UDC 517 . 946 + 517 . 958

ENVELOPE MODULATIONAL INSTABILITY IN THE NONLINEAR DISSIPATIVE TRANSMISSION LINE

МОДУЛЯЦІЙНА НЕСТАБІЛЬНІСТЬ ОБВІДНОЇ В НЕЛІНІЙНІЙ ДИСИПАТИВНІЙ ЛІНІЇ ЕЛЕКТРОПЕРЕДАЧІ

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Biinductance LC circuit is considered and envelope modulation is reduced to the complex nonlinear Schrödinger (CNLS) equation. The Benjamin – Feir (or modulational, as it is sometimes called) instability for the CNLS equation is investigated and reduced to the Lange – Newell criterion.

Розглядається LC-ланцюг, для якого модуляція обвідної зводиться до комплексного нелінійного рівняння Шредінгера (КНРШ). Вивчається нестійкість за Бенджаміном та Фейєром (що іноді називається модуляційною) для КНРШ і зводиться до критерію Ланга – Ньювелла.

1. Introduction. Our work deals with the derivation of a nonlinear wave equation for electromagnetic wave propagation on a nonlinear dissipative biinductance transmission line [1 – 4] shown in Fig. 1.

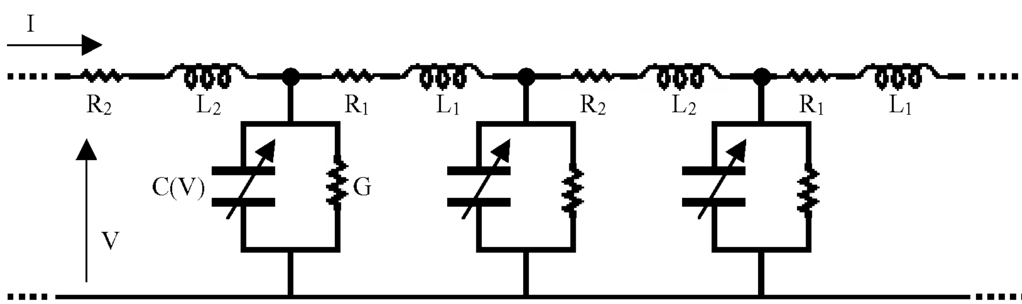

Fig. 1. A typical section for a distributed nonlinear dissipative transmission line.

In this transmission line, $C(V) \approx C_1 - C_N V$, where C_N is a nonlinear capacitor such as a "VARICAP" or a reverse-biased $p-n$ junction diode, the capacitance of which depends on the voltage applied across it. One can write the set of partial differential equations for the currents and voltages as

$$L_2 I_{2n,t} + R_2 I_{2n} = V_{2n-1} - V_{2n}, \quad (1)$$

$$L_1 I_{2n+1,t} + R_1 I_{2n+1} = V_{2n} - V_{2n+1}, \quad (2)$$

$$Q_{2n,t} = I_{2n} - I_{2n+1} - GV_{2n}, \tag{3}$$

$$Q_{2n+1,t} = I_{2n+1} - I_{2n+2} - GV_{2n+1}, \tag{4}$$

where I_{2n} is the current through the inductance L_2 and the resistance R_2, Q_{2n} being the charge density stored in the $2n$-th capacitor, V_{2n} is the ac voltage across it, subscript t denotes the differentiation with respect to t.

The dependence of $Q(V)$ (Q is a charge density, Coulomb/length) must be specified before we proceed. The simplest choice is to expand $Q(V)$ in a Taylor series as

$$Q(V) = C_0(V - aV^2). \tag{5}$$

From (1)–(4), we can eliminate the currents, and using (5), we obtain the following set of discrete equations:

$$C_0\frac{\partial^2}{\partial t^2}(V_{2n} - aV_{2n}^2) + \frac{R_2C_0}{L_2}\frac{\partial}{\partial t}(V_{2n} - aV_{2n}^2) + G\frac{\partial}{\partial t}V_{2n} =$$

