UDC 532.595

GRAVITY-CAPILLARY WAVES IN LAYERED FLUID

ГРАВІТАЦІЙНО-КАПІЛЯРНІ ХВИЛІ У РОЗШАРОВАНІЙ РІДИНІ

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In this work we analyse stability of a gravity wave generated on the separation surface of two immiscible liquids inside a moving container and perturbed by a capillary wave. Such a phenomenon is experimentally observed when the amplitude and the frequency of the motion imposed to the container attain certain values. Evolution of the system is described by the variational principle. We assume that motion of the system is decomposed into two modes: the gravity mode and the capillary mode. With suitable scaling assumptions it is possible to show that the evolution of the gravity mode is determined by the forcing motion, while the capillary mode is excited by the nonlinear interactions between the capillary and gravity modes. At last, an analytical dispersion relation is obtained for the pulsation of the capillary mode. This relation is a function of several quantities, all depending on the capillary wavenumber and the characteristics of the exciting motion.

Аналізується стійкість гравітаційних хвиль, що утворюються на поверхні поділу двох рідин, які не змішуються, в рухомому резервуарі та збуреної капілярної хвилі. Таке явище експериментально спостерігається, коли амплітуда і частота руху резервуара досягають деяких значень. Еволюція системи описується на основі варіаційного принципу. Ми припускаємо, що рух системи розкладається на дві форми: гравітаційну та капілярну. За допомогою належного масштабування можна показати, що еволюція гравітаційної форми руху визначається вимушеним рухом, а капілярна форма руху збурюється за рахунок нелінійної взаємодії між капілярною та гравітаційною формами руху. Одержано в аналітичному вигляді дисперсійну залежність для пульсацій капілярної форми руху. Таке співвідношення є функцією кількох величин, що залежать від капілярних хвильових чисел і характеристик збурення руху системи.

1. Introduction. Short scale wave phenomena, for which surface-tension effects are important, have been actively studied in literature. Considerable attention has been paid, in particular, to parasitic capillary wave trains — or capillary ripples — riding on steep gravity free surface waves, because such capillary ripples, although small, influence the dissipative and other dynamical properties of the surface waves [1]. The state of the art is well represented by the works of [1 – 4].

In this work we want to study the generation of capillary ripples riding on longer waves on the separation surface of two immiscible liquid layers, inside of a prismatic, squared-section, moving container. The phenomenon under consideration is analogous to the generation of capillary ripples on free-surface waves and its analysis is motivated, for example, by the necessity of understanding the dynamics of sloshing of stratified liquids in petro-chemical industry. Good tools for such analysis are both the Hamiltonian and the variational formulation of the problem, widely used in many recent studies. In particular, the works [5 – 8] are concerned with the Hamiltonian formulation of the waves in a stratified fluid, while in [9], the authors apply the variational formulation to the sloshing of a two-layer liquid without free-surface.

The variational formulation, based on the theoretical frameworks [10, 11], revealed some peculiar features that make it attractive for the application to the dynamics of interfacial waves. Among such features, one of them has to be mentioned: the possibility of taking account easily of both surface-tension and dissipative effects.

The above mentioned studies however are not directly focused on the generation and the stability of gravity-capillary waves. In this work the variational formulation presented in [9] will be applied to the study of the generation and the stability of gravity-capillary waves on the separation surface existing between two immiscible liquid layers subject to sloshing. The variational approach permits to obtain a nonlinear dynamical system for the evolution of the gravity-capillary wave. The nonlinear interactions are crucial for the generation and the stability of the capillary wave, considered as a perturbation of the gravity wave. Adopting a suitable scaling of the variables, it is possible to model the evolution of the dominant gravity wave independently from the evolution of the capillary wave. The latter, to the contrary, is influenced by the former via nonlinear interactions. In this work, considering the interaction between a dominant gravity wave and a capillary perturbation wave (both characterized by a single wavenumber), a dispersion relation depending on the capillary wavenumber, the frequency and the amplitude of the dominant wave is found. Such a dispersion relation permits to predict if, given the amplitude and the frequency of the dominant wave, the capillary wave is unstable or not. Experimental observations confirm quite well the theoretical predictions.

