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DETERMINATION OF THE BOUNDARIES BETWEEN THE DOMAINS OF STABILITY AND INSTABILITY FOR THE HILL'S EQUATION

ВИЗНАЧЕННЯ МЕЖ МІЖ ОБЛАСТЯМИ СТАБІЛЬНОСТІ ТА НЕСТАБІЛЬНОСТІ ДЛЯ РІВНЯННЯ ХІЛЛА

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Stability problem for the Hill's equation containing two parameters is analyzed with computer algebra system $Mathematica$. The characteristic constant is found as a series expansion in powers of a small parameter e. It has been shown that the domains of instability are located only between the curves $a = a(e)$ on the $a-e$ plane crossing the $e = 0$ axis at the points $a = (2k-1)^2/4,\ k = 1, 2, 3\ldots$. The corresponding curves are found as power series in e with accuracy $O(e^6)$.

Проблема стабільності для рівняння Хілла, яке містить два параметри, вивчається за допомогою комп'ютерної системи $Mathematica$. Знайдено характеристичну константу в термінах ряду відносно степенів малого параметра e. Показано, що області стабільності і нестабільності знаходяться тільки на площині $a - e$ між кривими $a = a(e)$, що перетинають вісь $e = 0$ в точках $a = (2k-1)^2/4,\ k = 1, 2, 3\ldots$. Знайдено відповідні криві як ряди за степенями e з точністю $O(e^6)$.

Introduction. The Hill's equation is the second order linear differential equation of the form

$$z''(t) + p(t)\, z(t) = 0, \tag{1}$$

where $p(t)$ is a periodic function with a period T (i. e. $p(t+T) = p(t)$ for all t). It describes dynamical systems with intrinsic periodicity and has a lot of applications [1,2]. Our interest to the Hill's equation arises because an equation of this type occurs in the study of stability of exact symmetrical solutions of Newton's gravitational many-body problem [3–5]. In this case equation (1) has the form

$$z''(t) + \frac{a + e\cos t}{1 + e\cos t}\, z(t) = 0, \tag{2}$$

where a and e are some nonnegative parameters. Equation (2) is just a homogeneous equation and, hence, all its solutions have the same stability, which is just the stability of its trivial solution. In this sense we can refer to stability of the Hill's equation.

If $e = 0$ equation (2) reduces to the equation for a simple harmonic oscillator of frequency $\sqrt{a}$ and its general solution is

$$z = C_1 \cos(\sqrt{a}\, t) + C_2 \sin(\sqrt{a}\, t). \tag{3}$$

This solution is bounded and oscillatory, and we can say that in case where $e = 0$ the trivial solution of (2) is stable. For $e > 0$ equation (2) may be regarded as a model of a mechanical system which can be described by a simple harmonic oscillator of natural frequency $\sqrt{a}$ that is subjected to an internally imposed forcing of period $T = 2\pi$. It is known that in such systems a parametric resonance may occur when the function $z(t)$ increases unboundedly as $t \to \infty$. Therefore, for some values of the parameter a, the trivial solution of (2) may be unstable even for very small e. To find the corresponding values of a and e, we should calculate the characteristic exponents for equation (2). This problem for a general equation (1) is discussed in detail in [1], where a number of stability criteria have been formulated. But some of these criteria, for instance, Liapunov's integral criterion or Zhukovskii criterion give too large a domain of instability for equation (2). Another ones are quite cumbersome and it is difficult to use them. So the main aim of the present paper is to find the domains of instability for equation (2) in the $a - e$ plane and to determine their boundaries. Solving this problem is connected with cumbersome analytical calculations that can be reasonably done only with a computer software. In the present paper, all calculations are done with computer algebra system Mathematica that is a very powerful tool for doing both numerical and symbolic calculations [6].

General stability theory for the Hill's equation. It is easy to see that equation (2) is equivalent to the first order system of the form

$$\frac{dx}{dt} = A(t)\, x, \tag{4}$$

where x is a vector with components $x_1 = z$ and $x_2 = z'$ and $A(t)$ is the 2×2 periodic matrix

$$A(t) = \begin{pmatrix} 0 & 1 \\ -\dfrac{a + e\cos t}{1 + e\cos t} & 0 \end{pmatrix}.$$