$$= \frac{1}{L_1}(V_{2n-1} - V_{2n}) - \frac{1}{L_2}(V_{2n} - V_{2n+1}) - \frac{R_2G}{L_2}V_{2n}, \tag{6}$$

$$C_0\frac{\partial^2}{\partial t^2}(V_{2n+1} - aV_{2n+1}^2) + \frac{R_2C_0}{L_2}\frac{\partial}{\partial t}(V_{2n+1} - aV_{2n+1}^2) + G\frac{\partial}{\partial t}V_{2n+1} =$$

$$= \frac{1}{L_2}(V_{2n} - V_{2n+1}) - \frac{1}{L_1}(V_{2n+1} - V_{2n+2}) - \frac{R_2G}{L_2}V_{2n+1} \tag{7}$$

with the duresses

$$R_2L_1 = R_1L_2,\ L_1 \geq L_2. \tag{8}$$

If we denote by $W_n(t)$ the voltage of the even cells V_{2n} and by V_n the voltage of the odd cells V_{2n+1}, system (6), (7) becomes

$$C_0\frac{\partial^2 V_n}{\partial t^2} + \left(\frac{R_2C_0}{L_2} + G\right)\frac{\partial V_n}{\partial t} - aC_0\left(\frac{\partial^2 V_n^2}{\partial t^2} + \frac{R_2}{L_2}\frac{\partial V_n^2}{\partial t}\right) =$$

$$= \frac{1}{L_1}(W_{n-1} - V_n) - \frac{1}{L_2}(V_n - W_n) - \frac{R_2G}{L_2}V_n, \tag{9}$$

$$C_0\frac{\partial^2 W_n}{\partial t^2} + \left(\frac{R_2C_0}{L_2} + G\right)\frac{\partial W_n}{\partial t} - aC_0\left(\frac{\partial^2 W_n^2}{\partial t^2} + \frac{R_2}{L_2}\frac{\partial W_n^2}{\partial t}\right) =$$

$$= \frac{1}{L_2}(V_n - W_n) - \frac{1}{L_1}(W_n - V_{n+1}) - \frac{R_2G}{L_2}W_n \tag{10}$$

with $n = 1, 2, ..., N$, where N is the number of cells considered.

2. A CNLS equation. In this section we consider the propagation of a group of waves centered around the wavenumber k and the frequency ω. To accomplish this, we use the semi-discrete approximation method [5–7]. We follow Taniuti and Yajima [8] and seek the solution $V_n(t)$ of the odd cells in the form

$$V_n(t) = \varepsilon V_{11}(n,t)\exp\{i\theta\} + \varepsilon^2 V_{22}(n,t)\exp\{2i\theta\} + (*) \tag{11}$$

where ε is a small, dimensionless parameter related to the amplitudes, $\theta = 2kn - \omega t = \theta(n,t)$. Here $(*)$ stands for the complex conjugate of the preceding expression. We use the following ansatz [7] to decouple the two equations (9),(10):

$$W_n(t) = \sigma_1 \exp\{ik\}\left(V_{11} + \varepsilon b_1 V_{11,x} + \varepsilon^2 \frac{b_2}{2} V_{11,xx} + c_8 \varepsilon^2 V_{11}^* V_{22}\right)\exp\{i\theta\} + (*) +$$

$$+ \sigma_2 \varepsilon^2 \exp\{2ik\}\left(V_{22} + \varepsilon c_1 V_{22,x} + C_9 V_{11} V_{11,x}\right)\exp\{2i\theta\} + (*), \tag{12}$$

where σ_1, σ_2, b_1, b_2, c_8, c_1, and c_9 are complex constants to be determined; here the subscript x denotes the differentiation with respect to x. We order the damping coefficients so that the effects of the damping and the nonlinearity appear in the same perturbation equations. Thus we let

$$\frac{R_2 C_0}{L_2} + G = \varepsilon^2 \mu. \tag{13}$$

Substituting (13) into (9) and (10) we obtain

$$C_0 V_{n,tt} + \varepsilon^2 \mu V_{n,t} - aC_0\left(V_{n,tt}^2 + \frac{R_2}{L_2} V_{n,t}^2\right) =$$

