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FIXED POINT THEORY FOR WEAKLY SEQUENTIALLY UPPER SEMICONTINUOUS MAPS WITH APPLICATIONS TO DIFFERENTIAL INCLUSIONS

ТЕОРІЯ НЕРУХОМОЇ ТОЧКИ ДЛЯ СЛАБКО СЕКВЕНЦІЙНО НАПІВНЕПЕРЕРВНИХ ВІДОБРАЖЕНЬ ІЗ ЗАСТОСУВАННЯМИ ДО ДИФЕРЕНЦІАЛЬНИХ ВКЛЮЧЕНЬ

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We present new fixed point theorems for weakly sequentially upper semicontinuous maps. These results are then used to establish existence principles for second order differential equations and inclusions.

Наведено нові теореми про нерухому точку для слабко секвенційно напівнеперервних відображень. Ці результати застосовано для доведення принципів існування для диференціальних рівнянь та включень другого порядку.

1. Introduction. The aim of this paper is twofold. First we present new fixed point results for weakly sequentially upper semicontinuous maps. Secondly we use these results to obtain existence principles for second order differential inclusions. In the literature (see [1–3] and the references therein) almost all papers establish existence for differential inclusions using the theory of compact (strong) operators. However it is known [4] that the appropriate Niemytzki operator in this situation is weakly completely continuous, and it is our opinion that it is more natural to obtain existence criteria via the theory of weakly sequentially upper semicontinuous maps. With this in mind we establish existence principles in Section 3 using a nonlinear alternative of Leray – Schauder (or a Furi – Pera fixed point theorem) for weakly sequentially upper semicontinuous maps. We present the details fully in Section 3, and the reader can see that the results and ideas extend to higher order differential inclusions or indeed to operator inclusions where the operator is weakly sequentially upper semicontinuous.

2. Fixed point theory. In this section we present the fixed point theory which will be needed in Section 3. First we state a fixed point result due to Arino, Gautier and Penot [5].

Theorem 2.1. *Let E be a metrizable locally convex linear topological space and let C be a weakly compact, convex subset of E. Then any weakly sequentially upper semicontinuous map $F : C \to K(C)$ has a fixed point (here $K(C)$ denotes the family of nonempty, convex, weakly compact subsets of C).*

Remark 2.1. Recall $F : C \to K(C)$ is weakly sequentially upper semicontinuous if for any weakly closed set A of C, $F^{-1}(A)$ is sequentially closed for the weak topology on C.

Notice the proof of Theorem 2.1 is immediate from Himmelberg's fixed point theorem and the next result [6].

Theorem 2.2. *Let E be a metrizable locally convex linear topological space with D a weakly compact subset of E. If $F : D \to K(E)$ is a weakly sequentially upper semicontinuous map, then $F : D \to K(E)$ is a weakly upper semicontinuous map.*

Our next result replaces the weak compactness of the space C with a weak compactness assumption on the operator F.

Theorem 2.3. *Let E be a Banach space with C a closed, convex subset of E. Then any weakly compact, weakly sequentially upper semicontinuous map $F : C \to K(C)$ has a fixed point.*

Proof. There exists a weakly compact subset K of C with $F(C) \subseteq K \subseteq C$. The Krein – Šmulian theorem [7, p. 434] guarantees that $\overline{\text{co}}\,(K)$ is weakly compact. Notice also that $F : \overline{\text{co}}\,(K) \to \overline{\text{co}}\,(K)$, so Theorem 2.1 guarantees that there exists $x \in \overline{\text{co}}\,(K)$ with $x \in F(x)$.

Remark 2.2. In Theorem 2.3, E Banach can be replaced by any metrizable locally convex linear topological space where the Krein – Šmulian theorem holds; for examples see [8, p. 553, 9, p. 82].

In applications to construct a set C so that F takes C back into C is very difficult and sometimes impossible. As a result it makes sense to discuss maps $F : C \to K(E)$. Our first result in this direction is the so called nonlinear alternative of Leray – Schauder (see [10]).

