq t_M \quad \text{and} \quad u(\alpha_3) = 0 \quad \text{if } \alpha_3 > a. \tag{2.33}$$

From (2.14), (2.33) and the assumption $\lambda \in \,]0, 1]$ we get

$$u(\alpha_3) \leq \lambda[u(b)]_- + c \leq \lambda m + c. \tag{2.34}$$

The integration of (2.11) from α_3 to t_M and from α_4 to t_m, in view of (1.3), (2.13), (2.18), (2.19), (2.32), (2.33) and (2.34), results in

$$M - \lambda m - c \leq M - u(\alpha_3) \leq M \int\limits_{\alpha_3}^{t_M} \ell_0(1)(s)ds + m \int\limits_{\alpha_3}^{t_M} \ell_1(1)(s)ds + \int\limits_{\alpha_3}^{t_M} q^*(s)ds,$$

$$m \leq M \int\limits_{\alpha_4}^{t_m} \ell_1(1)(s)ds + m \int\limits_{\alpha_4}^{t_m} \ell_0(1)(s)ds + \int\limits_{\alpha_4}^{t_m} q^*(s)ds.$$

From the last two inequalities we obtain

$$M(1 - C_2) \leq m(A_2 + \lambda) + \|q^*\|_L + c, \qquad m(1 - D_2) \leq MB_2 + \|q^*\|_L, \tag{2.35}$$

where

$$A_2 = \int\limits_{\alpha_3}^{t_M} \ell_1(1)(s)ds, \qquad B_2 = \int\limits_{\alpha_4}^{t_m} \ell_1(1)(s)ds,$$

$$C_2 = \int\limits_{\alpha_3}^{t_M} \ell_0(1)(s)ds, \qquad D_2 = \int\limits_{\alpha_4}^{t_m} \ell_0(1)(s)ds.$$

Due to the first inequality in (1.5), $C_2 < 1$, $D_2 < 1$. Consequently, (2.35) implies

$$\begin{aligned} 0 < M(1-C_2)(1-D_2) &\leq (A_2+\lambda)(MB_2 + \|q^*\|_L) + \|q^*\|_L + c \leq \\ &\leq M(A_2+\lambda)B_2 + (\|q^*\|_L + c)(\|\ell_1(1)\|_L + 1 + \lambda), \\ 0 < m(1-C_2)(1-D_2) &\leq B_2(m(A_2+\lambda) + \|q^*\|_L + c) + \|q^*\|_L \leq \\ &\leq m(A_2+\lambda)B_2 + (\|q^*\|_L + c)(\|\ell_1(1)\|_L + 1 + \lambda). \end{aligned} \tag{2.36}$$

Obviously,

$$(1-C_2)(1-D_2) \geq 1 - (C_2 + D_2) \geq 1 - \|\ell_0(1)\|_L > 0. \tag{2.37}$$

If $\|\ell_0(1)\|_L \geq 1 - \lambda^2$, then according to (1.6) and the second inequality in (1.5), we obtain $\|\ell_1(1)\|_L < \lambda$. Hence, $B_2 < \lambda$ and

$$(A_2+\lambda)B_2 = A_2B_2 + \lambda B_2 \leq \lambda(A_2 + B_2) \leq \lambda\|\ell_1(1)\|_L.$$

By the last inequality and (2.37), (2.36) implies (2.28), where r_0 is defined by (2.29). Therefore, the estimate (2.5) is valid.

If $\|\ell_0(1)\|_L < 1 - \lambda^2$, then by the inequalities

$$4(A_2+\lambda)B_2 \leq (A_2 + B_2 + \lambda)^2 \leq (\|\ell_1(1)\|_L + \lambda)^2$$

and (2.37), (2.36) implies (2.30), where r_1 is defined by (2.31). Therefore, the estimate (2.5) holds.

The lemma is proved.

Lemma 2.5. *Let $\lambda \in\ ]0,1]$, the operator ℓ admit the representation $\ell = \ell_0 - \ell_1$, where ℓ_0 and ℓ_1 satisfy the conditions (1.3) and (1.9). Then ℓ belongs to the set $U_2(\lambda)$.*

Proof. Let $q^* \in L([a,b];R_+)$, $c \in R_+$ and $u \in \widetilde{C}([a,b];R)$ satisfy (2.3) and (2.4) for $i = 2$. We show that (2.5) holds, where

$$r = \frac{\lambda\|\ell_0(1)\|_L + 1 + \lambda}{\lambda - \lambda\|\ell_1(1)\|_L - \|\ell_0(1)\|_L}. \tag{2.38}$$