$$= \frac{1}{L_1}(W_{n-1} - V_n) - \frac{1}{L_2}(V_n - W_n) - \varepsilon^2 \delta V_n, \tag{14}$$

$$C_0 W_{n,tt} + \varepsilon^2 \mu W_{n,t} - aC_0\left(W_{n,tt}^2 + \frac{R_2}{L_2} W_{n,t}^2\right) =$$

$$= \frac{1}{L_2}(V_n - W_n) - \frac{1}{L_1}(W_n - V_{n+1}) - \varepsilon^2 \delta W_n. \tag{15}$$

In order to determine V_{11} and V_{22}, and the constants σ_1, σ_2, b_1, b_2, c_8, c_1, and c_9 we insert $V_n(t)$ from (10) and $W_n(t)$ from (11) and their derivatives into (14) and (15), and impose to the resulting equations to be equivalent. These operations yield many equations distinguished by the powers of ε and the powers of $\exp\{i\theta\}$.

Equations of $(\varepsilon, \exp\{i\theta\})$ give

$$\begin{gathered}\left(-C_0\omega^2 + \frac{1}{L} + \frac{R_2 G}{L_2}\right)V_{11} = \sigma_1\left(\frac{\exp\{-ik\}}{L_1} + \frac{\exp\{ik\}}{L_2}\right)V_{11}, \\ \left(\frac{\exp\{ik\}}{L_1} + \frac{\exp\{-ik\}}{L_2}\right)V_{11} = \sigma_1\left(-C_0\omega^2 + \frac{1}{L} + \frac{R_2 G}{L_2}\right)V_{11},\end{gathered} \tag{16}$$

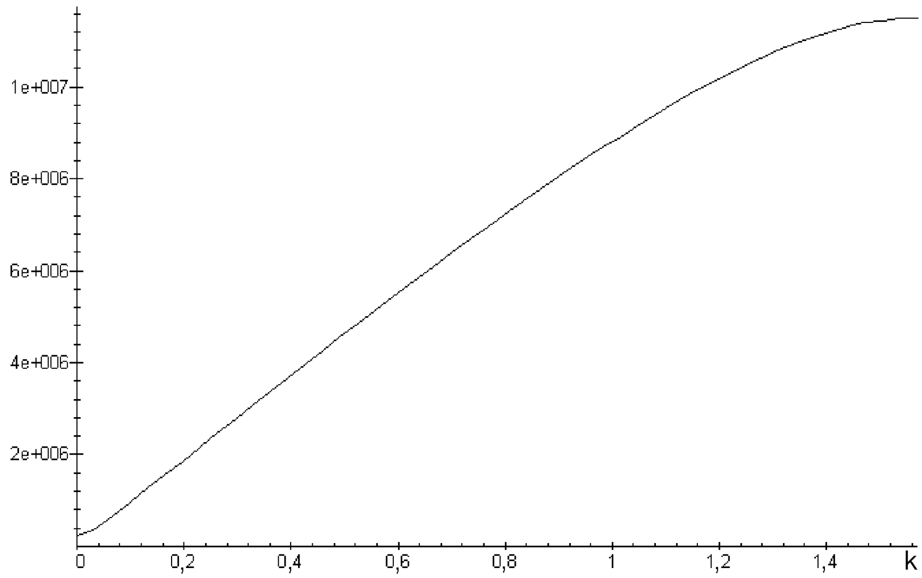

Fig. 2. Dispersion curve for the linearized version of the transmission line shown in Fig. 1 for the Low Frequency mode.

where

$$\frac{1}{L} = \frac{1}{L_1} + \frac{1}{L_2}.$$

From (16) one derives the linear dispersion relation

$$\omega^2 = \frac{1}{C_0}\left[\frac{1}{L} + \frac{R_2G}{L_2} \pm \sqrt{\frac{1}{L^2} - \frac{4}{L_1L_2}\sin^2 k}\right] \tag{17}$$

which is illustrated in Fig. 2 and Fig. 3 for the line parameters $L_1 = 28\mu H$; $L_2 = 14\mu H$; $C_0 = 540pF$; $R_2 = 10\Omega$; and $G = 38,6 \cdot 10^{-6}\Omega^{-1}$, when $0 \le k \le \pi/2$.

