UDC 517.929.51

ON STABILITY OF A NONLINEAR PENDULUM

ПРО СТІЙКІСТЬ НЕЛІНІЙНОГО МАЯТНИКА

E. Kengne

Univ. Dschang
P.O. Box 173, Dschang, Cameroon
e-mail: ekengne6@yahoo.fr.

By means of Liapounov's method, we establish a sufficient condition for stability of a nonlinear pendulum.

Методом Ляпунова знайдено достатню умову стійкості руху нелінійного маятника.

1. Introduction. Various interesting dynamical processes described by ordinary differential equations with many-dimensional rapid and slow variables have been studied by several authors [1–4]. Using the methods of works [5–8], Matviitchuk [9, 10] established and proved sufficient criteria of stability for the class of the mentioned dynamical processes.

The purpose of this paper is to establish by the means of Liapounov's method [11] a sufficient criterion for stability of a nonlinear pendulum. In Section 2, we obtain a differential system for oscillations of a nonlinear pendulum. The obtained system is a dynamic system with two rapid variables and one slow variable. In Section 3, we find the interval of stability of nonlinear pendulum described in Section 2.

2. Mathematical considerations. Throughout this paper we use the following notations: g is the acceleration of gravity; α the angular deviation from vertical; $x(\tau)$ the slowly variation of the length of the thread; $\tau = \mu t$, where μ is a small positive quantity [2]. We assume that the length of the thread changes not by means of external forces, but by means of the self energy of the system, that is, the rate of change of the length of the thread depends only on α, $\dot{\alpha}$ and x. We also assume that the rate of the plastic deformation of the thread is very small and proportional to the tension of the thread, that is, $\dot{x} = \lambda\mu P$, where P is the tension of the thread and $\lambda\mu$ a small coefficient of deformation. If we denote by x_0 the initial length of the thread then we can write $x(\tau) = x_0 + \xi(\tau)$, with $0 \leq \xi \leq K - x_0$, where K is a positive constant. The oscillations of such a pendulum are governed by the dynamic system [1]

$$\begin{aligned}
\frac{d\alpha}{dt} &= \frac{p}{m(x_0+\xi)^2}, \\
\frac{dp}{dt} &= -mg(x_0+\xi)\alpha, \\
\frac{d\xi}{dt} &= \lambda\mu\left[mg\cos\alpha + \frac{p^2}{m(x_0+\xi)^3}\right],
\end{aligned} \tag{1}$$

defined in the region

$$D = \{\alpha, \xi, p, \mu, t : \ -\pi < \alpha < \pi,\ 0 \leq \xi \leq K - x_0,\ -p_0 \leq p \leq p_0,\ p_0 = \text{const} > 0,$$

$$0 < \mu < \mu_0,\ \mu_0 = \cos nt > 0,\ 0 \leq t \leq \mu^{-1}\} .$$

A pendulum whose oscillations are governed by system (1) is called a nonlinear pendulum.

By setting $\varepsilon = \lambda\mu$, system (1) under the transformation $t = t/\varepsilon$ is reduced to the following system with a small positive parameter along the two first derivatives:

$$\begin{aligned} \varepsilon\frac{d\alpha}{dt} &= \frac{p}{m(x_0+\xi)^2}, \\ \varepsilon\frac{dp}{dt} &= -mg(x_0+\xi)\alpha, \\ \frac{d\xi}{dt} &= mg\cos\alpha + \frac{p^2}{m(x_0+\xi)^3}. \end{aligned} \tag{2}$$

At every point (α, p, ξ) of the phase space (or the region where system (2) is defined) system (2) defines the vector of the phase velocity

$$v(\alpha, p, \xi) = \left(\frac{1}{\varepsilon}\frac{p}{m(x_0+\xi)^2}, \frac{-mg(x_0+\xi)\alpha}{\varepsilon}, mg\cos\alpha + \frac{p^2}{m(x_0+\xi)^3}\right). \tag{3}$$

As it is seen, the third component of the phase velocity (3) possesses a finite value, and, generally speaking, the two first components are infinitely large. Therefore for the phase portrait of system (2) we notice the presence of rapid movements and slow movements. Nevertheless the rapid movements occur far from the region [12]

$$\Gamma = \{\alpha, p, \xi : \alpha = 0,\ p = 0,\ 0 \leq \xi \leq K - x_0\}$$

almost in parallel to the plane αp, and the slow movements occur near the region Γ.