Obviously, u satisfies (2.11), where $\widetilde{q}$ is defined by (2.12). Clearly,

$$-\widetilde{q}(t)\operatorname{sgn} u(t) \leq q^*(t) \qquad \text{for } t \in [a,b], \tag{2.39}$$

and

$$[u(a) + \lambda u(b)]\operatorname{sgn} u(b) \leq c. \tag{2.40}$$

First suppose that u does not change its sign. According to (2.40) and the assumption $\lambda \in$ $\in\ ]0,1]$, we obtain

$$|u(b)| \leq \frac{c}{\lambda}. \tag{2.41}$$

Choose $t_0 \in [a,b]$ such that (2.16) holds. Due to (1.3) and (2.41), (2.11) implies

$$-|u(t)|' \leq \|u\|_C\ \ell_1(1)(t) + q^*(t) \qquad \text{for } t \in [a,b]. \tag{2.42}$$

The integration of (2.42) from t_0 to b, on account of (1.3), (2.41) and (2.16), results in

$$\|u\|_C - \frac{c}{\lambda} \leq \|u\|_C - |u(b)| \leq \|u\|_C \int\limits_{t_0}^{b} \ell_1(1)(s)ds + \int\limits_{t_0}^{b} q^*(s)ds \leq$$

$$\leq \|u\|_C \|\ell_1(1)\|_L + \|q^*\|_L.$$

Thus

$$\|u\|_C \left(1 - \|\ell_1(1)\|_L\right) \leq \frac{c + \|q^*\|_L}{\lambda},$$

and, consequently, the estimate (2.5) holds.

Now suppose that u changes its sign. Define numbers M and m by (2.18) and choose $t_M, t_m \in [a,b]$ such that (2.19) is fulfilled. Obviously, $M > 0$, $m > 0$, and either (2.20) or (2.21) is valid.

First suppose that (2.21) holds. It is clear that there exists $\alpha_1 \in\ ]t_M, t_m[$ such that

$$u(t) > 0 \quad \text{for } t_M \leq t < \alpha_1, \qquad u(\alpha_1) = 0. \tag{2.43}$$

Let

$$\alpha_2 = \sup\{t \in [t_m, b] : u(s) < 0 \text{ for } t_m \leq s \leq t\}.$$

Obviously,

$$u(t) < 0 \quad \text{for } t_m \leq t < \alpha_2 \quad \text{and} \quad u(\alpha_2) = 0 \quad \text{if } \alpha_2 < b. \tag{2.44}$$

From (2.40), (2.44) and the assumption $\lambda \in\ ]0,1]$ we obtain

$$u(\alpha_2) \geq -\frac{1}{\lambda}[u(a)]_+ - \frac{c}{\lambda} \geq -\frac{M}{\lambda} - \frac{c}{\lambda}. \tag{2.45}$$

The integration of (2.11) from t_M to α_1 and from t_m to α_2, in view of (1.3), (2.18), (2.19), (2.39), (2.43), (2.44) and (2.45), implies

$$M \le M \int\limits_{t_M}^{\alpha_1} \ell_1(1)(s)ds + m \int\limits_{t_M}^{\alpha_1} \ell_0(1)(s)ds + \int\limits_{t_M}^{\alpha_1} q^*(s)ds,$$

$$m - \frac{M}{\lambda} - \frac{c}{\lambda} \le m + u(\alpha_2) \le M \int\limits_{t_m}^{\alpha_2} \ell_0(1)(s)ds + m \int\limits_{t_m}^{\alpha_2} \ell_1(1)(s)ds + \int\limits_{t_m}^{\alpha_2} q^*(s)ds.$$

From the last two inequalities we get

$$M(1 - A_1) \le mC_1 + \|q^*\|_L, \qquad m(1 - B_1) \le M\left(D_1 + \frac{1}{\lambda}\right) + \|q^*\|_L + \frac{c}{\lambda}, \tag{2.46}$$

where

$$A_1 = \int\limits_{t_M}^{\alpha_1} \ell_1(1)(s)ds, \qquad B_1 = \int\limits_{t_m}^{\alpha_2} \ell_1(1)(s)ds,$$

$$C_1 = \int\limits_{t_M}^{\alpha_1} \ell_0(1)(s)ds, \qquad D_1 = \int\limits_{t_m}^{\alpha_2} \ell_0(1)(s)ds.$$