The dispersion relation (17) provides two types of frequency modes: the Low Frequency (LF) mode in the region $\omega_1 \le \omega \le \omega_2$ (Fig. 2) and the High Frequency (HF) mode for $\omega_3 \le \le \omega \le \omega_4$ (Fig. 3), where the cut-off frequencies $\omega_1, \omega_2, \omega_3$, and ω_4 are defined by

$$\omega_1 = \sqrt{\frac{R_2G}{C_0L_2}}, \quad \omega_2 = \sqrt{\frac{1}{C_0}\left(\frac{2}{L_1} + \frac{R_2G}{L_2}\right)},$$

$$\omega_3 = \sqrt{\frac{2 + R_2G}{C_0L_2}}, \quad \omega_4 = \sqrt{\frac{1}{C_0}\left(\frac{2}{L} + \frac{R_2G}{L_2}\right)}.$$

From (16) we deduce

$$\sigma_1 = \frac{1}{H_0}\left[\frac{\cos k}{L} + i\frac{\sin k}{L_0}\right],$$

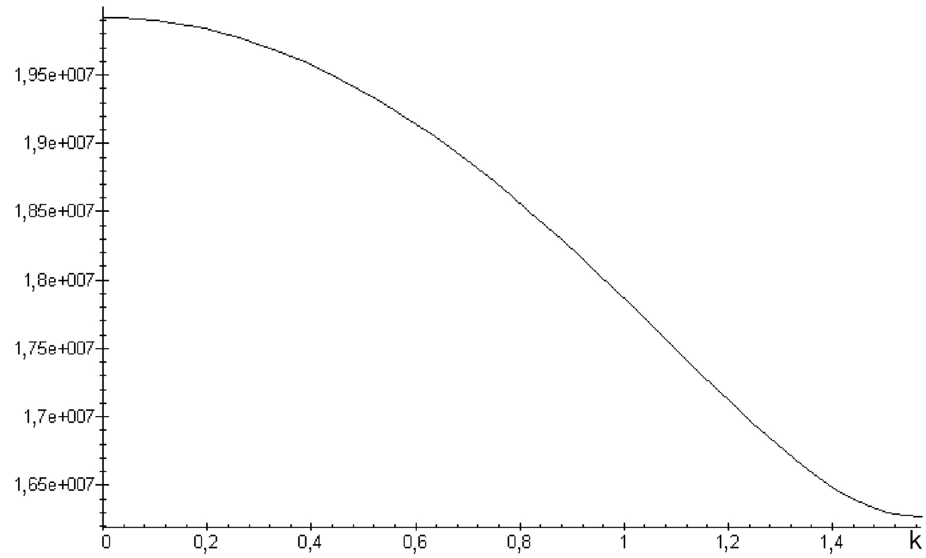

Fig. 3. Dispersion curve for the linearized version of the transmission line shown in Fig. 1 for the High Frequency mode.

where

$$H_0 = -C_0\omega^2 + \frac{1}{L} + \frac{R_2 G}{L_2}, \text{ and } \frac{1}{L_0} = \frac{1}{L_1} - \frac{1}{L_2}. \text{ From (8) we have } \frac{1}{L_0} \leq 0.$$

From equations of $(\varepsilon^2, \exp\{i\theta\})$ we have

$$-2iC_0\omega V_{11,t} = \sigma_1 \left[\left(\frac{\exp\{-ik\}}{L_1} + \frac{\exp[-ik\}}{L_2} \right) b_1 - \frac{2\exp\{-ik\}}{L_1} \right] V_{11,x} -$$

$$- 2iC_0\omega V_{11,t} = \left(\frac{2\sigma_1^* \exp\{ik\}}{L_1} - H_0 \right) V_{11,x},$$

which gives

$$b_1 = \frac{4}{L_1 H_0^2} \left(\frac{1}{L_1} + \frac{\cos(2k)}{L_2} \right) - 1.$$