Theorem 2.4. *Let E be a Banach space, C a closed convex subset of E, U a weakly open subset of C, $0 \in U$ and $\overline{U^w}$ weakly compact (here $\overline{U^w}$ denotes the weak closure of U in C). Suppose $F : \overline{U^w} \to K(C)$ is a (weakly compact) weakly sequentially upper semicontinuous map which satisfies the following property:*

$$x \notin \lambda F x \quad \textit{for every} \quad x \in \partial U \quad \textit{and} \quad \lambda \in (0,1); \tag{2.1}$$

here ∂U denotes the weak boundary of U in C. Then F has a fixed point in $\overline{U^w}$.

Proof. Suppose F does not have a fixed point in ∂U (otherwise we are finished), so $x \notin \lambda F x$ for every $x \in \partial U$ and $\lambda \in [0,1]$. Consider

$$A = \{x \in \overline{U^w} : \ x \in t F(x) \ \text{ for some } \ t \in [0,1]\}.$$

Now $A \neq \emptyset$ since $0 \in U$. Also Theorem 2.2 guarantees that $F : \overline{U^w} \to K(C)$ is weakly upper semicontinuous. Thus A is weakly closed and in fact weakly compact since $\overline{U^w}$ is weakly compact.

Also $A \cap \partial U = \emptyset$ so there exists (since (E, w), the space E endowed with the weak topology, is completely regular) a weakly continuous map $\mu : \overline{U^w} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(A) = 1$. Let

$$J(x) = \begin{cases} \mu(x)\, F(x), & x \in \overline{U^w}; \\ \{0\}, & x \in C \setminus \overline{U^w}. \end{cases}$$

Clearly $J : C \to K(C)$ is a weakly compact, weakly sequentially upper semicontinuous map. Theorem 2.3 guarantees that there exists $x \in C$ with $x \in J(x)$. Notice $x \in U$ since $0 \in U$. As a result $x \in \mu(x) F(x)$, so $x \in A$. Thus $\mu(x) = 1$ and so $x \in F(x)$.

Next we present a Furi – Pera theorem for weakly sequentially upper semicontinuous maps. This result can be found in [10] (Theorem 3.4); we note that one of the conditions is stated incorrectly and that the proof there has to be adjusted slightly.

Theorem 2.5. *Let E be a separable and reflexive Banach space, C and Q are closed bounded convex subsets of E with $Q \subseteq C$ and $0 \in Q$. Suppose $F : Q \to K(C)$ is a weakly sequentially upper semicontinuous map and assume the following condition is satisfied:*

$$\begin{aligned}
&\textit{if } \{(x_j, \lambda_j)\}_1^\infty \textit{ is a sequence in } Q \times [0,1] \textit{ with}\\
&x_j \rightharpoonup x \in \partial Q \textit{ and } \lambda_j \to \lambda \textit{ and if } x \in \lambda F(x) \textit{ for}\\
&0 \le \lambda < 1, \textit{ then there exists } j_0 \in \{1, 2, ...\} \textit{ with}\\
&\{\lambda_{j_0} F(x_{j_0})\} \subseteq Q; \textit{ here } \partial Q \textit{ denotes the weak boundary}\\
&\textit{of } Q \textit{ relative to } C \textit{ and } \rightharpoonup \textit{ denotes weak convergence.}
\end{aligned} \tag{2.2}$$

Then F has a fixed point in Q.

Remark 2.3. A special case of (2.2) (which is all we need in Section 3) is the following condition:

$$\begin{aligned}
&\text{if } \{(x_j, \lambda_j)\}_1^\infty \text{ is a sequence in } Q \times [0,1] \text{ with}\\
&x_j \rightharpoonup x \text{ and } \lambda_j \to \lambda \text{ and if } x \in \lambda F(x) \text{ for } 0 \le \lambda < 1,\\
&\text{then there exists } j_0 \in \{1, 2, ...\} \text{ with } \{\lambda_{j_0} F(x_{j_0})\} \subseteq Q.
\end{aligned} \tag{2.3}$$