Due to (1.9), $A_1 < 1$, $B_1 < 1$. Consequently, (2.46) implies

$$\begin{aligned} 0 < M(1 - A_1)(1 - B_1) &\le C_1\left(M\left(D_1 + \frac{1}{\lambda}\right) + \|q^*\|_L + \frac{c}{\lambda}\right) + \|q^*\|_L \le \\ &\le MC_1\left(D_1 + \frac{1}{\lambda}\right) + \left(\|\ell_0(1)\|_L + 1 + \frac{1}{\lambda}\right)\left(\|q^*\|_L + c\right), \\ 0 < m(1 - A_1)(1 - B_1) &\le \left(D_1 + \frac{1}{\lambda}\right)\left(mC_1 + \|q^*\|_L\right) + \|q^*\|_L + \frac{c}{\lambda} \le \\ &\le mC_1\left(D_1 + \frac{1}{\lambda}\right) + \left(\|\ell_0(1)\|_L + 1 + \frac{1}{\lambda}\right)\left(\|q^*\|_L + c\right). \end{aligned} \tag{2.47}$$

Obviously,

$$(1 - A_1)(1 - B_1) \ge 1 - (A_1 + B_1) \ge 1 - \|\ell_1(1)\|_L > 0. \tag{2.48}$$

According to (1.9) and the assumption $\lambda \in \,]0, 1]$, we obtain $\|\ell_0(1)\|_L < \dfrac{1}{\lambda}$. Hence, $C_1 < \dfrac{1}{\lambda}$ and

$$C_1\left(D_1 + \frac{1}{\lambda}\right) = C_1 D_1 + \frac{1}{\lambda}C_1 \le \frac{1}{\lambda}(C_1 + D_1) \le \frac{1}{\lambda}\|\ell_0(1)\|_L.$$

By the last inequality, (2.48) and the assumption $\lambda \in \,]0,1]$, from (2.47) we get

$$M \le r_0 \left(\lambda \|\ell_0(1)\|_L + 1 + \lambda\right)\left(c + \|q^*\|_L\right),$$
$$m \le r_0 \left(\lambda \|\ell_0(1)\|_L + 1 + \lambda\right)\left(c + \|q^*\|_L\right), \tag{2.49}$$

where

$$r_0 = \left(\lambda - \lambda\|\ell_1(1)\|_L - \|\ell_0(1)\|_L\right)^{-1}. \tag{2.50}$$

Therefore, the estimate (2.5) holds.

Now suppose that (2.20) is valid. Obviously there exists $\alpha_3 \in \,]t_m, t_M[$ such that

$$u(t) < 0 \quad \text{for } t_m \le t < \alpha_3, \qquad u(\alpha_3) = 0. \tag{2.51}$$

Let

$$\alpha_4 = \sup\{t \in [t_M, b] : u(s) > 0 \text{ for } t_M \le s \le t\}.$$

It is clear that

$$u(t) > 0 \quad \text{for } t_M \le t < \alpha_4 \quad \text{and} \quad u(\alpha_4) = 0 \quad \text{if} \ \ \alpha_4 < b. \tag{2.52}$$

From (2.40), (2.52), and the assumption $\lambda \in \,]0,1]$ it follows that

$$u(\alpha_4) \le \frac{1}{\lambda}[u(a)]_- + \frac{c}{\lambda} \le \frac{m}{\lambda} + \frac{c}{\lambda}. \tag{2.53}$$

The integration of (2.11) from t_m to α_3 and from t_M to α_4, in view of (1.3), (2.18), (2.19), (2.39), (2.51), (2.52) and (2.53), yields

$$m \le M \int\limits_{t_m}^{\alpha_3} \ell_0(1)(s)ds + m \int\limits_{t_m}^{\alpha_3} \ell_1(1)(s)ds + \int\limits_{t_m}^{\alpha_3} q^*(s)ds,$$

$$M - \frac{m}{\lambda} - \frac{c}{\lambda} \le M - u(\alpha_4) \le M \int\limits_{t_M}^{\alpha_4} \ell_1(1)(s)ds + m \int\limits_{t_M}^{\alpha_4} \ell_0(1)(s)ds + \int\limits_{t_M}^{\alpha_4} q^*(s)ds.$$

From the last two inequalities we get

$$m(1 - A_2) \le MC_2 + \|q^*\|_L, \qquad M(1 - B_2) \le m\left(D_2 + \frac{1}{\lambda}\right) + \|q^*\|_L + \frac{c}{\lambda}, \tag{2.54}$$

where

$$A_2 = \int\limits_{t_m}^{\alpha_3} \ell_1(1)(s)ds, \qquad B_2 = \int\limits_{t_M}^{\alpha_4} \ell_1(1)(s)ds,$$

$$C_2 = \int\limits_{t_m}^{\alpha_3} \ell_0(1)(s)ds, \qquad D_2 = \int\limits_{t_M}^{\alpha_4} \ell_0(1)(s)ds.$$