Equations of $(\varepsilon^2, \exp\{2i\theta\})$ yield

$$\left[H_0 - 3C_0\omega^2 - \left(\frac{\exp\{2ik\}}{L_2} + \frac{\exp\{-2ik\}}{L_1} \right) \sigma_2 \right] V_{22} = aC_0 \left(-4\omega^2 - 2i\omega \frac{R_2}{L_2} \right) V_{11}^2,$$

$$\left[\left(H_0 - 3C_0\omega^2 \right) \sigma_2 - \left(\frac{\exp\{-2ik\}}{L_2} + \frac{\exp\{2ik\}}{L_1} \right) \right] V_{22} = aC_0 \left(-4\omega^2 - 2i\omega \frac{R_2}{L_2} \sigma_1 \right) V_{11}^2,$$

from where we obtain

$$\sigma_2 = \frac{\left(H_0 - 3C_0\omega^2\right)\left(2\omega^2\sigma_1 + i\sigma_1^2\frac{R_2}{L_2}\right) + \left(2\omega^2 + i\frac{R_2}{L_2}\right)\left(\frac{\exp\{-2k\}}{L_2} + \frac{\exp\{2ik\}}{L_1}\right)}{\left(2\omega^2 + i\frac{R_2}{L_2}\right)\left(H_0 - 3C_0\omega^2\right) + \left(2\omega^2\sigma_1 + i\sigma_1^2\frac{R_2}{L_2}\right)\left(\frac{\exp\{2k\}}{L_2} + \frac{\exp\{-2ik\}}{L_1}\right)}$$

and

$$V_{22} = \frac{-2aC_0\omega\left(2\omega + i\frac{R_2}{L_2}\right)}{H_0 - 3C_0\omega^2 - \left(\frac{\exp\{2ik\}}{L_2} + \frac{\exp\{-2ik\}}{L_1}\right)\sigma_2} V_{11}^2.$$

Equations of $(\varepsilon^3, \exp\{i\theta\})$ give

$$\begin{aligned}
&C_0\bigl(V_{11,tt} - 2i\omega V_{11,t}\bigr) - i\omega\mu V_{11} + \left[2aC_0\omega\left(\omega + i\frac{R_2}{L_2}\right) - H_0 c_8\right] V_{11}^* V_{22} + \\
&\quad + \left[\sigma_1 \frac{\exp\{-ik\}}{L_1}(2b_1 - 1) - \frac{H_0}{2} b_2\right] V_{11,xx} = 0, \\
&C_0\bigl(V_{11,tt} - 2i\omega V_{11,t}\bigr) - i\omega\mu V_{11} + \left[aC_0\omega^2 {\sigma_1^*}^2 \sigma_2 + \left(H_0 + aC_0 i\omega\frac{R_2}{L_2}\right) c_8\right] \times \\
&\quad \times V_{11}^* V_{22} + \left[\left(H_0 + aC_0 i\omega\right)\frac{b_2}{2} - \frac{\sigma_1^* \exp\{ik\}}{L_1}\right] V_{11,xx} - \\
&\quad - C_0 b_1 \left[2i\omega + a\frac{R_2}{L_2}\right] V_{11,xt} = 0.
\end{aligned} \tag{18}$$

The compatibility condition for the nonlinear term $V_{11}^* V_{22}$ gives

$$c_8 = \frac{aC_0\left[2\omega\left(\omega + i\frac{R_2}{L_2}\right) - \omega^2\frac{\sigma_2}{\sigma_1^2}\right]}{2H_0 + aC_0 i\omega\frac{R_2}{L_2}}$$

and the compatibility condition for $V_{11,xx}$ gives

$$b_2 = \frac{2\left[\sigma_1\left(2b_1 - 1\right)\exp\{-ik\} + \sigma_1^*\exp\{ik\}\right]}{L_1\left[2H_0 + aC_0 i\omega\right]}.$$