Proof. Let $r : E \to Q$ be a weakly continuous retraction (see [11]) and let

$$B = \{x \in E : \ x \in \ F r(x)\}.$$

Note $B \subseteq C$ since $F : Q \to K(C)$. It is easy to see that $B \neq \emptyset$ is weakly closed and weakly compact (note C is weakly compact since C is closed and convex (so weakly closed) and bounded in the norm topology). It remains to show $B \cap Q \neq \emptyset$. To do so we argue by contradiction. Suppose $B \cap Q = \emptyset$. Also since E is separable we know from [7] that the weak topology on C is metrizable; let $d^\star$ denote the metric. With respect to $(C, d^\star)$ note Q is closed, B is compact, $B \cap Q = \emptyset$ so there exists $\varepsilon > 0$ with

$$d^\star(B, Q) = \inf\{d^\star(x, y) : \ x \in B, \ y \in Q\} \ > \ \varepsilon.$$

For $i \in \{1, 2, ...\}$ let

$$U_i = \left\{x \in C : \ d^\star(x, Q) < \frac{\varepsilon}{i}\right\}.$$

Fix $i \in \{1, 2, ...\}$. Now U_i is $d^\star$-open in C, so U_i is weakly open in C. Also

$$\overline{U_i}^w = \overline{U_i}^{d^\star} = \left\{ x \in C : \; d^\star(x, Q) \le \frac{\varepsilon}{i} \right\}$$

and

$$\partial U_i = \left\{ x \in C : \; d^\star(x, Q) = \frac{\varepsilon}{i} \right\}.$$

Now $B \cap \overline{U_i}^w = \emptyset$ (since $d^\star(B, Q) > \varepsilon$) and Theorem 2.4 (with $F = F r$ and $U = U_i$) guarantees that there exists $\lambda_i \in (0, 1)$ and $y_i \in \partial U_i$ with $y_i \in \lambda_i F r\,(y_i)$. We can do this argument for each $i \in \{1, 2, ...\}$. Notice in particular since $y_i \in \partial U_i$ that

$$\{\lambda_i F r(y_i)\} \not\subseteq Q \quad \text{for each} \quad i \in \{1, 2, ...\}. \tag{2.4}$$

Now look at

$$D = \{x \in E : \; x \in \lambda F r\,(x) \;\; \text{for some} \;\; \lambda \in [0, 1]\,\}.$$

Now D is weakly compact (so weakly sequentially compact by the Eberlein – Šmulian theorem). This together with

$$d^\star(y_j, Q) = \frac{\varepsilon}{j}, \quad |\lambda_j| \le 1 \;\; \text{for} \;\; j \in \{1, 2, ...\},$$

implies that we may assume without loss of generality that

$$\lambda_j \to \lambda^\star \;\; \text{and} \;\; y_j \rightharpoonup y^\star \in \overline{Q^w} \cap \overline{C \setminus Q^w} = \partial Q.$$

Also since $y_j \in \lambda_j F r\,(y_j)$ we have that $y^\star \in \lambda^\star F r\,(y^\star)$ (recall $F r : C \to K(C)$ is weakly upper semicontinuous). If $\lambda^\star = 1$ then $y^\star \in F r\,(y^\star)$ which contradicts $B \cap Q = \emptyset$. Thus $0 \le \lambda^\star < 1$. But in this case (2.2), with

$$x_j = r(y_j) \;\;\; \text{and} \;\; x = y^\star = r(y^\star),$$

implies there exists $j_0 \in \{1, 2, ...\}$ with $\{\lambda_{j_0} F r\,(y_{j_0})\} \subseteq Q$. This contradicts (2.4). Thus $B \cap \cap Q \neq \emptyset$. As a result there exists $x \in Q$ with $x \in F r\,(x) = F(x)$.