Due to (1.9), $A_2 < 1$, $B_2 < 1$. Consequently, (2.54) implies

$$\begin{aligned} 0 < m(1-A_2)(1-B_2) &\leq C_2\left(m\left(D_2+\frac{1}{\lambda}\right)+\|q^*\|_L+\frac{c}{\lambda}\right)+\|q^*\|_L \leq \\ &\leq mC_2\left(D_2+\frac{1}{\lambda}\right)+\left(\|\ell_0(1)\|_L+1+\frac{1}{\lambda}\right)\left(\|q^*\|_L+c\right), \\ 0 < M(1-A_2)(1-B_2) &\leq \left(D_2+\frac{1}{\lambda}\right)\left(MC_2+\|q^*\|_L\right)+\|q^*\|_L+\frac{c}{\lambda} \leq \\ &\leq MC_2\left(D_2+\frac{1}{\lambda}\right)+\left(\|\ell_0(1)\|_L+1+\frac{1}{\lambda}\right)\left(\|q^*\|_L+c\right). \end{aligned} \tag{2.55}$$

Obviously,

$$(1-A_2)(1-B_2) \geq 1-(A_2+B_2) \geq 1-\|\ell_1(1)\|_L > 0. \tag{2.56}$$

According to (1.9) and the assumption $\lambda \in \,]0,1]$, we obtain $\|\ell_0(1)\|_L < \dfrac{1}{\lambda}$. Hence, $C_2 < \dfrac{1}{\lambda}$ and

$$C_2\left(D_2+\frac{1}{\lambda}\right) = C_2D_2+\frac{1}{\lambda}C_2 \leq \frac{1}{\lambda}(C_2+D_2) \leq \frac{1}{\lambda}\|\ell_0(1)\|_L.$$

By the last inequality and (2.56), (2.55) implies (2.49), where r_0 is defined by (2.50). Therefore, the estimate (2.5) is valid.

The lemma is proved.

3. Proofs of the main results. Theorem 1.1 follows from Lemmas 2.2 and 2.4, Theorem 1.2 follows from Lemmas 2.2 and 2.5, Theorem 1.5 follows from Lemmas 2.3 and 2.4, and Theorem 1.6 follows from Lemmas 2.3 and 2.5.

Proof of Corollary 1.1. Obviously, the conditions (1.18) and (1.19), with γ defined by (1.20), yield the conditions (1.4) and (1.5) with α defined by (1.6), where

$$\begin{gathered} F(v)(t) \stackrel{\mathrm{df}}{=} p(t)v(\tau(t)) - g(t)v(\mu(t)) + f(t, v(t), v(\nu(t))), \\ \ell_0(v)(t) \stackrel{\mathrm{df}}{=} p(t)v(\tau(t)), \qquad \ell_1(v)(t) \stackrel{\mathrm{df}}{=} g(t)v(\mu(t)). \end{gathered} \tag{3.1}$$

Consequently, all the assumptions of Theorem 1.1 are fulfilled.

Proof of Corollary 1.5. Obviously, the conditions (1.26) and (1.19), with γ defined by (1.20), yield the conditions (1.15) and (1.5) with α defined by (1.6), where F, ℓ_0 and ℓ_1 are defined by (3.1). Consequently, all the assumptions of Theorem 1.5 are fulfilled.

Corollaries 1.2 – 1.4 and 1.6 – 1.8 can be proved analogously.

4. On remarks 1.1 and 1.2. ***On Remark 1.1.*** Let $\lambda \in \,]0,1]$ (for the case $\lambda = 0$, see [5]). Denote by G the set of pairs $(x,y) \in R_+ \times R_+$ such that either

$$x < 1-\lambda^2, \qquad y < 2\sqrt{1-x}-\lambda,$$

or

$$1-\lambda^2 \le x < 1, \qquad y < \frac{1-x}{\lambda}.$$

According to Theorem 1.1, if (1.2) is fulfilled and there exist $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ such that $(\|\ell_0(1)\|_L,$ $\|\ell_1(1)\|_L) \in G$, and on the set $B^1_{\lambda c}([a,b];R)$ the inequality (1.4) holds, then the problem (0.1), (0.2) is solvable.