Finally, it can be stated that the followed approach is promising and future work could be done in order to account for dissipative effects and interactions of a more complex wave (i. e. waves characterized by more than a single wavenumber).

2. Variational formulation. A closed, prismatic container of height H and side B is completely filled with two immiscible liquids whose densities are ρ_1 and ρ_2 ($\rho_1 > \rho_2$). At rest, the layer of the liquid, whose density is ρ_1, reaches the level H_1, measured with respect to the bottom of the tank, while the thickness of the second liquid layer is H_2. Let the frame of reference $Oxyz$ be attached to the container, with $\mathbf{i}, \mathbf{j}, \mathbf{k}$ unit vectors of x, y, z axes respectively. The level $z = 0$ coincides with the separation surface between the two layers at rest.

The container is subject to a rigid motion consisting in a rotation θ around an axis $\Re$ parallel to the y-axis, where $\theta = \theta(t)$ is a given function of the time t. As a consequence of such rigid motion the fluid system is set in motion, which — for the sake of simplicity — will be assumed two-dimensional and irrotational in the absolute frame of reference.

The purpose of the present work is to study the generation and the stability of gravity-capillary waves on the separation surface $z = \eta(x, t)$ by means of a variational formulation.

As it is well known, the crucial point in a variational formulation is the definition of a suitable functional F whose extrema coincide with the solution of the examined problem. In this work, the following definition for F is adopted:

$$F \equiv \int_{t_0}^{t_1} L dt \equiv \int_{t_0}^{t_1} \left(\left\langle \int_{-H_1}^{\eta} p_1 dz \right\rangle + \left\langle \int_{\eta}^{H_2} p_2 dz \right\rangle + \left\langle \tau \left(\sqrt{1+\eta_x^2} - 1 \right) \right\rangle \right) dt \tag{1}$$

where τ is the surface-tension at the separation surface between the two liquid layers and the operator $\langle \bullet \rangle$ is defined as

$$\langle \bullet \rangle \equiv \int_0^B \bullet dx \tag{2}$$

and L is the Lagrangian of the motion [11]. Such a definition is nothing else but that of [9] with the addition of the "surface-tension" term $\left\langle \tau \left(\sqrt{1+\eta_x^2} - 1 \right) \right\rangle$ which accounts for surface-tension effects. As the fluid motion was assumed to be irrotational in the absolute frame of reference, it is possible to obtain analytical expressions of the pressure fields p_1, p_2 of the two liquid layers depending on the respective velocity fields [9]. Substituting such pressure fields in (1) and performing the integration respect to z, the functional F depends on the unknown functions $\varphi_1, \varphi_2, \eta$, i. e., $F = F(\varphi_1, \varphi_2, \eta)$. The functions $\varphi_1, \varphi_2, \eta$ which make the first variation of F, calculated with respect to $\varphi_1, \varphi_2, \eta$, equal to zero are also solutions of the motion equations [9].

The equivalence between the variational formulation and the differential formulation for the water wave dynamics is shown in [12]. The extension to the two-liquid sloshing is shown in [9] when the surface tension effects are considered negligible: they can be however accounted in the following way. Let us consider the first variation of $\left\langle \tau \left(\sqrt{1+\eta_x^2} - 1 \right) \right\rangle$ with respect to η,

$$\delta \left\langle \tau \left(\sqrt{1+\eta_x^2} - 1 \right) \right\rangle = \left\langle \tau \frac{\partial}{\partial x} \left(\frac{\eta_x \delta\eta}{\left(1+\eta_x^2\right)^{1/2}} \right) \right\rangle - \left\langle \tau \frac{\eta_{xx} \delta\eta}{\left(1+\eta_x^2\right)^{3/2}} \right\rangle . \tag{3}$$