3. Applications. In this section we present existence principles for the second order differential inclusion

$$\begin{gathered} y'' \in f(t, y, y') \;\;\; \text{a.e. on} \;\; [0, 1], \\ y(0) = y(1) = 0 \end{gathered} \tag{3.1}$$

where $f : [0, 1] \times \mathbf{R}^2 \to CK(\mathbf{R})$ is a L^p-Carathéodory function (here $p > 1$ and $CK(\mathbf{R})$ denotes the family of nonempty, convex, compact subsets of $\mathbf{R}$); by this we mean

(a) $t \mapsto f(t, x, y)$ is measurable for every $(x, y) \in \mathbf{R}^2$,

(b) $(x, y) \mapsto f(t, x, y)$ is upper semicontinuous for a.e. $t \in [0,1]$,

and

(c) for each $r > 0$, $\exists\, h_r \in L^p[0,1]$ with $|f(t,x,y)| \le h_r(t)$ for a.e. $t \in [0,1]$ and every $(x,y) \in \mathbf{R}^2$ with $|x| \le r$ and $|y| \le r$.

This section presents an existence principle for (3.1) using Theorem 2.4 and Theorem 2.5. We remark that these existence principles could also be established using the theory of compact (strong) operators (see [3], Chapter 3). However the proofs given in this section have the advantage of automatically yielding new and general existence principles for nonlinear operator equations where the operator is weakly sequentially upper semicontinuous. For notational purposes, for appropriate functions u, let

$$\|u\|_0 = \sup_{[0,1]} |u(t)|, \ \ \|u\|_1 = \max\{\|u\|_0, \|u'\|_0\} \text{ and } \|u\|_{L^p} = \left(\int_0^1 |u(t)|^p\, dt\right)^{\frac{1}{p}}.$$

Recall $W^{k,p}[0,1]$, $1 \le p < \infty$, denotes the space of functions $u : [0,1] \to \mathbf{R}^n$ with $u^{(k-1)} \in$ $\in AC[0,1]$ and $u^{(k)} \in L^p[0,1]$. Note $W^{k,p}[0,1]$ is reflexive if $1 < p < \infty$.

We begin by using Theorem 2.4 to establish an existence principle for (3.1).

Theorem 3.1. *Let $f : [0,1] \times \mathbf{R}^2 \to CK(\mathbf{R})$ be a L^p-Carathéodory function ($1 < p < \infty$) and assume there exists a constant M_0 (independent of λ) with $\|y\|_1 \ne M_0$ for any solution $y \in W^{2,p}[0,1]$ to*

$$y'' \in \lambda f(t, y, y') \quad \text{a.e. on} \quad [0,1],$$
$$y(0) = y(1) = 0$$

for $0 < \lambda < 1$. Then (3.1) *has a solution in* $W^{2,p}[0,1]$.

Proof. Since f is L^p-Carathéodory, there exists $h_{M_0} \in L^p[0,1]$ with

$$\begin{aligned} &|f(t,u,v)| \le h_{M_0}(t) \ \text{ for a.e. } \ t \in [0,1] \ \text{ and} \\ &\text{every } \ (u,v) \in \mathbf{R}^2 \ \text{ with } \ |u| \le M_0 \ \text{ and } \ |v| \le M_0. \end{aligned} \tag{3.2}$$

Let

$$G(t,s) = \begin{cases} (t-1)\,s, & 0 \le s \le t \le 1; \\ (s-1)\,t, & 0 \le t \le s \le 1, \end{cases}$$

and $N = \max\{N_0, N_1, M_0\}$ where $\left(\text{here } \dfrac{1}{p} + \dfrac{1}{q} = 1\right)$,

$$N_0 = \|h_{M_0}\|_{L^p} \sup_{t\in[0,1]} \left(\int_0^1 |G(t,s)|^q\, ds\right)^{\frac{1}{q}}$$

and

$$N_1 = \|h_{M_0}\|_{L^p} \sup_{t\in[0,1]} \left(\int_0^1 |G_t(t,s)|^q \, ds \right)^{\frac{1}{q}} .$$

Also we let

$$N_2 = \|h_{M_0}\|_{L^p}.$$

We will apply Theorem 2.4 with $E = W^{2,p}[0,1]$,

$$C = \left\{ u \in W^{2,p}[0,1] : \ \|u\|_1 \le N \ \text{ and } \ \|u''\|_{L^p} \le N_2 \right\}$$

and

$$U = \left\{ u \in W^{2,p}[0,1] : \ \|u\|_1 < M_0 \ \text{ and } \ \|u''\|_{L^p} \le N_2 \right\}.$$