Using Galilea transformation

$$\xi = x - \frac{d\omega}{dk}t,\ \tau = t$$

in the second equation of (18) and by going into the reference frame moving with the group velocity $V_g = d\omega/dt$, we derive the resulting equation for V_{11} that describes the evolution of the wave packet

$$iV_{11,\tau} + PV_{11,\xi\xi} + Q\,|V_{11}|^2\,V_{11} + i\Gamma V_{11} = 0 \tag{19}$$

where

$$P = P(k) = \frac{1}{2C_0\omega}\left[\frac{\exp\{ik\}}{\sigma_1 L_1} - (H_0 + aC_0 i\omega)\,\frac{b_2}{2} - C_0\left(\frac{d\omega}{dk}\right)^2 - \right.$$

$$\left. -\frac{d\omega}{dk}\left(2C_0 b_1 i\omega + aC_0 b_1\frac{R_2}{L_2}\right)\right],$$

$$Q = Q(k) = \frac{a\left(2\omega + i\dfrac{R_2}{L_2}\right)\left[aC_0\omega^2\dfrac{\sigma_2}{\sigma_1^2} + \left(H_0 + aC_0 i\omega\dfrac{R_2}{L_2}\right)c_8\right]}{H_0 - 3C_0\omega^2 - \left(\dfrac{\exp\{2ik\}}{L_2} + \dfrac{\exp\{-2ik\}}{L_1}\right)\sigma_2},$$

$$\Gamma = \frac{\mu}{2C_0}.$$

Equation (19 is a CNLS equation. If we denote by P_r, Q_r, and Γ_r the real parts of P, Q, and Γ, respectively and by P_i, Q_i, and Γ_i the imaginary parts of P, Q, and Γ, respectively, then we can write

$$P = P_r + iP_i, \qquad Q = Q_r + iQ_i, \qquad \Gamma = \Gamma_r + i\Gamma_i.$$

3. Envelope modulational instability for the CNLS equation. In this section we consider the modulation of the envelope of the unstable periodic solution of Eq. (19) by considering the first perturbation of the amplitude of a plane wave [7, 9, 10].

First we investigate periodic solution of Eq. (19). Thus we seek a solution in the form

$$V_{11} = A\exp\left\{i\left(\tilde{k}\xi - \tilde{\omega}\tau\right)\right\} \tag{20}$$

where A is a complex constant, $\tilde{\omega}$ and $\tilde{k}$ are real constants. Substituting (20) into (19), we obtain

$$\tilde{\omega} - \tilde{k}^2 P + |A|^2\,Q + \Gamma = 0. \tag{21}$$

Next we perturb V_{11}. That is, we let

$$V_{11} = [1 + b]\,A\exp\left\{i\left(\tilde{k}\xi - \tilde{\omega}\tau\right)\right\} + (^*) \tag{22}$$

where $b = b(\xi, \tau)$ is assumed to be infinitesimal. Substituting (22) into (19), using (21) and keeping only linear terms in the perturbation quantity, we obtain

$$ib_\tau + Pb_{\xi\xi} + 2i\tilde{k}Pb_\xi + |A|^2\,Q\,(b + b^*)\,. \tag{23}$$

Since (23) has constant coefficients, one can represent its solutions in the form

$$b(\xi,\tau) = b_1 \exp\{i(K\xi + \Omega\tau)\} + b_2^* \exp\{-i(K\xi + \Omega^*\tau)\} \tag{24}$$

where b_1, b_2, K, and Ω are constants. Substituting (24) into (23) yield

$$\begin{gathered}\left(\Omega + K^2P + 2\tilde{k}KP - |A|^2 Q\right) b_1 - |A|^2 Q b_2 = 0,\\ |A|^2 Q^* b_1 + \left(\Omega - K^2P^* + 2\tilde{k}KP^* + |A|^2 Q^*\right) b_2 = 0.\end{gathered} \tag{25}$$