The term $\left\langle \tau \frac{\partial}{\partial x} \left(\frac{\eta_x \delta\eta}{\left(1+\eta_x^2\right)^{1/2}} \right) \right\rangle$ in (3) gives

$$\tau \left(\frac{\eta_x \delta\eta}{\left(1+\eta_x^2\right)^{1/2}} \right) \delta\eta \Bigg|_{x=B} - \tau \left(\frac{\eta_x \delta\eta}{\left(1+\eta_x^2\right)^{1/2}} \right) \delta\eta \Bigg|_{x=0}$$

which can be assumed to be equal to zero if $\delta\eta|_{x=B} = \delta\eta|_{x=0} = 0$, i. e., if the arbitrary variation of the function η is zero at $x = 0, x = B$. This condition is fulfilled by a suitable choice of the spatial structure of the function $\eta(x,t)$. Then, adding the term $-\left\langle \tau \frac{\eta_{xx} \delta\eta}{\left(1+\eta_x^2\right)^{3/2}} \right\rangle$ to the first variation respect to η of $\left\langle \int_{-H_1}^{\eta} p_1 dz \right\rangle + \left\langle \int_{\eta}^{H_2} p_2 dz \right\rangle$, the following result is obtained:

$$\left((p_1 - p_2)|_{z=\eta} + \tau \frac{\eta_{xx}}{\left(1+\eta_x^2\right)^{3/2}} \right) \delta\eta = 0 \tag{4}$$

which is zero for every arbitrary $\delta\eta$, if the quantity $(p_1 - p_2)|_{z=\eta} + \tau \dfrac{\eta_{xx}}{(1+\eta_x^2)^{3/2}}$ is equal to zero for $z = \eta(x,t)$. Such a condition coincides with the so-called dynamical boundary condition on the separation surface.

The following modal expansions for the functions φ_1, φ_2 are now introduced:

$$\begin{aligned} \varphi_1(x,z,t) &= \theta'(t)\,\varphi_{p1}(x,z) + \sum_{n=1}^{\infty} A_{1n}(t)\,\frac{\cosh[\kappa_n(H_1+z)]}{\cosh[\kappa_n H_1]}\cos(\kappa_n x), \\ \varphi_2(x,z,t) &= \theta'(t)\,\varphi_{p2}(x,z) + \sum_{n=1}^{\infty} A_{2n}(t)\,\frac{\cosh[\kappa_n(z-H_2)]}{\cosh[\kappa_n H_2]}\cos(\kappa_n x). \end{aligned} \tag{5}$$

Where $\kappa_n = \dfrac{n\pi}{B}$. The definitions of $\varphi_{p1}(x,z), \varphi_{p2}(x,z)$ are given in [9].

Substituting the expansions (5) into (1) and integrating with respect to z, the Lagrangian function L is obtained as a function of the vectors $\mathbf{A}_1(t) \equiv \{A_{1n}(t)\}$, $\mathbf{A}_2(t) \equiv \{A_{2n}(t)\}$, i. e., it has the following structure:

$$\begin{aligned} L \equiv & \rho_1 \left\{ \sum\nolimits_n \langle M_{1n}(\eta,\, x, y, t)\rangle \frac{dA_{1n}}{dt} + \langle N_1(\mathbf{A}_1, \eta,\, x, y, t)\rangle \right\} - \\ & - \rho_2 \left\{ \sum\nolimits_n \langle M_{2n}(\eta,\, x, y, t)\rangle \frac{dA_{2n}}{dt} + \langle N_2(\mathbf{A}_2, \eta,\, x, y, t)\rangle \right\} + \left\langle \tau\left(\sqrt{1+\eta_x^2} - 1\right)\right\rangle, \end{aligned} \tag{6}$$

where M_{1n}, M_{2n}, N_1, N_2 are nonlinear functions of the variables $\mathbf{A}_1, \mathbf{A}_2, \eta,\, x, y, t$. The functions M_{1n}, M_{2n}, N_1, N_2 are defined in [9].