Below we give examples which show that for any pair $(x_0,y_0) \notin \overline{G}$, $x_0 \ge 0$, $y_0 \ge 0$, there exist functions $p_0 \in L([a,b];R)$, $-p_1 \in L([a,b];R_+)$, and $\tau \in \mathcal{M}_{ab}$ such that

$$\int\limits_a^b [p_0(s)]_+ ds = x_0, \quad \int\limits_a^b [p_0(s)]_- ds = y_0, \tag{4.1}$$

and the problem

$$u'(t) = p_0(t)u(\tau(t)) + p_1(t)u(t), \quad u(a) + \lambda u(b) = 0 \tag{4.2}$$

has a nontrivial solution. Then by Remark 0.1, there exist $q_0 \in L([a,b];R)$ and $c_0 \in R$ such that the problem (0.1), (0.2) with

$$F(v)(t) \overset{\text{df}}{=} p_0(t)v(\tau(t)) + p_1(t)v(t) + q_0(t), \quad h(v) \overset{\text{df}}{=} c_0 \tag{4.3}$$

has no solution, while the conditions (1.2) and (1.4) are fulfilled with $\ell_0(v)(t) \overset{\text{df}}{=} [p_0(t)]_+ v(\tau(t))$, $\ell_1(v)(t) \overset{\text{df}}{=} [p_0(t)]_- v(\tau(t))$, $q \equiv |q_0|$ and $c = |c_0|$.

It is clear that if $x_0, y_0 \in R_+$ and $(x_0,y_0) \notin \overline{G}$, then (x_0,y_0) belongs to at least one of the following sets

$$G_1 = \left\{(x,y) \in R_+ \times R_+ \ : \ 1 < x, \ 0 \le y\right\},$$

$$G_2 = \left\{(x,y) \in R_+ \times R_+ \ : \ 1-\lambda^2 \le x \le 1, \ \frac{1-x}{\lambda} < y\right\},$$

$$G_3 = \left\{(x,y) \in R_+ \times R_+ \ : \ 0 \le x < 1-\lambda^2, \ 2\sqrt{1-x}-\lambda < y\right\}.$$

Example 4.1. Let $(x_0, y_0) \in G_1$, and $\varepsilon > 0$ be such that $x_0 - \varepsilon \geq 1$, $\lambda - \varepsilon > 0$. Put $a = 0$, $b = 4$, $t_0 = 3 + \dfrac{\varepsilon}{1+\varepsilon}$,

$$p_0(t) = \begin{cases} 0 & \text{for } t \in [0,1[; \\ -y_0 & \text{for } t \in [1,2[; \\ x_0 - 1 - \varepsilon & \text{for } t \in [2,3[; \\ 1+\varepsilon & \text{for } t \in [3,4], \end{cases}$$

$$p_1(t) = \begin{cases} -\dfrac{\lambda - \varepsilon}{\lambda - (\lambda-\varepsilon)t} & \text{for } t \in [0,1[; \\ 0 & \text{for } t \in [1,4], \end{cases}$$

$$\tau(t) = \begin{cases} t_0 & \text{for } t \in [0,3[; \\ 4 & \text{for } t \in [3,4]. \end{cases}$$

Then (4.1) holds, and the problem (4.2) has the nontrivial solution

$$u(t) = \begin{cases} -(\lambda-\varepsilon)t + \lambda & \text{for } t \in [0,1[; \\ \varepsilon & \text{for } t \in [1,3[; \\ -(1+\varepsilon)(t-3) + \varepsilon & \text{for } t \in [3,4]. \end{cases}$$

Example 4.2. Let $(x_0, y_0) \in G_2$, and $\varepsilon > 0$ be such that $\dfrac{1 - x_0 + \varepsilon}{\lambda} \leq y_0$, $\lambda - \varepsilon > 0$. Put $a = 0$, $b = 4$, $t_0 = 2 + \dfrac{\varepsilon}{1 - x_0 + \varepsilon}$,

$$p_0(t) = \begin{cases} 0 & \text{for } t \in [0,1[; \\ -y_0 + \dfrac{1 - x_0 + \varepsilon}{\lambda} & \text{for } t \in [1,2[; \\ -\dfrac{1 - x_0 + \varepsilon}{\lambda} & \text{for } t \in [2,3[; \\ x_0 & \text{for } t \in [3,4], \end{cases}$$

$$p_1(t) = \begin{cases} -\dfrac{\lambda - \varepsilon}{\lambda - (\lambda-\varepsilon)t} & \text{for } t \in [0,1[; \\ 0 & \text{for } t \in [1,4], \end{cases}$$

$$\tau(t) = \begin{cases} t_0 & \text{for } t \in [0,2[; \\ 0 & \text{for } t \in [2,3[; \\ 4 & \text{for } t \in [3,4]. \end{cases}$$

