UDC 517.9

ON MULTI-FREQUENCY SYSTEMS WITH IMPULSES*

ПРО БАГАТОЧАСТОТНІ СИСТЕМИ З ІМПУЛЬСАМИ

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We study exponential dichotomy for a linear quasiperiodic system with impulses. The piecewise smoothness of separatrix manifolds is proved. For nonlinear quasiperiodic impulsive system with linearized system that posseses the property of exponential dichotomy, the conditions for the existence of discontinuous quasiperiodic solutions are obtained.

Вивчається експоненціально дихотомічна лінійна квазіперіодична імпульсна система. Отримано умови кускової неперервності та кускової гладкості сепаратрисних множин. Для слабконелінійної імпульсної квазіперіодичної системи з експоненціально дихотомічною лінійною частиною доведено існування розривних квазіперіодичних розв'язків.

Introduction. We will consider an impulsive system of the form

$$\frac{d\varphi}{dt} = \omega, \tag{1}$$

$$\frac{dx}{dt} = A(\varphi)x, \quad \varphi \in \mathbb{T}_m \setminus \Gamma, \tag{2}$$

$$\Delta x \Big|_{\varphi \in \Gamma} = B(\varphi)x, \tag{3}$$

where $x \in \mathbb{R}^n$, $\varphi \in \mathbb{T}_m$, $\mathbb{T}_m$ is an m-dimensional torus, Γ is a smooth compact submanifold of $\mathbb{T}_m$ of codimension 1, and $\omega = (\omega_1, ..., \omega_m)$ is a constant vector with rationaly independent coordinates. Δx stands for the jump of the functions x at the point φ obtained during the motion along the trajectory of equation (1). Suppose that $\det(I + B(\varphi)) = 0$ (I is the identity matrix) for some or all $\varphi \in \Gamma$. Therefore, solutions of system (1) – (3) cannot be continued on the negative semi-axis $t < 0$ or can be continued ambiguously.

In this paper, we introduce the concept of exponential dichotomy for the system (1) – (3) and investigate analytic properties of separatrix manifolds. For a nonlinear quasiperiodic impusive system with linearized system (1) – (3) that possesses the property of exponential dichotomy, sufficient conditions for the existence of invariant set are obtained. Trajectories on the invariant set are quasi-periodic with the first-order discontinuities. We continue the investigations of [1–7]. Piecewise smooth quasiperiodic impulsive systems were studied in [5, 6] under the assumption that $\det(I + B(\varphi)) \neq 0$ for all $\varphi \in \Gamma$.

* Partially supported by INTAS Grant 96-0915 and Ukrainian Ministry on Science and Technology Grant 1.4/269.

1. Linear system. Denote by $C^s(\mathbb{T}_m)$ the space of s times continuously differentiable functions or matrices on $\mathbb{T}_m$. For $f(\varphi) \in C^s(\mathbb{T}_m)$, we denote the norm

$$\|f(\varphi)\|_s = \max_{0\le|j|\le s} \sup_{\varphi\in\mathbb{T}_m} \|\partial^{|j|} f(\varphi)/\partial\varphi^j\|,$$

where $j = (j_1, ..., j_m)$, $\varphi^j = (\varphi_1^{j_1} ... \varphi_m^{j_m})$, $|j| = j_1 + ... + j_m$, and $\|\cdot\|$ is the norm in $\mathbb{R}^n$ or in the space of matrices.

Equation (1) has solutions $\sigma(t,\varphi) = \varphi \cdot t = \omega t + \varphi$. Suppose that solutions $\varphi \cdot t$ intersect the manifold Γ transversally. We denote by $t_j(\varphi)$, $j \in \mathbb{Z}$, the sequence of points t where $\varphi \cdot t$ intersects the manifold Γ.

For fixed φ, system (1) – (3) has the following form:

$$\frac{dx}{dt} = A(\varphi\cdot t)x, \qquad t \ne t_i(\varphi), \tag{4}$$

$$\Delta x\Big|_{t=t_i(\varphi)} = B(\sigma_i(\varphi))x, \tag{5}$$

where $\sigma_i(\varphi) = \sigma(t_i(\varphi), \varphi)$. Let $x(t,\varphi,x_0)$ be the solution of the initial-value problem for (4), (5) with initial value $x(0,\varphi,x_0) = x_0$. The solution $x(t,\varphi,x_0)$ is piecewise continuous and has discontinuities in t where solution $\varphi\cdot t$ intersects the manifold Γ. Denote by $X(t,\varphi)$, $t \ge 0$, the fundamental solution for the system (4), (5), $X(0,\varphi) = I$. We assume that $x(t,\varphi,x_0)$ and $X(t,\varphi)$ are left-side continuous.

Lemma 1. *The sequence $\{t_k(\varphi), k \in \mathbb{Z}\}$ has equipotentially almost periodic differences $t_k^j(\varphi) = t_{k+j}(\varphi) - t_k(\varphi)$, i.e., for an arbitrary $\varepsilon > 0$ there exists a relatively dense set of ε-almost periods that are common to all sequences t_k^j.*

Proof. 1) Let Θ_ε be the set of real numbers ω for which there exists an integer h_ω such that

$$|t_k^{h_\omega} - \omega| = |t_{k+h_\omega} - t_k - \omega| < \varepsilon \quad \text{for all} \quad k \in \mathbb{Z}.$$

By Lemma 25 [3], the sequences $\{t_k^j(\varphi), k \in \mathbb{Z}\}$ are equipotentially almost periodic if and only if the set Θ_ε is relatively dense for every $\varepsilon > 0$. We prove that Θ_ε is relatively dense in our case.