The principal fundamental matrix for the system (4) is

$$X(t) = \begin{pmatrix} z_1 & z_2 \\ z_1' & z_2' \end{pmatrix},$$

where $z_1(t)$ and $z_2(t)$ are linearly independent solutions of equation (2) satisfying the following initial conditions:

$$\begin{aligned} z_1(0) &= 1, \qquad z_1'(0) = 0, \\ z_2(0) &= 0, \qquad z_2'(0) = 1. \end{aligned} \tag{5}$$

The characteristic multipliers ρ for the system (4) are defined as eigenvalues of the matrix $X(T) \equiv X(2\pi)$ and, hence, are given by the characteristic equation

$$\det(X(2\pi) - \rho I_2) = 0, \tag{6}$$

where I_2 is a 2×2 identity matrix. Using (2) and (5) it is easy to prove that

$$\det X(t) = z_1 z_2' - z_1' z_2 \equiv 1.$$

So characteristic equation (6) can be rewritten as

$$\rho^2 - 2B\rho + 1 = 0,$$

where the characteristic constant B is

$$B = (z_1(2\pi) + z_2'(2\pi))/2. \tag{7}$$

Thus, the characteristic multipliers $\rho_{1,2}$ are functions of the single parameter B and are given by

$$\rho_{1,2} = B \pm \sqrt{B^2 - 1}. \tag{8}$$

As soon as the parameter B has been found, the characteristic exponents $\mu_{1,2}$ are easily defined from the relationship

$$\rho_{1,2} = \exp(\mu_{1,2}\, T) = \exp(2\pi\, \mu_{1,2}). \tag{9}$$

It is evident that the next relations are true:

$$\rho_1 \rho_2 = 1, \ \rho_1 + \rho_2 = 2B. \tag{10}$$

As a consequence, we may write the next relationships for the characteristic exponents $\mu_{1,2}$

$$\mu_1 + \mu_2 = 0, \ \cosh(\mu_1 T) = \cosh(2\pi\mu_1) = B. \tag{11}$$

Thus, to determine $\rho_{1,2}$ and $\mu_{1,2}$ we should find two linear independent solutions $z_1(t)$ and $z_2(t)$ of equation (2) satisfying the initial conditions (5). Although these solutions are not found yet, we can characterize properties of $\rho_{1,2}$ and $\mu_{1,2}$ in terms of the characteristic constant B.

a) If $B > 1$, then, according to (8), the characteristic multipliers $\rho_{1,2}$ are both real and positive ($\rho_1 > 1 > \rho_2 > 0$). Consequently (see (9)–(11)), $\mu_1 = \dfrac{\ln \rho_1}{2\pi}$ is real and positive too, while $\mu_2 = -\mu_1$ is real and negative. From the theory of linear differential equations with periodic coefficients [1,7] we can deduce that the general solution of equation (2), in this case, is

$$z = C_1 \exp(\mu_1 t) f_1(t) + C_2 \exp(-\mu_1 t) f_2(t), \tag{12}$$

where $f_1(t)$ and $f_2(t)$ are periodic functions with period $T = 2\pi$. The solution (12) is not periodic and, in general, $|z| \to \infty$ as $t \to \infty$. So, in the case of $B > 1$, the trivial solution of equation (2) is unstable.

b) If $B < -1$ then the characteristic multipliers $\rho_{1,2}$ are both real and negative ($\rho_2 < -1 < \rho_1 < 0$) and again there is one characteristic exponent with positive real part $\left(\mu_2 = \beta - i\pi/T = \beta - i/2 \text{ and } \beta = \operatorname{Re}\mu_2 = \dfrac{\ln|\rho_2|}{2\pi} > 0\right)$. Now, the general solution of equation (2) is

$$z = C_1 \exp(-\beta t) g_1(t) + C_2 \exp(\beta t) g_2(t), \tag{13}$$

where $g_1(t)$ and $g_2(t)$ are periodic functions with period $2T = 4\pi$. The solution (13) is not periodic and, in general, $|z| \to \infty$ as $t \to \infty$. So, in the case of $B < -1$, the trivial solution of equation (2) is also unstable.

c) If $|B| < 1$, then $\rho_{1,2}$ are both complex-valued with unit magnitude (i. e. $|\rho_{1,2}| = 1$) and can be represented as

$$\rho_{1,2} = \exp(\pm 2\pi\sigma i). \tag{14}$$

Consequently, the characteristic exponents $\mu_{1,2}$ are both imaginary, i. e., $\mu_{1,2} = \pm i\sigma$, where σ is a real number. The corresponding solution of equation (2) is

$$z = C_1 \operatorname{Re}(\exp(i\sigma t) f(t)) + C_2 \operatorname{Im}(\exp(i\sigma t) f(t)), \tag{15}$$

where $f(t)$ is a complex-valued periodic function with period $T = 2\pi$. Solution (15) is bounded and oscillatory, and so the trivial solution of (2) is stable.