Then (4.1) holds, and the problem (4.2) has the nontrivial solution

$$u(t) = \begin{cases} -(\lambda-\varepsilon)t+\lambda & \text{for } t \in [0,1[; \\ \varepsilon & \text{for } t \in [1,2[; \\ -(1-x_0+\varepsilon)(t-2)+\varepsilon & \text{for } t \in [2,3[; \\ -x_0(t-3)-(1-x_0) & \text{for } t \in [3,4]. \end{cases}$$

Example 4.3. Let $(x_0, y_0) \in G_3$, and $\varepsilon > 0$ be such that $y_0 \geq 2\sqrt{1-x_0} - \lambda + \varepsilon$, $\varepsilon < 1 - \sqrt{1-x_0}$. Put $a = 0$, $b = 5$,

$$p_0(t) = \begin{cases} -\sqrt{1-x_0}+\lambda & \text{for } t \in [0,1[; \\ 0 & \text{for } t \in [1, 3-\sqrt{1-x_0}-\varepsilon[; \\ -1 & \text{for } t \in [3-\sqrt{1-x_0}-\varepsilon, 3[; \\ -y_0+2\sqrt{1-x_0}-\lambda+\varepsilon & \text{for } t \in [3,4[; \\ x_0 & \text{for } t \in [4,5], \end{cases}$$

$$p_1(t) = \begin{cases} 0 & \text{for } t \in [0,1[\cup[3-\sqrt{1-x_0}-\varepsilon, 5]; \\ -\dfrac{1-x_0}{(1-x_0)(1-t)+\sqrt{1-x_0}} & \text{for } t \in [1,2[; \\ -\dfrac{\sqrt{1-x_0}}{\sqrt{1-x_0}(3-t)-(1-x_0)} & \text{for } t \in [2, 3-\sqrt{1-x_0}-\varepsilon[, \end{cases}$$

$$\tau(t) = \begin{cases} 5 & \text{for } t \in [0,1[; \\ 1 & \text{for } t \in [1,3[; \\ 3-\sqrt{1-x_0} & \text{for } t \in [3,4[; \\ 5 & \text{for } t \in [4,5]. \end{cases}$$

Then (4.1) holds, and the problem (4.2) has the nontrivial solution

$$u(t) = \begin{cases} (\sqrt{1-x_0}-\lambda)t+\lambda & \text{for } t \in [0,1[; \\ (1-x_0)(1-t)+\sqrt{1-x_0} & \text{for } t \in [1,2[; \\ \sqrt{1-x_0}(3-t)-(1-x_0) & \text{for } t \in [2,3[; \\ -(1-x_0) & \text{for } t \in [3,4[; \\ x_0(5-t)-1 & \text{for } t \in [4,5]. \end{cases}$$

On Remark 1.2. Let $\lambda \in\ ]0,1]$. Denote by H the set of pairs $(x,y) \in R_+ \times R_+$ such that

$$x + \lambda y < \lambda.$$

By Theorem 1.2, if (1.7) is fulfilled and there exist $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ and $q \in L([a,b];R_+)$ such that $(\|\ell_0(1)\|_L, \|\ell_1(1)\|_L) \in H$, and on the set $B^2_{\lambda c}([a,b];R)$ the inequality (1.8) holds, then the problem (0.1), (0.2) is solvable.

Below we give examples which show that for any pair $(x_0,y_0) \notin \overline{H}$, $x_0 \geq 0$, $y_0 \geq 0$, there exist functions $p_0 \in L([a,b];R)$, $p_1 \in L([a,b];R_+)$, and $\tau \in \mathcal{M}_{ab}$ such that (4.1) is fulfilled and the problem (4.2) has a nontrivial solution. Then by Remark 0.1, there exist $q_0 \in L([a,b];R)$ and $c_0 \in R$ such that the problem (0.1), (0.2), where F and h are defined by (4.3), has no solution, while the conditions (1.7) and (1.8) are fulfilled with $\ell_0(v)(t) \stackrel{\text{df}}{=} [p_0(t)]_+ v(\tau(t))$, $\ell_1(v)(t) \stackrel{\text{df}}{=}$ $\stackrel{\text{df}}{=} [p_0(t)]_- v(\tau(t))$, $q \equiv |q_0|$ and $c = |c_0|$.