Now let

$$F = L \circ N_f : C \to 2^E$$

where $L : L^p[0,1] \to W^{2,p}[0,1]$ and $N_f : W^{2,p}[0,1] \to 2^{L^p[0,1]}$ are given by

$$L\, y(t) = \int_0^1 G(t,s)\, y(s)\, ds$$

and

$$N_f\, u = \left\{ y \in L^p[0,1] : \ y(t) \in f(t, u(t), u'(t)) \ \text{ a.e. } \ t \in [0,1] \right\}.$$

Note N_f is well defined since if $x \in C$ then [1, 2] guarantees that $N_f\, x \ne \emptyset$.

Notice C is a convex, closed, bounded subset of E. We first show U is weakly open in C. To do this we will show that $C \setminus U$ is weakly closed. Let $x \in \overline{C \setminus U}^w$. Then there exists $x_n \in C \setminus U$ (see [12, p. 81]) with $x_n \rightharpoonup x$ (here $W^{2,p}[0,1]$ is endowed with the weak topology and $\rightharpoonup$ denotes weak convergence). We must show $x \in C \setminus U$. Now since the embedding $j : W^{2,p}[0,1] \to C^1[0,1]$ is completely continuous [13], there is a subsequence S of integers with

$$x_n \to x \ \text{ in } \ C^1[0,1] \ \text{ and } \ x_n'' \rightharpoonup x'' \ \text{ in } \ L^p[0,1]$$

as $n \to \infty$ in S. Also

$$\|x\|_1 = \lim_{n\to\infty} \|x_n\|_1 \ \text{ and } \ \|x''\|_{L^p} \le \liminf \|x_n''\|_{L^p} \le N_2.$$

Note $M_0 \le \|x\|_1 \le N$ since $M_0 \le \|x_n\|_1 \le N$ for all n. As a result $x \in C \setminus U$, so $\overline{C \setminus U^w} = C \setminus U$. Thus U is weakly open in C. Also

$$\partial U = \{u \in C: \ \|u\|_1 = M_0\} \ \text{ and } \ \overline{U^w} = \{u \in C: \ \|u\|_1 \le M_0\}.$$

To see this let $x \in \overline{U^w}$. Then [12, p. 81] guarantees that there exists $x_n \in U$ with $x_n \rightharpoonup x$. Essentially the same reasoning as above yields $\|x\|_1 \le M_0$ and $\|x''\|_{L^p} \le N_2$, so $\overline{U^w} \subseteq \{u \in C: \ \|u\|_1 \le M_0\} \equiv A$. On the other hand if $x \in A$ (note A is closed), then there exists $x_n \in U$ with $x_n \to x$ in $W^{2,p}[0,1]$, so in particular $x_n \rightharpoonup x$ in $W^{2,p}[0,1]$. Thus $x \in \overline{U^w}$, so $\overline{U^w} = \{u \in C: \ \|u\|_1 \le M_0\}$.

Next we note $\overline{U^w}$ is weakly compact (note $W^{2,p}[0,1]$ is reflexive). Notice also that $F: \overline{U^w} \to 2^C$ since if $y \in \overline{U^w}$ then from (3.2) we have

$$\|F\,y\|_0 \le \|h_{M_0}\|_{L^p} \sup_{t\in[0,1]} \left(\int_0^1 |G(t,s)|^q \, ds \right)^{\frac{1}{q}} = N_0,$$

$$\|(F\,y)'\|_0 \le \|h_{M_0}\|_{L^p} \sup_{t\in[0,1]} \left(\int_0^1 |G_t(t,s)|^q \, ds \right)^{\frac{1}{q}} = N_1,$$