d) In the case of $B = 1$ there is a single characteristic multiplier $\rho_1 = 1$ and a single characteristic exponent $\mu_1 = 0$. It can be regarded as the limit $\mu_1 \to 0$ in case a) or $\sigma \to 0$ in case c). According to (12), (15), there should exist at least one periodic solution of equation (2) with period $T = 2\pi$. But, in general, there may exist an additional solution growing linearly with $t \to \infty$.

e) In case of $B = -1$ there is again a single characteristic multiplier $\rho_1 = -1$ and a single characteristic exponent $\mu_1 = i\pi/T = i/2$. It can be regarded as the limit $\mu_1 \to i/2$ in case b) or $\sigma \to 1/2$ in case c). Consequently, there exists a periodic solution of equation (2) with period $2T = 4\pi$. In addition there may exist a solution growing linearly with $t \to \infty$.

Thus, the inequality $|B| < 1$ is a necessary and sufficient condition for stability of the trivial solution of equation (2). The equations $B = \pm 1$ determine boundaries of the domains of instability in the $a - e$ plane.

Calculation of the characteristic constant with the method of a small parameter. The function $p(t) = \dfrac{a + e\cos t}{1 + e\cos t}$ may be represented as a series expansion in powers of e

$$p(t) = \sum_{k=0}^{\infty} p_k(t) e^k, \tag{16}$$

where

$$p_0 = a,\ p_k(t) = (a-1)(-\cos t)^k,\ \ k = 1, 2, 3,\ \dots\ .$$

The series (16) converges for any t in the domain $|e| < 1$ and $p_k(t)$ are continuous functions. So, according to Poincare – Liapunov theorem [8 – 10], the general solution of equation (2) may be also represented as the power series

$$z(t) = \sum_{k=0}^{\infty} z_k(t) e^k \tag{17}$$

that converges in the domain $|e| < 1$ for any t and $z_k(t)$ are continuous functions.

To find differential equations determining the functions $z_k(t)$, let us substitute expansions (16), (17) into equation (2). Then, equating the coefficients of e^k, $k = 0, 1, 2, \ldots$, in the left and right-hand sides of the equation we obtain the following system of differential equations:

$$z_0'' + a z_0 = 0, \quad z_k'' + a z_k = (1-a) \sum_{n=1}^{k} (-\cos t)^n \, z_{k-n}, \qquad k = 1, 2, \ldots . \tag{18}$$

Two linearly independent solutions z_0 of the first equation in (18) must satisfy the initial conditions (5). The corresponding functions are easily found from (3) and are given by

$$z_0(t) = \cos(\sqrt{a}\, t), \quad z_0(t) = \frac{1}{\sqrt{a}} \sin(\sqrt{a}\, t). \tag{19}$$

Initial conditions for the functions z_k, $k = 1, 2, \ldots$, may be written then as

$$z_k(0) = z_k'(0) = 0. \tag{20}$$

Solving the second equation in (18) with initial conditions (20) we find the recurrence relationship for calculating the functions z_k, $k = 1, 2, \ldots$,

$$z_k = \frac{1-a}{\sqrt{a}} \int_0^t \sin(\sqrt{a}\,(t-\tau)) \sum_{n=1}^{k} (-\cos\tau)^n \, z_{k-n}\, d\tau. \tag{21}$$

Realizing the calculations according to (21) with initial functions z_0 given in (19) we find the parameter B as a power series in e

$$\begin{aligned} B = \; & \cos(2\pi\sqrt{a}) - \frac{3(a-1)\pi\sqrt{a}}{2(4a-1)} \sin(2\pi\sqrt{a})\, e^2 - \\ & - \frac{3(a-1)\pi\sqrt{a}}{32(4a-1)^3} \left(12\sqrt{a}\,(1-5a+4a^2)\,\pi\cos(2\pi\sqrt{a}) + \right. \\ & + \left.(14 - 55a + 140a^2)\, \sin(2\pi\sqrt{a})\right) e^4 + \frac{3(a-1)\pi\sqrt{a}}{128(4a-9)(4a-1)^5} \left(-6\sqrt{a}\,(-126 + \right. \\ & + 1181a - 4739a^2 + 10164a^3 - 8720a^4 + 2240a^5)\,\pi\cos(2\pi\sqrt{a}) + (462 + 1536a^6\pi^2 + \\ & + 47a^2(455+48\pi^2) - 64a^5(385+114\pi^2) - a(3457+216\pi^2) - 12a^3(4725+674\pi^2) + \\ & + \left. 16a^4(4795+738\pi^2)\, \sin(2\pi\sqrt{a})\right) e^6, \end{aligned} \tag{22}$$