System

$$\begin{aligned} \varepsilon\frac{d\alpha}{dt} &= \frac{p}{m(x_0+\xi)^2}, \\ \varepsilon\frac{dp}{dt} &= -mg(x_0+\xi)\alpha \end{aligned}$$

in which ξ is considered as a parameter is called the system of rapid movements, corresponding to system (2), and the variables α and p are called rapid variables [12] of system (2). The variable ξ is called the slow variable [12] of system (2). Therefore, (1) is a dynamic system with many-dimensional rapid and slow variables (two rapid variables, α and p, and one slow variable, ξ).

Because

$$f_1(\alpha, p, \xi) = \frac{p}{m(x_0+\xi)^2}, \quad f_2(\alpha, p, \xi) = -mg(x_0+\xi)\alpha$$

and

$$f_3(\alpha, p, \xi) = mg\cos\alpha + \frac{p^2}{m(x_0+\xi)^3}$$

are continuous functions in $\tilde{\Omega} = \{-\pi < \alpha < \pi, 0 \leq \xi \leq K - x_0, -p_0 \leq p \leq p_0, p_0 = \text{const} > > 0\}$, we have the following conclusion [12–14]: if $\varphi(t, \varepsilon) = (\alpha(t, \varepsilon), p(t, \varepsilon), \xi(t, \xi))$ is a solution of system (1) (with $\varepsilon = \lambda\mu$) verifying the initial condition $\varphi(0, \varepsilon) = \varphi_0$ and $\varphi_0(t)$ a solution of the singular system

$$\begin{aligned} \frac{d\alpha}{dt} &= \frac{p}{m(x_0+\xi)^2}, \\ \frac{dp}{dt} &= -mg(x_0+\xi)\alpha, \\ \frac{d\xi}{dt} &= 0 \end{aligned} \tag{4}$$

(this system is obtained from (1) by setting $\mu = 0$) corresponding to system (1), defined in the region $0 \leq t \leq T$ and verifying the same initial condition $\varphi_0(0) = \varphi_0$, then

$$\varphi(t, \varepsilon) = \varphi_0(t) + R_0(t, \varepsilon),$$

where $R_0(t, \varepsilon)$ tends uniformly to zero with respect to $t \in [0, T]$ as $\varepsilon \to 0$.

If we consider α and $\dot{\alpha}$ to be very small, then $\cos\alpha \approx 1 - \dfrac{\alpha^2}{2}$. Therefore, system (1) relatively coincides with the nonlinear system

$$\begin{aligned} \frac{d\alpha}{dt} &= \frac{p}{m(x_0+\xi)^2}, \\ \frac{dp}{dt} &= -mg(x_0+\xi)\alpha, \\ \frac{d\xi}{dt} &= \lambda\mu\left[mg - \frac{1}{2}mg\alpha^2 + \frac{p^2}{m(x_0+\xi)^3}\right]. \end{aligned} \tag{5}$$

Systems (4) and (5) play an important role in the investigation of the stability of nonlinear pendulum whose oscillations are governed by system (1).

3. Stability of nonlinear pendulum. The solutions of singular system (4) read

$$\alpha(t) = a\cos(\omega t + \beta), \; p(t) = -ma(x_0+\xi)^2\omega\sin(\omega t + \beta), \; \xi(t) = \text{const}, \tag{6}$$

where a is the amplitude of oscillations of the pendulum; $\omega = \sqrt{g/(x_0+\xi)}$, $\beta = \text{const}$. With the help of the singular solution (6), we write system (1) in the form [15]

$$\begin{aligned} \frac{d\xi}{dt} &= \tilde{\mu}Z(t, \xi, \varphi(t, \xi, a, \beta)) + R_1(t, \xi, \varphi(t, \xi, a, \beta)), \\ \frac{d\alpha}{dt} &= \tilde{\mu}Z_1(t, \xi, \varphi(t, \xi, a, \beta)) + R_2(t, \xi, \varphi(t, \xi, a, \beta)), \\ \frac{dp}{dt} &= \tilde{\mu}Z_2(t, \xi, \varphi(t, \xi, a, \beta)) + R_3(t, \xi, \varphi(t, \xi, a, \beta)), \end{aligned} \tag{7}$$

where

$$\varphi(t,\xi,a,\beta) = (a\cos(\omega t+\beta), -ma(x_0+\xi)^2\omega\sin(\omega t+\beta)),$$

$$Z(t,\xi,\varphi(t,\xi,a,\beta)) = \lambda m\{g\cos[a\cos(\omega t+\beta)] + (x_0+\xi)a^2\omega^2\sin^2(\omega t+\beta)\},$$