3. Gravity-capillary waves. When the rigid motion is imposed to the container, waves are generated on the separation surface. The characteristics of such waves depend of course on the amplitude and the frequency of the imposed motion. In particular, experimental simulations showed that when the frequency of the exciting motion is close to the value of the first resonance a wave is generated on the separation surface which appears to be the superposition of a longer wave (whose wavelength can be assumed $\lambda_g \propto 2B = 1m$) and a shorter one (whose wavelength can be assumed $\lambda_c \propto 2B \times 10^{-2} = 0,01m$). When the exciting frequency is made closer and closer to the value of the first resonance, the amplitude of the longer wave grows and, as a consequence, breaking of the shorter wave is observed, causing the mixing of the two liquid layers.

Assuming that the capillary wave is a perturbation of the longer gravity wave, it is interesting to determine the conditions that make such a perturbation growing or exstinguishing. In order to do this in the framework of the variational formulation, the Taylor expansion of L with respect to η, η_x, up to the second order [13], has to be performed: such an expansion makes L depending explicitly on η, η_x. Then let us assume that the separation surface $\eta(x,t)$ can be represented as the sum of a gravity wave and a capillary wave;

$$\eta(x,t) = Q_1(t)\cos(\kappa_1 x) + Q_{n_c}(t)\cos(\kappa_{n_c} x), \tag{7}$$

where κ_{n_c} is a suitable capillary wavenumber. Analogous truncated sums are adopted for the velocity potentials (5). After having substituted (7) in the Taylor expansion of L, the operator

$\langle\cdot\rangle$ can finally be applied in order to express L as a function of the unknown time depending functions $Q_1(t), Q_{n_c}(t), A_{11}(t), A_{1n_c}(t), A_{21}(t), A_{2n_c}(t)$. The first variation of the functional $F(\varphi_1, \varphi_2, \eta)$ is equal to zero if the functions $Q_1(t), Q_{n_c}(t), A_{11}(t), A_{1n_c}(t), A_{21}(t), A_{2n_c}(t)$, considered as generalized coordinates of the motion, satisfy the Lagrange equations [10, 11, 14]:

$$
\begin{gathered}
\frac{dQ_1}{dt} - \kappa_1 \tanh(\kappa_1 H_1) A_{11} = 2B\theta_0\Omega\Im_{11}e^{j\Omega t} + \Psi_{11} + cc, \\
\frac{dQ_1}{dt} + \kappa_1 \tanh(\kappa_1 H_2) A_{21} = 2B\theta_0\Omega\Im_{21}e^{j\Omega t} + \Psi_{21} + cc, \\
\rho_1\frac{dA_{11}}{dt} - \rho_2\frac{dA_{21}}{dt} + \left[(\rho_1-\rho_2)g + \tau\kappa_1^2\right]Q_1 = -j2B\theta_0 I_1(\rho_1-\rho_2)ge^{j\Omega t} + \Gamma_1 + cc, \\
\frac{dQ_{n_c}}{dt} - \kappa_{n_c} \tanh(\kappa_{n_c} H_1) A_{11} = 2B\theta_0\Omega\Im_{1n_c}e^{j\Omega t} + \Psi_{1n_c} + cc, \\
\frac{dQ_{n_c}}{dt} + \kappa_{n_c} \tanh(\kappa_{n_c} H_2) A_{2n_c} = 2B\theta_0\Omega\Im_{2n_c}e^{j\Omega t} + \Psi_{2n_c} + cc, \\
\rho_1\frac{dA_{1n_c}}{dt} - \rho_2\frac{dA_{2n_c}}{dt} + \left[(\rho_1-\rho_2)g + \tau\kappa_{n_c}^2\right]Q_{n_c} == -j2B\theta_0 I_{n_c}(\rho_1-\rho_2)ge^{j\Omega t} + \Gamma_{n_c} + cc,
\end{gathered}
\tag{8}
$$