It is clear that if $x_0, y_0 \in R_+$ and $(x_0,y_0) \notin \overline{H}$, then (x_0,y_0) belongs to at least one of the following sets

$$H_1 = \{(x,y) \in R_+ \times R_+ \ :\ \lambda < x,\ 0 \leq y\},$$

$$H_2 = \left\{(x,y) \in R_+ \times R_+ \ :\ 0 \leq x \leq \lambda,\ -\frac{x}{\lambda} + 1 < y\right\}.$$

Example 4.4. Let $(x_0,y_0) \in H_1$, and $\varepsilon > 0$ be such that $x_0 - \lambda \geq \varepsilon$, $1 - \varepsilon > 0$. Put $a = 0$, $b = 4$, $t_0 = \dfrac{\lambda}{\lambda+\varepsilon}$,

$$p_0(t) = \begin{cases} \lambda + \varepsilon & \text{for } t \in [0,1[; \\ -y_0 & \text{for } t \in [1,2[; \\ x_0 - \lambda - \varepsilon & \text{for } t \in [2,3[; \\ 0 & \text{for } t \in [3,4], \end{cases}$$

$$p_1(t) = \begin{cases} 0 & \text{for } t \in [0,3[; \\ \dfrac{1-\varepsilon}{(1-\varepsilon)(t-4)+1} & \text{for } t \in [3,4], \end{cases}$$

$$\tau(t) = \begin{cases} 4 & \text{for } t \in [0,1[; \\ t_0 & \text{for } t \in [1,4]. \end{cases}$$

Then (4.1) holds, and the problem (4.2) has the nontrivial solution

$$u(t) = \begin{cases} (\lambda+\varepsilon)t - \lambda & \text{for } t \in [0,1[; \\ \varepsilon & \text{for } t \in [1,3[; \\ (1-\varepsilon)(t-4)+1 & \text{for } t \in [3,4]. \end{cases}$$

Example 4.5. Let $(x_0, y_0) \in H_2$, and $\varepsilon > 0$ be such that $\dfrac{\lambda - x_0 + \varepsilon}{\lambda} \le y_0$, $1 - \varepsilon > 0$. Put $a = 0, b = 4, t_0 = 2 - \dfrac{\varepsilon}{\lambda - x_0 + \varepsilon}$,

$$p_0(t) = \begin{cases} x_0 & \text{for } t \in [0,1[; \\ -\dfrac{\lambda - x_0 + \varepsilon}{\lambda} & \text{for } t \in [1,2[; \\ -y_0 + \dfrac{\lambda - x_0 + \varepsilon}{\lambda} & \text{for } t \in [2,3[; \\ 0 & \text{for } t \in [3,4], \end{cases}$$

$$p_1(t) = \begin{cases} 0 & \text{for } t \in [0,3[; \\ \dfrac{1-\varepsilon}{1-(1-\varepsilon)(4-t)} & \text{for } t \in [3,4], \end{cases}$$

$$\tau(t) = \begin{cases} 4 & \text{for } t \in [0,1[; \\ 0 & \text{for } t \in [1,2[; \\ t_0 & \text{for } t \in [2,4]. \end{cases}$$

Then (4.1) holds, and the problem (4.2) has the nontrivial solution

$$u(t) = \begin{cases} -x_0 t + \lambda & \text{for } t \in [0,1[; \\ (\lambda - x_0 + \varepsilon)(2-t) - \varepsilon & \text{for } t \in [1,2[; \\ -\varepsilon & \text{for } t \in [2,3[; \\ (1-\varepsilon)(4-t) - 1 & \text{for } t \in [3,4]. \end{cases}$$

Acknowledgement. For the first author this work was supported by the Grant № 201/99/0295 of the Grant Agency of the Czech Republic and for the second author by the RI No. J07/98:143100001.

1. *Azbelev N. V., Maksimov V. P., and Rakhmatullina L. F.* Introduction to the theory of functional differential equations. — Moscow: Nauka, 1991 (in Russian).
2. *Azbelev N. V., Rakhmatullina L. F.* Theory of linear abstract functional differential equations and applications // Mem. Different. Equat. Math. Phys. — 1996. — **8**. — P. 1–102.
3. *Bravyi E.* A note on the Fredholm property of boundary-value problems for linear functional differential equations // Ibid. — 2000. — **20**. — P. 133–135.
4. *Bravyi E., Hakl R., and Lomtatidze A.* Optimal conditions for unique solvability of the Cauchy problem for first order linear functional differential equations // Czech. Math. J. — 2002. — **52**, № 3. — P. 513–530.
5. *Bravyi E., Hakl R., and Lomtatidze A.* On Cauchy problem for the first order nonlinear functional differential equations of non-Volterra's type // Ibid. — №. 4. — P. 673–690.