$$Z_1(t,\xi,\varphi(t,\xi,a,\beta)) = \frac{-3a}{2\sqrt{x_0+\xi}}\sin^2(\omega t+\beta)Z(t,\xi,\varphi(t,\xi,a,\beta)),$$

$$Z_2(t,\xi,\varphi(t,\xi,a,\beta)) = \frac{\omega t - \frac{3}{2}\sin(\omega t+\beta)}{2(x_0+\xi)}Z(t,\xi,\varphi(t,\xi,a,\beta)),$$

$$R_1(t,\xi,\varphi(t,\xi,a,\beta)) = \lambda\mu\left(mg\cos\alpha + \frac{p^2}{m(x_0+\xi)^3}\right) - \tilde{\mu}Z(t,\xi,\varphi(t,\xi,a,\beta)),$$

$$R_2(t,\xi,\varphi(t,\xi,a,\beta)) = \frac{p}{m(x_0+\xi)^2} + \frac{3a\mu}{2(x_0+\xi)^2}\sin^2(\omega t+\beta)Z(t,\xi,\varphi(t,\xi,a,\beta)),$$

$$R_3(t,\xi,\varphi(t,\xi,a,\beta)) = -\Bigg\{mg(x_0+\xi)\alpha + \frac{\tilde{\mu}}{2(x_0+\xi)}\left(\omega t - \frac{3\sin(\omega t+\beta)}{2}\right)\times$$

$$\times Z(t,\xi,\varphi(t,\xi,a,\beta))\Bigg\}.$$

Let us group, together with system

$$\begin{aligned}\frac{d\xi}{dt} &= \tilde{\mu}Z(t,\xi,\varphi(t,\xi,a,\beta)),\\ \frac{d\alpha}{dt} &= \tilde{\mu}Z_1(t,\xi,\varphi(t,\xi,a,\beta)),\\ \frac{dp}{dt} &= \tilde{\mu}Z_2(t,\xi,\varphi(t,\xi,a,\beta)),\end{aligned} \tag{8}$$

Liapounov's function

$$V(t,\xi,\alpha,p,\tilde{\mu}) = e^{-\sin(e^{-t})}\left[2^{-1}\xi^2 + \tilde{\mu}(\sin^2\alpha + \sin^2 p)\right]$$

in the domain D for which the derivative along the system (6), under the conditions

$$|\mu-\tilde{\mu}| < \mu_1,\ \mu_1 \le \lambda mK\frac{\gamma-\tilde{\mu}K_0}{g+Kp_0^3},\ K_0 = mK[g(\pi+3)+Kp_0^2],\ \gamma = \text{const} > 0,$$

has the estimate

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial \xi}\frac{d\xi}{dt} + \frac{\partial V}{\partial \alpha}\frac{d\alpha}{dt} + \frac{\partial V}{\partial p}\frac{dp}{dt} \leq$$

$$\leq e^{-t}\cos(e^{-t})V(t,\xi,\alpha,p,\tilde{\mu}) + e^{-\sin(e^{-t})}(\mu K_1 + \gamma),$$

$$K_1 = \lambda mg(2+a^2)\left[K + (2x_0)^{-1}\left(2\sqrt{g} + 5\tilde{\mu}\right)\right] + \frac{p_0}{mx_0}.$$

That is,

$$\frac{dV}{dt} \leq e^{-t}\cos(e^{-t})V(t,\xi,\alpha,p,\tilde{\mu}) + e^{-\sin(e^{-t})}(\mu K_1 + \gamma). \tag{9}$$

Let $s(t,\tilde{\mu})$ be a function defined by

$$s(t,\tilde{\mu}) = \int_{t_0}^{t} e^{\sin e^{-t_0} - \sin e^{-s}}(\tilde{\mu}K_1 + \gamma)ds.$$

Let us consider the following Cauchy problem for the congruence equation [9, 10, 16]:

$$\begin{aligned} \frac{dy}{dt} &= e^{-t}\cos(e^{-t})[y + s(t,\tilde{\mu})], \\ y(t_0) &= y_0 \geq V_0 = V(t_0,\xi(t_0),\alpha(t_0),p(t_0),\tilde{\mu}). \end{aligned} \tag{10}$$

Cauchy problem (10) possesses a bounded and continuous solution

$$\tilde{y}(t) = e^{\sin e^{-t_0} - \sin e^{-t}}\left[y_0 + (\tilde{\mu}K_1 + \gamma)(t - t_0)\right] - s(t,\tilde{\mu}).$$