where $I_n \equiv \dfrac{\cos(n\pi)-1}{(n\pi)^2}, \Im_{1n} \equiv I_n\dfrac{\cosh(\kappa_n H_1)-1}{\cosh(\kappa_n H_1)}, \Im_{2n} \equiv I_n\dfrac{\cosh(\kappa_n H_2)-1}{\cosh(\kappa_n H_2)}$.

$\Psi_{11}, \Psi_{21}, \Gamma_1, \Psi_{1n_c}, \Psi_{2n_c}, \Gamma_{n_c}$ are nonlinear functions of $Q_1(t), Q_{n_c}(t), A_{11}(t), A_{1n_c}(t)$, $A_{21}(t), A_{2n_c}(t)$. Moreover, it has been assumed that $\theta(t) = -j\theta_0 e^{j\Omega t} + cc$, being θ_0 the amplitude and Ω the frequency of the imposed motion. Now let us assume that the forcing terms of the second three equations (8) are negligible because of the frequency Ω being very far from the capillary resonance frequency ω_{n_c}. ω_n is the resonance frequency of the n^{th} mode given by the expression $\omega_n = \sqrt{\dfrac{\tanh(\kappa_n H_1)\tanh(\kappa_n H_2)\left[(\rho_1-\rho_2)g + \tau\kappa_n^2\right]\kappa_n}{\rho_1\tanh(\kappa_n H_2) + \rho_2\tanh(\kappa_n H_1)}}$.

Let us consider the following scaling assumptions, respectively for the longer and the shorter wave:

$$
\begin{gathered}
t = \frac{t^*}{\Omega}, Q_1 = B\theta_0 Q_1^*, \frac{dQ_1}{dt} = B\theta_0\Omega\frac{dQ_1^*}{dt^*}, \\
Q_{n_c} = B\theta_0^2 Q_{n_c}^*, \frac{dQ_{n_c}}{dt} = B\theta_0^2\Omega\frac{dQ_{n_c}^*}{dt^*}, \\
A_{i1} = B^2\theta_0\Omega A_{i1}^*, \frac{dA_{i1}}{dt} = B^2\theta_0\Omega^2\frac{dA_{i1}^*}{dt^*}, \\
A_{in_c} = B^2\theta_0^2\Omega A_{in_c}^*, \frac{dA_{in_c}}{dt} = B^2\theta_0^2\Omega^2\frac{dA_{in_c}^*}{dt^*},
\end{gathered}
\tag{9}
$$

where $i = 1, 2$. It can be observed that the nonlinear terms are the sum of many quadratic and cubic terms, whose coefficients are different from zero if the wavenumbers of the considered modes satisfy the following conditions: $\kappa_n \pm \kappa_m \pm \kappa_l = 0, \kappa_n \pm \kappa_m \pm \kappa_l \pm \kappa_i = 0$, respectively for the quadratic and cubic terms. In these conditions n is the index of the equation, i. e., is the index of the evolving mode, while m, l, i are the indices of the modes interacting with the n^{th} mode. In the present case only two modes interact, then only the "cubic" condition can be satisfied with $n = m = n_c, l = i = 1$ or $n = m = 1, l = i = n_c$. Cubic interactions with $n = m = l = i$ (i. e. cubic auto-interactions) are also present but negligible. It follows that in the first three equations (8) there are terms like

$$\mathcal{C}_{11n_cn_c} A_{11} Q_{n_c}^2 = \theta_0^5 B^4 \Omega \mathcal{C}_{11n_cn_c} A_{11}^* Q_{n_c}^{*2}, \quad \mathcal{C}_{1111} A_{11} Q_1^2 = \theta_0^3 B^4 \Omega \mathcal{C}_{1111} A_{11}^* Q_1^{*2}$$