where the error term is $O(e^7)$. It should be noticed that series (22) converges in the domain $|e| < 1$ for any a. Calculating the parameter B we can easily find the characteristic multipliers $\rho_{1,2}$ according to (8).

Domains of instability of the trivial solution. It follows from (22) that for $e = 0$, when equation (2) reduces to a simple harmonic oscillator equation, the characteristic constant is $B = \cos(2\pi\sqrt{a})$. This means that $|B| < 1$ for all values of a, except for

$$a = \frac{k^2}{4}, \qquad k = 0, 1, 2, \ \ldots \ , \tag{23}$$

and there is stability of equation (2). The points (23) also correspond to stable behavior of the trivial solution of (2) because there are two linearly independent oscillatory solutions (3) in every such point. But, in these points, we have $B = \pm 1$ and so they can belong to the boundaries between the domains of stability and instability.

Since $B = B(a, e)$ is an analytic function of a and e, we may expect that, if an instability occurs for $e > 0$, it must do so in a vicinity of points (23). And the boundaries of the domains of instability on the $a - e$ plane are some curves $a = a(e)$ crossing the $e = 0$ axis at these points. So, for sufficiently small e we can represent the stability boundaries $a = a(e)$ as a power series in e,

$$a = \frac{k^2}{4} + a_1 e + a_2 e^2 + \ \ldots \ , \qquad k = 0, 1, 2 \ \ldots \ . \tag{24}$$

To find the coefficients a_n, let us substitute (24) into (22) and expand the characteristic constant B in powers of e. Then the coefficients a_n can be determined from the equations $B = 1$ and $B = -1$. Equating the coefficients of e^n, $n = 1, 2, 3, \ \ldots$, to zero, we obtain a system of algebraic equations giving two curves crossing the $e = 0$ axis in every point $a = (2k - 1)^2/4$, $k = 1, 2, 3, \ \ldots$. They are

$$\begin{gathered}
a = \frac{1}{4} \mp \frac{3}{8} e + \frac{15}{128}\, e^2 \mp \frac{45}{2048}\, e^3 + \frac{885}{32768}\, e^4 \mp \frac{6105}{524288}\, e^5 + \frac{220305}{16777216}\, e^6, \\
a = \frac{9}{4} - \frac{135}{256}\, e^2 \mp \frac{45}{2048}\, e^3 - \frac{34695}{262144}\, e^4 \mp \frac{23895}{2097152}\, e^5 - \frac{8975205}{134217728}\, e^6, \\
a = \frac{25}{4} - \frac{525}{256}\, e^2 - \frac{141225}{262144}\, e^4 \mp \frac{525}{2097152}\, e^5 - \frac{37465575}{134217728}\, e^6, \ \ldots \ ,
\end{gathered} \tag{25}$$

where the error term is $O(e^7)$. It is easy to see from (25) that, as the number k increases, the curves, in a point where they intersect the axis $e = 0$, differ in a term that has a higher order of e. This means that the width of the domain between these curves becomes smaller for larger numbers k. At the same time two corresponding curves crossing the $e = 0$ axis in every point $a = k^2$, $k = 0, 1, 2, \ \ldots$, coincide giving one curve of the form

$$\begin{gathered}
a = 0, \ \ a = 1, \ \ a = 4 - \frac{6}{5}\, e^2 - \frac{39}{125}\, e^4 - \frac{7023}{43750}\, e^6, \\
a = 9 - \frac{108}{35}\, e^2 - \frac{139617}{171500}\, e^4 - \frac{177754233}{420175000}\, e^6, \ \ldots \ .
\end{gathered} \tag{26}$$

So we may conclude that the zones of instability are only between the curves crossing the $e = 0$ axis in the points $a = (2k-1)^2/4,\ \ k = 1, 2, 3, \dots$), and the width of these zones decreases as the number k increases. This conclusion coincides with the results obtained in [11].