Using Theorem 9.5 of work [8] on the differential inequalities and by virtue of the inequality (9), the following estimate holds for the Liapounov's function (8) along the solution of system (1) for the values of $t \in \left[0,\mu^{-1}\right] \cap \left[0,\tilde{\mu}^{-1}\right]$:

$$V(t,\xi(t),\alpha(t),p(t),\tilde{\mu}) \leq \tilde{y}(t) + s(t,\tilde{\mu}).$$

That is,

$$V(t,\xi(t),\alpha(t),p(t),\tilde{\mu}) \leq e^{\sin e^{-t_0} - \sin e^{-t}}\left[y_0 + (\tilde{\mu}K_1 + \gamma)(t - t_0)\right].$$

By virtue of the fact that $s(t,\tilde{\mu})$ is a bounded quantity on $\left[0,\mu^{-1}\right] \cap \left[0,\tilde{\mu}^{-1}\right]$ and

$$y(t) \underset{t\to+\infty}{\rightarrow} y_0 e^{\sin e^{-t_0}},$$

we finally have for $t \in \left[0, \mu^{-1}\right] \cap \left[0, \tilde{\mu}^{-1}\right]$ the estimate

$$0 \leq V(t, \xi(t), \alpha(t), p(t), \tilde{\mu}) \leq e^{\sin e^{-t_0}} \left[y_0 + (\tilde{\mu} K_1 + \gamma) \left(\frac{1}{\breve{\mu}} - t_0 \right) \right],$$

where $\breve{\mu} = \max(\mu, \tilde{\mu})$. Consequently, the processes (1), subordinate to condition (5), is stable [10] on the segment $I = \left[0, \mu^{-1}\right] \cap \left[0, \tilde{\mu}^{-1}\right]$. In other words, the nonlinear pendulum described in Section 2 is stable on the region $I = \left[0, \mu^{-1}\right] \cap \left[0, \tilde{\mu}^{-1}\right]$.

1. *Bogolioubov N.N., Zubarev D.N.* Method of asymptotic approximations for systems with revolutionary phase and its application to the motion of charged particles in the magnetic field // Ukr. Math. J. — 1955. — **7**, № 1. — P. 5–17.

2. *Volossof V.M.* Averaging in the systems of ordinary differential equations // Ibid. — 1962. — **17**, № 6. — P. 3–126.

3. *Alfven H.* On the motion of charged particle in a magnetic field // Ark. mat., astron. och fys. — 1940. — **37A**, № 22. — P. 3–10.

4. *Mitchenko E. F., Pontriaguin L.S.* Relaxation oscillations and differential equations containing a small parameter with the senior derivative // Calcutta Math. Soc. The Golden Jubilee Commemoration Volume (1958–1959). — Calcutta, 1963. — P. 605–626.

5. *Kamke E.* Differentialgleichungen reeler Funktionen. — Leipzig: Akad. Verlagsgesellschaft, 1930. — 365 S.

6. *Lakshmikantham V., Leela S.* Differential and integral inequalities. Theory and applications. — New York; London: Acad. Press, 1969. — Vols 1, 2. — 569 p.

7. *Rao M.R.M.* A note on an integral inequality // J. Indian Math. Soc. — 1963. — **24**, № 2. — P. 69–71.

8. *Sharski J.* Differential inequalities. — Warzawa: Pánst. wyd-wo nauk., 1967. — 256 p.

9. *Matviitchuk K.S.* Remarks on the method of comparison for systems of differential equations with rapidly rotating phase // Ukr. Math. J. — 1982. — **34**, № 4. — P. 456–462.

10. *Matviitchuk K.S.* On the question of the stability of systems connecting solids with the shock elements // Ibid. — 1983. — **19**, № 5. — P. 100–106.

11. *Pouch N., Abetsc P., and Lalia M.* Direct method of Liapounov in the theory of stability. — Moscow, 1980. — 300 p.

12. *Mitchenko E.F., Rosov N.K.* Differential equations with a small parameter and relaxation oscillations. — Moscow, 1975. — 148 p.

13. *Lefschetz S.* Geometrical theory of differential equations. — Moscow, 1961.

14. *Andronov A.A., Vitt A.A., and Haikin S.E.* Theory of vibrations. — Moscow: Fizmatgiz, 1959.

15. *Filatov A.N.* Averaging methods in differential and integro-differential equations. — Tashkient: Fan, 1971. — 279 p.

16. *Belmann R., Cook K.L.* Difference differential equations. — Moscow: Mir, 1967. — 548 p.

Received 05.07.2002