($\mathcal{C}_{nmli}$ are given coefficients), while in the second three there are terms like

$$\mathcal{C}_{n_cn_c11} A_{1nc} Q_1^2 = \theta_0^2 B^4 \Omega \mathcal{C}_{n_cn_c11}^* A_{1nc}^* Q_1^{*2}, \quad \mathcal{C}_{n_cn_cn_cn_c} A_{1n_c} Q_{n_c}^2 = \theta_0^6 B^4 \Omega \mathcal{C}_{n_cn_cn_cn_c} A_{1n_c}^* Q_{n_c}^{*2}$$

(having assumed $\mathcal{C}_{n_cn_c11}^* = \theta_0^2 \mathcal{C}_{n_cn_c11}$). Such terms give raise to leading order terms which are $O\left(\theta_0^2\right)$ in the first three equations (8),

$$\frac{dQ_1^*}{dt^*} - B\kappa_1 \tanh\left(\kappa_1 H_1\right) A_{11}^* = 2\Im_{11} e^{jt^*} + \theta_0^2 \Psi_{11}^* + cc,$$

$$\frac{dQ_1^*}{dt^*} + B\kappa_1 \tanh\left(\kappa_1 H_2\right) A_{21}^* = 2\Im_{21} e^{jt^*} + \theta_0^2 \Psi_{21}^* + cc, \tag{10}$$

$$\frac{dA_{11}^*}{dt^*} - \frac{\rho_2}{\rho_1}\frac{dA_{21}^*}{dt^*} + \left[\left(1 - \frac{\rho_2}{\rho_1}\right)\frac{g}{B\Omega^2}\right] Q_1^* = -j\frac{2I_1}{B\Omega^2}\left(1 - \frac{\rho_2}{\rho_1}\right) g e^{jt^*} + \theta_0^2 \Gamma_1^* + cc.$$

Assuming $Q_1^* = Q_1^{*0} + \theta_0^2 Q_1^{*2}, A_{11}^* = A_{11}^{*0} + \theta_0^2 A_{11}^{*2}, A_{21}^* = A_{21}^{*0} + \theta_0^2 A_{21}^{*2}$ in (10), three linear equations are obtained for the amplitude coefficients $Q_1^{*0}, A_{11}^{*0}, A_{21}^{*0}$ of the longer wave, which give the following periodic solutions:

$$Q_1^{*0} = \mathcal{Q}_1^{*0} e^{jt^*} + cc, \quad A_{11}^{*0} = \mathcal{A}_{11}^{*0} e^{jt^*} + cc, \quad A_{21}^{*0} = \mathcal{A}_{21}^{*0} e^{jt^*} + cc, \tag{11}$$

where $\mathcal{Q}_1^{*0}, \mathcal{A}_{11}^{*0}, \mathcal{A}_{21}^{*0}$ are known complex coefficients depending on the characteristic of the imposed motion. The second three equations (8) become, accounting for nonlinear cubic terms and scaling assumptions previously described,

$$\frac{dQ_{n_c}^*}{dt^*} - \kappa_{n_c}^* \tanh\left(\kappa_{n_c}^* \frac{H_1}{B}\right) A_{1n_c}^* = \alpha_{n_c}^* A_{1n_c}^* + \beta_{n_c}^* Q_{n_c}^*,$$

$$\frac{dQ_{n_c}^*}{dt^*} + \kappa_{n_c}^* \tanh\left(\kappa_{n_c}^* \frac{H_2}{B}\right) A_{2n_c}^* = \gamma_{n_c}^* A_{2n_c}^* + \delta_{n_c}^* Q_{n_c}^*, \tag{12}$$

$$\frac{dA_{1n_c}^*}{dt^*} - \frac{\rho_2}{\rho_1}\frac{dA_{2n_c}^*}{dt^*} + \left[\left(1 - \frac{\rho_2}{\rho_1}\right)\frac{g}{B\Omega^2} + \frac{\tau\kappa_{n_c}^{*2}}{\rho_1 B^3 \Omega^2}\right] Q_{n_c}^* = \varepsilon_{n_c}^* A_{1n_c}^* + \zeta_{n_c}^* A_{2n_c}^* + \eta_{n_c}^* Q_{n_c}^*.$$