Of course, we may suppose that this conclusion is true only with an accuracy up to e^6, and the distinction of curves crossing the $e = 0$ axis in the points $a = k^2,\ \ k = 0, 1, 2, \dots,$ will appear in higher orders of e. To check this hypothesis we can calculate the characteristic constant B numerically for some fixed value of e and different a and compare it with the result obtained in (22). It should be emphasized that with the system Mathematica we can do numerical calculations with an arbitrary precision. And our calculations, which were made with an accuracy of 25 digits, confirmed that the domains where $|B| > 1$ really exist only near the points $a = \dfrac{(2k-1)^2}{4},\ \ k = 1, 2, \dots$. We have found that for every fixed e the characteristic constant B, as function of a, oscillates and becomes less than -1 only in a vicinity of these points. At the same time the inequality $B \le 1$ is true for any a and the curve $B = B(a)$ only touches the line $B = 1$ near the points $a = k^2,\ \ k = 0, 1, 2, \dots$. Positions of the corresponding points coincide with the results obtained in (26). So, if we show that there are two bounded linearly independent solutions of equation (2) in every point of the $a-e$ plane belonging to the curves (26) then we'll prove that these curves belong to the domain of stability of equation (2).

Infinite determinant method for stability analysis. There is another possibility to prove the conclusion obtained above. According to the general theory [1, 7, 12], the boundaries between the domains of stability and instability in the $a-e$ plane are some curves $a = a(e)$ which are characterized by the presence of periodic solutions with the periods 2π and 4π . Hence, we can attempt to determine these boundaries directly by seeking a solution of equation (2) in the form

$$z = c_0 + \sum_{k=1}^{\infty} \left(c_k \, \cos\left(\frac{k}{2}\, t\right) + d_k \, \sin\left(\frac{k}{2}\, t\right) \right). \tag{27}$$

Although this is a Fourier series for the function $z = z(t)$ of period 4π, it can also be used to obtain a solution with period 2π by setting to zero the Fourier coefficients corresponding to k being an odd integer. On substituting (27) into equation (2) and setting the coefficients of $\cos\left(\dfrac{k}{2}\, t\right)$ and $\sin\left(\dfrac{k}{2}\, t\right)$ to zero we obtain the following infinite sequence of equations for determining the coefficients of the Fourier series (27):

$$\begin{gathered} a\, c_0 \;=\; 0, \\ e c_0 + (a-1) c_2 - \frac{3}{2}\, e c_4 \;=\; 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ -\frac{k(k-2)}{2}\, e c_{2k-2} + (a-k^2) c_{2k} - \frac{k(k+2)}{2}\, e c_{2k+2} \;=\; 0; \end{gathered} \tag{28}$$

$$\left(a-\frac{1}{4}+\frac{3}{8}\,e\right)c_1-\frac{5}{8}\,ec_3=0,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots \tag{29}$$

$$-\frac{(2k-5)(2k-1)}{8}\,ec_{2k-3}+\left(a-\left(k-\frac{1}{2}\right)^2\right)c_{2k-1}-\frac{(2k-1)(2k+3)}{8}\,ec_{2k+1}=0;$$

$$(a-1)\,d_2-\frac{3}{2}\,ed_4=0,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots \tag{30}$$

$$-\frac{k(k-2)}{2}\,ed_{2k-2}+(a-k^2)d_{2k}-\frac{k(k+2)}{2}\,ed_{2k+2}=0;$$

$$\left(a-\frac{1}{4}-\frac{3}{8}\,e\right)\,d_1-\frac{5}{8}\,e\,d_3=0,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots \tag{31}$$

$$-\frac{(2k-5)(2k-1)}{8}\,e\,d_{2k-3}+\left(a-\left(k-\frac{1}{2}\right)^2\right)\,d_{2k-1}-\frac{(2k-1)(2k+3)}{8}\,e\,d_{2k+1}=0.$$

It can be seen that in fact there are four infinite subsequences of linear homogeneous equations (28) – (31). Two of these (28) and (30) are for the coefficients $c_0,\ c_2,\ \dots\ ,\ c_{2k}$ and $d_2,\ \dots\ ,\ d_{2k}$, respectively, and represent solution (27) with period 2π. For a solution to exist, the corresponding determinants of infinite systems (28), (30) must vanish, thus determining the stability boundaries in the $a-e$ plane. These boundaries obviously reduce to $a=k^2,\ k=0,1,2,\dots$ when $e\to 0$. The remaining two subsequences of equations (29) and (31) are for $c_1,c_3,\ \dots\ ,\ c_{2k+1}$ and $d_1,\ \dots\ ,\ d_{2k+1}$, and correspond to those stability boundaries which reduce to $a=\dfrac{(2k-1)^2}{4},\ k=1,2,\ \dots,$ when $e\to 0$.