6. *Bravyi E., Lomtatidze A., and Půža B.* A note on the theorem on differential inequalities // Georg. Math. J. — 2000. — **7**, № 4. — P. 627–631.

7. *Hakl R.* On some boundary-value problems for systems of linear functional differential equations // E. J. Qualitative Theory Different. Equat. — 1999. — № 10. — P. 1–16.

8. *Hakl R., Kiguradze I., and Půža B.* Upper and lower solutions of boundary-value problems for functional differential equations and theorems on functional differential inequalities // Georg. Math. J. — 2000. — **7**, № 3. — P. 489–512.

9. *Hakl R., Lomtatidze A.* A note on the Cauchy problem for first order linear differential equations with a deviating argument // Arch. Math. (Brno). — 2002. — **38**, № 1. — P. 61–71.

10. *Hakl R., Lomtatidze A., and Půža B.* On a periodic boundary-value problem for the first order scalar functional differential equation // Differentsial'nye Uravneniya (to appear).

11. *Hakl R., Lomtatidze A., and Půža B.* On periodic solutions of first order linear functional differential equations // Nonlinear Anal.: Theory, Meth. and Appl. — 2001. — **23**. — P. 51–84.

12. *Hakl R., Lomtatidze A., and Půža B.* New optimal conditions for unique solvability of the Cauchy problem for first order linear functional differential equations // Math. Bohemica. — 2002. — **127**, № 4. — P. 509–524.

13. *Hakl R., Lomtatidze A., and Šremr J.* On an antiperiodic type boundary-value problem for first order linear functional differential equations // Arch. Math. (Brno). — 2002. — **38**, № 2. — P. 149–160.

14. *Hakl R., Lomtatidze A., and Šremr J.* On a periodic type boundary-value problem for first order linear functional differential equations // Nonlinear Oscillations. — 2002. — **5**, № 3. — P. 416–432.

15. *Hakl R., Lomtatidze A., and Šremr J.* On a periodic type boundary-value problem for first order nonlinear functional differential equations // Nonlinear Anal.: Theory, Meth. and Appl. — 2002. — **51**, № 3. — P. 425–447.

16. *Hale J.* Theory of functional differential equations. — New York etc.: Springer, 1977.

17. *Kiguradze I.* On periodic solutions of first order nonlinear differential equations with deviating arguments // Mem. Different. Equat. Math. Phys. — 1997. — **10**. — P. 134–137.

18. *Kiguradze I.* Initial and boundary-value problems for systems of ordinary differential equations I. — Tbilisi: Metsniereba, 1997 (in Russian).

19. *Kiguradze I., Půža B.* On boundary-value problems for systems of linear functional differential equations // Czech. Math. J. — 1997. — **47**, № 2. — P. 341–373.

20. *Kiguradze I., Půža B.* On periodic solutions of systems of linear functional differential equations // Arch. Math. — 1997. — **33**, № 3. — P. 197–212.

21. *Kiguradze I., Půža B.* Conti–Opial type theorems for systems of functional differential equations // Differentsial'nye Uravneniya. — 1997. — **33**, № 2. — P. 185–194 (in Russian).

22. *Kiguradze I., Půža B.* On boundary-value problems for functional differential equations // Mem. Different. Equat. Math. Phys. — 1997. — **12**. — P. 106–113.

23. *Kiguradze I., Půža B.* On periodic solutions of nonlinear functional differential equations // Georg. Math. J. — 1999. — **6**, № 1. — P. 47–66.

24. *Kiguradze I., Půža B.* On periodic solutions of systems of differential equations with deviating arguments // Nonlinear Anal.: Theory, Meth. and Appl. — 2000. — **42**, № 2. — P. 229–242.

25. *Kolmanovskii V., Myshkis A.* Introduction to the theory and applications of functional differential equations. — Dordrecht: Kluwer Acad. Publ., 1999.

26. *Mawhin J.* Periodic solutions of nonlinear functional differential equations // J. Different. Equat. — 1971. — **10**. — P. 240–261.

27. *Schwabik Š., Tvrdý M., and Vejvoda O.* Differential and integral equations: boundary-value problems and adjoints. — Praha: Academia, 1979.

Received 08.09.2003