The stability of the shorter wave can now be analysed substituting in (12) a solution like

$$Q^*_{n_c} = q^*_{n_c} e^{j\omega^* t^*}, \quad A^*_{1n_c} = a^*_{1n_c} e^{j\omega^* t^*}, \quad A^*_{2n_c} = a^*_{2n_c} e^{j\omega^* t^*} \tag{13}$$

obtaining then a dispersion relation between ω^* (i. e., the pulsation of the shorter wave) and the nondimensional wave number $\kappa^*_{n_c} = B\kappa_{n_c}$. The real coefficients $\alpha^*_{n_c}, \beta^*_{n_c}, \gamma^*_{n_c}, \delta^*_{n_c}, \varepsilon^*_{n_c}, \zeta^*_{n_c}, \eta^*_{n_c}$ depend on the coefficients $\mathcal{Q}^{*0}_1, \mathcal{A}^{*0}_{11}, \mathcal{A}^{*0}_{21}$ and their conjugate and influence the stability of the shorter wave.

4. Results and conclusions. The dispersion relation obtained substituting the solutions (13) in (12) can be expressed formally by

$$\Theta\left(\omega^*, \kappa^*_{n_c}; \alpha^*_{n_c}, \beta^*_{n_c}, \gamma^*_{n_c}, \delta^*_{n_c}, \varepsilon^*_{n_c}, \zeta^*_{n_c}, \eta^*_{n_c}\right) = 0. \tag{14}$$

In this paper we focus on the influence of the term $\eta^*_{n_c}$, i. e., we suppose that the parameters $\alpha^*_{n_c}, \beta^*_{n_c}, \gamma^*_{n_c}, \delta^*_{n_c}, \varepsilon^*_{n_c}, \zeta^*_{n_c}$ are negligible with respect to $\eta^*_{n_c}$. The following dispersion relation is then obtained:

$$\omega^* = \pm\sqrt{(\omega^*_{nc})^2 - \eta^*_{n_c}\frac{\kappa^*_{n_c}\tanh\left(\kappa^*_{n_c}\frac{H_1}{B}\right)\tanh\left(\kappa^*_{n_c}\frac{H_2}{B}\right)}{\frac{\rho_2}{\rho 1}\tanh\left(\kappa^*_{n_c}\frac{H_1}{B}\right)+\tanh\left(\kappa^*_{n_c}\frac{H_2}{B}\right)}}, \tag{15}$$

where ω^*_{nc} is the nondimensional resonance frequency for the n_c^{th} mode (the mode of the capillary wave). The expression for $\eta^*_{n_c}$ is

$$\eta^*_{n_c} = \kappa^*_{n_c}\frac{\pi^3}{2}\left(\mathcal{A}^{*0}_{11}\overline{\mathcal{A}}^{*0}_{11}\tanh\left(\frac{\pi}{B}H_1\right) + \frac{\rho_2}{\rho 1}\mathcal{A}^{*0}_{21}\overline{\mathcal{A}}^{*0}_{21}\tanh\left(\frac{\pi}{B}H_2\right)\right)\theta_0^2. \tag{16}$$