For a nontrivial solution the determinant of the coefficient matrix (25) must vanish. That is,

$$\left[\Omega + \left(2\tilde{k}KP_r + i\left(K^2P_i - |A|^2 Q_i\right)\right)\right]^2 = X + iY, \tag{26}$$

where

$$X = K^4P_r^2 - 4\tilde{k}^2K^2P_i^2 - |A|^4 Q_i^2 + 2K^2 |A|^2 P_iQ_i - 2K^2 |A|^2 (P_rQ_r + P_iQ_i),$$

$$Y = 4\tilde{k}KP_i\left(K^2P_r - |A|^2 Q_r\right).$$

If we introduce the notations

$$H_1 = \pm\sqrt{\frac{1}{2}\left(X + \sqrt{X^2+Y^2}\right)} \text{ and } H_2 = \pm\sqrt{\frac{1}{2}\left(-X + \sqrt{X^2+Y^2}\right)} \tag{27}$$

then from (26) we have

$$\Omega = \left(-2\tilde{k}KP_r \pm H_1\right) + i\left(|A|^2 Q_i - K^2P_i \pm H_2\right). \tag{28}$$

We deduce from (24), (27), and (28) that for the boundedness of $b(\xi,\tau)$ (as $\tau \to +\infty$) it is necessary and sufficient that at least one of the following conditions occurs

a) K is a solution of equation

$$|A|^2 Q_i - K^2P_i \pm H_2 = 0 \tag{29}$$

b)

$$|A|^2 Q_i - K^2P_i \pm H_2 > 0. \tag{30}$$

We should note that condition (29) means that Ω is real and condition (30) means that the imaginary part of Ω is positive.

It follows from (28) that for Ω to be complex so that $b(\xi,\tau)$ is unbounded (as $\tau \to +\infty$) it is necessary and sufficient that the wavenumber of the perturbation K should be a solution of system

$$\begin{gathered}\sqrt{\frac{1}{2}\left(-X + \sqrt{X^2+Y^2}\right)} + |A|^2 Q_i - K^2P_i < 0,\\ -\sqrt{\frac{1}{2}\left(-X + \sqrt{X^2+Y^2}\right)} + |A|^2 Q_i - K^2P_i < 0.\end{gathered} \tag{31}$$

A necessary condition for (31) to have a solution is that

$$|A|^2 Q_i - K^2 P_i < 0. \tag{32}$$

It follows from (32) that

$$K^2 > |A|^2 Q_i P_i^{-1}, \text{ if } P_i > 0 \tag{33}$$

and

$$K^2 \acute{<} |A|^2 Q_i P_i^{-1}, \text{ if } P_i < 0. \tag{34}$$

From (33) we have $K > |A| \sqrt{Q_i P_i^{-1}}$, if $P_i Q_i > 0$ and $K > 0$ if $\mathrm{P}_i Q_i < 0$, and from (34) we have $0 < K < |A| \sqrt{Q_i / P_i}$ if $P_i Q_i > 0$. We have the following conclusions:

i) if $P_i Q_i > 0$ and either $P_i > 0$ and $K > |A| \sqrt{Q_i P_i^{-1}}$ or $P_i < 0$ and $0 < K < < |A| \sqrt{Q_1 P_i^{-1}}$ then for the modulational instability for the plane wave in the nonlinear dissipative transmission line, it is necessary that

$$P_r Q_r + P_i Q_i > \frac{K^4 (2P_i^2 + P_r^2) + |A|^4 Q_i^2}{2K^2 |A|^2} > 0; \tag{35}$$

ii) if $P_i > 0$, $P_i Q_i < 0$ and $K > 0$, then for the modulational instability for the plane wave in the nonlinear dissipative transmission line, it is necessary that

$$P_r Q_r + P_i Q_i > \frac{K^4 (2P_i^2 + P_r^2) + |A|^4 Q_i^2 - 2K^2 |A|^2 P_i Q_i}{2K^2 |A|^2} > 0. \tag{36}$$

From (35) and (36) we obtain

$$P_r Q_r + P_i Q_i > 0$$

which is the well known Lange and Newell's criterion [10] of the modulational instability.

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Received 01.08.2001