Of course, it's impossible to calculate the determinant of an infinite matrix. So to find the stability boundaries $a=a(e)$ we should truncate the infinite subsequences of equations (28) – (31) after the k-th term, where k is a suitably large number. The corresponding determinant of the system (28), for example, can be written as

$$D_k=\begin{vmatrix} a & 0 & 0 & 0 & \dots & 0\\ c & a-1 & -\dfrac{3}{2}\,e & 0 & \dots & 0\\ 0 & 0 & a-4 & -4e & \dots & 0\\ 0 & 0 & -\dfrac{3}{2}\,e & a-9 & \dots & 0\\ \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & 0 & -\dfrac{k(k-2)}{2}\,e & a-k^2 \end{vmatrix}. \tag{32}$$

Setting determinant (32) to zero we obtain an algebraic equation giving an approximation for the stability boundary $a = a(e)$. An exact expression for the boundary is obtained when $k \to \infty$. Determinant (32) is best evaluated from the following recurrence relation

$$D_k = (a - k^2)D_{k-1} - \frac{e^2}{2}(k-2)(k-1)k(k+1)D_{k-2}, \qquad k = 3, 4, \dots, \tag{33}$$

which is readily established from (32). To start the iterative process we observe that

$$D_1 = a, \quad D_2 = a(a-1).$$

A similar procedure can be followed for the other systems (29) – (31). For instance, the determinant of system (30) is just the same as (32) with the first row and column deleted. The recurrence relation is again (33) for $k \geq 3$, but the starting values are now given by

$$D_1 = a - 1, \quad D_2 = (a-1)(a-4).$$

The corresponding recurrence relation for the determinants of systems (29), (31) is

$$D_k = \left(a - \frac{(2k-1)^2}{4}\right) D_{k-1} - \frac{e^2}{64}(2k-5)(2k-3)(2k-1)(2k+1)\, D_{k-2} \tag{34}$$

with the starting values

$$D_1 = a - \frac{1}{4} \pm \frac{3e}{8}, \quad D_2 = \left(a - \frac{1}{4} \pm \frac{3e}{8}\right)\left(a - \frac{9}{4}\right) + \frac{15e^2}{64}.$$

It's evident from (33), (34) that in the case of $e = 0$ the determinants of systems (28) – (31) will equal to zero when $a = \frac{1}{4}k^2,\ k = 0, 1, 2, \dots$, and the stability boundaries cross the $e = 0$ axis in the $a - e$ plane at these points. For sufficiently small e we can represent the stability boundaries $a = a(e)$ in a vicinity of the points $a = \frac{1}{4}k^2$ as a power series (24). It is easy to see from (33), (34) that to find the curves $a = a(e)$ in vicinity of the point $a = \frac{1}{4}k^2$ with accuracy up to e^{2n}, it is necessary to calculate the determinant D_{k+n}.

Substituting (24) into (33), (34) and setting the corresponding determinants D_k to zero we obtain just the same curves (25), (26) that we have obtained with the method of a small parameter. As the systems (28), (30) determine the same curves (26) crossing the $e = 0$ axis on the $a - e$ plane in the points $a = k^2,\ \ k = 0, 1, 2, \dots$, there exist two linearly independent periodic solutions of equation (2) in every point of these curves. One solution is an even function determined in (27) for $d_k = 0$ and another one is an odd function corresponding to the case of $c_k = 0$. So we can conclude that zones of instability for equation (2) exist only between the curves (25).

It should be noticed that in the case of the corresponding Mathieu equation of the form

$$z''(t) + (a + e\cos t)z(t) = 0,$$

the domains of instability exist near every point $a = \dfrac{k^2}{4}$, $k = 0, 1, 2, \ldots$.

Finally, we can formulate the next theorem.

Theorem. *The Hill's equation of the form*

$$z''(t) + \frac{a + e\cos t}{1 + e\cos t}\, z(t) = 0,$$

is unstable only if the parameters a *and* e *belong to the domains located between curves (25) on the* $a - e$ *plane crossing the* $e = 0$ *axis in the points* $a = \dfrac{(2k-1)^2}{4}$, $k = 1, 2, 3 \ldots$.

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