It is clear that if $(\omega^*_{nc})^2 - \eta^*_{n_c}\dfrac{\kappa^*_{n_c}\tanh\left(\kappa^*_{n_c}\frac{H_1}{B}\right)\tanh\left(\kappa^*_{n_c}\frac{H_2}{B}\right)}{\frac{\rho_2}{\rho 1}\tanh\left(\kappa^*_{n_c}\frac{H_1}{B}\right)+\tanh\left(\kappa^*_{n_c}\frac{H_2}{B}\right)} < 0$, the n_c^{th} mode excited by the longer wave is unstable. Then it is possible to obtain, for given amplitude and frequency of the longer wave, a set of the integers n_c related to unstable capillary wavemodes. Fig. 1 shows such set for the case $\theta_0 = 5^\circ$, $\Omega = 2{,}16\ rad/s$. It is interesting to see that the minimum of the curve is given near the value $\widetilde{n}_c \simeq 24$, corresponding to the wavelength $\lambda_c = \dfrac{2\pi}{\kappa_{n_c}} = 4cm$. This value coincides with the one given by the formula $\widetilde{n}_c \simeq \dfrac{B}{\pi}\sqrt{\dfrac{(\rho_1 - \rho_2)\,g}{\tau}}$ (with $\rho_1 = 1000\ kg/m^3, \rho_2 = 840\ kg/m^3, \tau = 0{,}07\ N/m, B = 0{,}5\ m$) obtained assuming that the quantity $\left(\dfrac{\omega^*_{nc}}{\kappa^*_{n_c}}\right)^2$ (the phase celerity of the capillary wave) attains a minimum. Then $\widetilde{n}_c$ indicates the most unstable mode [15].

From (15) it is possible to obtain, for a given mode and frequency of the exciting wave, the minimum value of the amplitude of the exciting wave in correspondence of which the capillary wave becomes unstable. This value is plotted in Fig. 2 for different values of the forcing

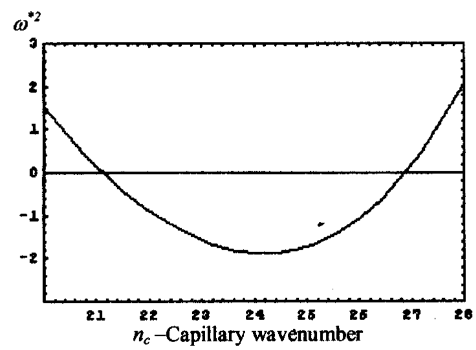

Fig. 1. Stability map for $\theta_0 = 5°, \Omega = 2,16\ rad/s$.

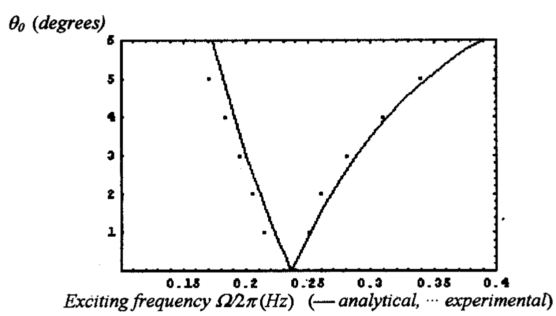

Fig. 2. Experimental and analytical stability map for $n = 24$ (maximum instability mode).

frequency and compared with experimental observations. The agreement between the experimental and the predicted values is very good. It can be observed that the closer is the frequency value to the resonance value, the lower is the minimum value of θ_0 which causes instability.

In conclusion it can be stated that the application of the variational formulation for studying the evolution of gravity-capillary waves permitted to obtain the dynamical system (8) which determines the evolution of the fluid motion. This dynamical system is interesting because it shows the different interactions which act during the fluid motion. In particular, it was shown that there are two interactions: the first is the energy transfer from the imposed rigid motion to the longer wave, the second is the energy transfer from the longer wave to the capillary one, due to the nonlinear cubic interactions. This energy transfer, for a given capillary mode and a given frequency of the rigid motion, occurs only when the amplitude of the rigid motion exceeds a certain threshold. This fact confirms a result which is well known for the gravity-capillary free-surface waves [1].

Future work has, however, to be made in order to account for dissipative effects and more complex wavemodes interactions.

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Received 15.11.2002