and

$$\|(F\,y)''\|_0 \le \|h_{M_0}\|_{L^p} = N_2.$$

To apply Theorem 2.4 it remains to show $F: \overline{U^w} \to K(C)$ is weakly sequentially upper semicontinuous. Let A be a weakly closed set in C and let $y_n \in F^{-1}(A)$ with $y_n \rightharpoonup y$ in $W^{2,p}[0,1]$. To show F is weakly sequentially upper semicontinuous we must show $y \in F^{-1}(A)$. Now since $y_n \in F^{-1}(A)$, there exists $x_n \in A$ with $x_n = F(y_n)$. Since C is weakly compact, we have that $\{x_n\} \subseteq A$ is relatively weakly compact, and so the Eberlein – Šmulian theorem [7, p. 430] (together with the fact that A is weakly closed) guarantees that there exists $x \in A$ and a subsequence S_0 of integers with $x_n \rightharpoonup x$ in $W^{2,p}[0,1]$ (as $n \to \infty$ in S_0). Now $y_n \rightharpoonup y$ in $W^{2,p}[0,1]$ together with the fact that $j: W^{2,p}[0,1] \to C^1[0,1]$ is completely continuous implies that there is a subsequence S of S_0 with

$$y_n \to y \ \text{ in } \ C^1[0,1] \ \ (\text{note also } \ x_n'' \rightharpoonup x'' \ \text{ in } \ L^p[0,1]) \tag{3.3}$$

as $n \to \infty$ in S. Now $x_n = F(y_n) = L \circ N_f(y_n)$, so

$$x_n''(t) \in f(t, y_n(t), y_n'(t)) \quad \text{for a.e. } \ t \in [0,1]. \tag{3.4}$$

Now (3.3), (3.4), together with a result of Pzusko [4] immediately yields

$$x''(t) \in f(t, y(t), y'(t)) \quad \text{for a.e. } \ t \in [0,1].$$

As a result $y \in F^{-1}(A)$ (note $x \in A$ from above). Thus $F : \overline{U^w} \to 2^C$ is weakly sequentially upper semicontinuous, and it is immediate that $F : \overline{U^w} \to K(C)$.

We now apply Theorem 2.4 to deduce our result; note (2.1) is satisfied since if there exists $x \in \partial U$ and $\lambda \in (0,1)$ with $x \in \lambda F x$ then $\|x\|_1 = M_0$ since $x \in \partial U$ and $\|x\|_1 \neq M_0$ by assumption. Thus F has a fixed point in $\overline{U^w}$.

Remark 3.1. It is easy to use the ideas in Theorem 3.1 to obtain an existence principle for the operator inclusion $y \in F y$ on $[0,1]$, when F is weakly sequentially upper semicontinuous; we leave the details to the reader.

Our final theorem in this section shows how Theorem 2.5 can be applied to obtain an existence principle for differential equations and inclusions. For convenience we discuss the differential equation

$$\begin{aligned} &y'' = f(t,y,y') \quad \text{a.e. on } [0,1], \\ &y(0) = y(1) = 0. \end{aligned} \tag{3.5}$$

Theorem 3.2. *Let $f : [0,1] \times \mathbf{R}^2 \to \mathbf{R}$ be a L^p-Carathéodory function ($1 < p < \infty$) and assume there exists a constant M_0 (independent of λ) with $\|y\|_1 \leq M_0$ for any solution $y \in W^{2,p}[0,1]$ to*

$$\begin{aligned} &y'' = \lambda f(t,y,y') \quad \textit{a.e. on } [0,1], \\ &y(0) = y(1) = 0 \end{aligned}$$

for $0 < \lambda < 1$. Then (3.5) has a solution in $W^{2,p}[0,1]$.

Proof. Since f is L^p-Carathéodory, there exists $h_r \in L^p[0,1]$ with

$$\begin{aligned} &|f(t,u,v)| \leq h_r(t) \quad \text{for a.e. } t \in [0,1] \text{ and} \\ &\text{every } (u,v) \in \mathbf{R}^2 \text{ with } |u| \leq r \text{ and } |v| \leq r; \end{aligned} \tag{3.6}$$

here $r = M_0$ or $r = M_0 + 1$. Let $G(t,s)$ be as in Theorem 3.1 and

$$N = \max\{N_2 + 1, K_2\}, \quad K = \max\{K_0, K_1, M_0 + 1\}$$

where

$$K_0 = \|h_{M_0+1}\|_{L^p} \sup_{t\in[0,1]} \left(\int_0^1 |G(t,s)|^q \, ds \right)^{\frac{1}{q}},$$

$$K_1 = \|h_{M_0+1}\|_{L^p} \sup_{t\in[0,1]} \left(\int_0^1 |G_t(t,s)|^q \, ds \right)^{\frac{1}{q}},$$

$$K_2 = \|h_{M_0+1}\|_{L^p},$$

and

$$N_2 = \|h_{M_0}\|_{L^p}.$$

We will apply Theorem 2.5 with $E = W^{2,p}[0,1]$,

$$Q = \left\{u \in W^{2,p}[0,1] : \ \|u\|_1 \le M_0 + 1 \ \text{ and } \ \|u''\|_{L^p} \le N\right\}$$

and

$$C = \left\{u \in W^{2,p}[0,1] : \ \|u\|_1 \le K \ \text{ and } \ \|u''\|_{L^p} \le N\right\}.$$

Let $F : W^{2,p}[0,1] \to W^{2,p}[0,1]$ be given by

$$F\,u(t) = \int_0^1 G(t,s)\, f(s, y(s), y'(s))\, ds.$$

Essentially the same reasoning as in Theorem 3.1 guarantees that $F : Q \to C$ is weakly sequentially continuous. We can apply Theorem 2.5 once we show (2.3) holds. Let $\{(x_j, \lambda_j)\}_1^\infty$ be a sequence in $Q \times [0,1]$ with $x_j \rightharpoonup x$, $\lambda_j \to \lambda$ and $x = \lambda\, F\, x$ for $0 \le \lambda < 1$. Now since the embedding $j : W^{2,p}[0,1] \to C^1[0,1]$ is completely continuous there is a subsequence S_1 of integers with

$$x_j \to x \ \text{ in } \ C^1[0,1] \ \text{ and } \ x_j'' \rightharpoonup x'' \ \text{ in } \ L^p[0,1] \tag{3.7}$$

as $j \to \infty$ in S_1. Also we know $F : Q \to C$ is weakly sequentially continuous, so $F\,x_j \rightharpoonup F\,x$ in $W^{2,p}[0,1]$. Now since the embedding $j : W^{2,p}[0,1] \to C^1[0,1]$ is completely continuous there is a subsequence S_2 of S_1 with

$$F\,x_j \to F\,x \ \text{ in } \ C^1[0,1] \ \text{ and } \ (F\,x_j)'' \to (F\,x)'' \ \text{ in } \ L^p[0,1]$$

as $j \to \infty$ in S_2; note $(F\,x_j)'' \to (F\,x)''$ in $L^p[0,1]$ as $j \to \infty$ in S_2 follows from (3.7) and the Lebesgue dominated convergence theorem. Thus for any $\varepsilon > 0$ (with $\varepsilon < \frac{1}{3}$ say), there exists $j_0 \in S_2$ with

$$\|F\,x_j\|_1 \le \|F\,x\|_1 + \varepsilon \ \text{ and } \ \|(F\,x_j)''\|_{L^p} \le \|(F\,x)''\|_{L^p} + \varepsilon \tag{3.8}$$

for $j \in S_2$ and $j \ge j_0$. Also $x = \lambda\, F\, x$ together with $\|x\|_1 \le M_0$ and (3.6) implies

$$\|\lambda\, F\,x\|_1 \le M_0 \ \text{ and } \ \|\lambda\, (F\,x)''\|_{L^p} \le N_2. \tag{3.9}$$

Now (3.8), (3.9) and $\lambda_j \to \lambda$ implies that there exists $j_1 \ge j_0$, $j_1 \in S_2$ with

$$\|\lambda_j\, F\,x_j\|_1 \le M_0 + 1 \ \text{ and } \ \|\lambda_j\, (F\,x_j)''\|_{L^p} \le N_2 + 1 \le N$$

for $j \in S_2,\ j \geq j_1$. Thus $\lambda_j F x_j \in Q$ for $j \in S_2$ sufficiently large. The result now follows from Theorem 2.5.

Remark 3.2. We refer the reader to [11] (Section 3) for another application of Theorem 2.5.

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