2) Denote $V_\delta(\varphi) = \{\varphi_1 \in \mathbb{T}_m : \rho(\varphi_1,\varphi) \le \delta\}$, where $\rho(\cdot,\cdot)$ is a metric on the torus $\mathbb{T}_m$. By Kronecker's theorem [8], for any $\varepsilon > 0$, there exists a relatively dense set of numbers τ_ε such that $\rho(\varphi\cdot\tau_\varepsilon, \varphi) < \varepsilon$ for all $\varphi \in \mathbb{T}_m$.

3) By assumption, the trajectory $\varphi\cdot t$ intersects the compact manifold Γ transversally; therefore, for $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\rho(\varphi, \tilde\varphi_1) < \varepsilon, \quad |\tilde\Delta(\varphi_1, \tilde\varphi_1)| < \varepsilon,$$

for all $\varphi \in \Gamma$, $\varphi_1 \in V_\delta(\varphi)$. Here, $\tilde\varphi_1$ is the point nearest to φ_1 at which the trajectory $\varphi\cdot t$ intersects the manifold Γ. The number $\tilde\Delta(\varphi_1,\tilde\varphi_1)$ is defined by the equality $\varphi_1\cdot\tilde\Delta(\varphi_1,\tilde\varphi_1) = \tilde\varphi_1$.

4) By 2), there exists a relatively dense set of numbers τ_δ such that

$$\rho((\varphi\cdot t_j)\cdot\tau_\delta, \varphi\cdot t_j) < \delta \quad \text{for all} \quad j \in \mathbb{Z}.$$

By 3), for fixed j, there exists a point $\varphi \cdot t_{j+l} \in \Gamma$ such that

$$\rho(\varphi \cdot t_{j+l}, \varphi \cdot t_j) < \varepsilon, \quad |\tilde{\Delta}(\varphi \cdot t_{j+l}, (\varphi \cdot t_j) \cdot \tau_\delta)| < \varepsilon.$$

We prove that the number l is the same for all $j \in \mathbb{Z}$. For the point $(\varphi \cdot t_{j+1}) \cdot \tau_\delta$, there exists a point $\varphi \cdot t_{j+1+m} \in \Gamma$ such that

$$\rho(\varphi \cdot t_{j+1+m}, \varphi \cdot t_{j+1}) < \varepsilon, \quad |\tilde{\Delta}(\varphi \cdot t_{j+1+m}, (\varphi \cdot t_{j+1}) \cdot \tau_\delta)| < \varepsilon.$$

We get

$$|\tilde{\Delta}((\varphi \cdot t_{j+l}, \varphi \cdot t_{j+1+m})| \leq |\tilde{\Delta}(\varphi \cdot t_{j+l}, \varphi \cdot (t_j + \tau_\delta))| +$$

$$+ |\tilde{\Delta}(\varphi \cdot (t_j + \tau_\delta), \varphi \cdot (t_{j+1} + \tau_\delta))| + |\tilde{\Delta}(\varphi \cdot (t_{j+1} + \tau_\delta), \varphi \cdot t_{j+1+m})| \leq$$

$$\leq \varepsilon + t_{j+1} - t_j + \varepsilon. \tag{6}$$

Therefore, $m = l$.

5) Using $|\tilde{\Delta}(\varphi \cdot t_{j+l}, (\varphi \cdot t_j) \cdot \tau_\delta)| < \varepsilon$, we get

$$|t_{j+l} - t_j - \tau_\delta| < \varepsilon \quad \text{for all} \quad j \in \mathbb{Z}. \tag{7}$$

The numbers τ_δ satisfying (7) form a relatively dense set. The lemma is proved.

Definition 1. *System (1) – (3) is said to be exponentially dichotomous if, for all $\varphi \in \mathbb{T}_m$, the space $\mathbb{R}^n$ can be represented in the form of the direct sum of the subspaces $U(\varphi)$ and $S(\varphi)$ so that*

1) any solution of system (4), (5) with $x_0 \in S(\varphi)$ satisfies the inequality

$$\|x(t, \varphi, x_0)\| \leq K \exp(-\alpha(t - \tau))\|x(\tau, \varphi, x_0)\|, \quad t \geq \tau \geq 0; \tag{8}$$

2) any solution with $x_0 \in U(\varphi)$ satisfies the inequality

$$\|x(t, \varphi, x_0)\| \geq K_1 \exp(\alpha(t - \tau))\|x(\tau, \varphi, x_0)\|, \quad t \geq \tau \geq 0; \tag{9}$$

where positive constants α, K, K_1 are independent of φ, x_0;

3) $X(t, \varphi)S(\varphi) \subseteq S(\varphi \cdot t), \quad X(t, \varphi)U(\varphi) \subseteq U(\varphi \cdot t), \quad t \geq 0$;

4) projectors $P(\varphi)$ and $Q(\varphi) = I - P(\varphi)$ corresponding to $S(\varphi)$ and $U(\varphi)$ are uniformly bounded

$$\sup_{\varphi \in \mathbb{T}_m} \|P(\varphi)\| + \sup_{\varphi \in \mathbb{T}_m} \|Q(\varphi)\| < \infty.$$

Theorem 1. *Assume that system (1) – (3) is exponentially dichotomous and the functions $A(\varphi), B(\varphi)$ are continuous on $\mathbb{T}_m$. Then the projector $P(\varphi)$ is continuous on the set $\mathbb{T}_m \setminus \Gamma$ and has discontinuities of the first kind on the set Γ and, moreover,*

$$P(\varphi + 0)(I + B(\varphi)) = (I + B(\varphi))P(\varphi - 0).$$

For this case, $(\varphi - 0)$ and $(\varphi + 0)$ are defined during the motion along the trajectory $\varphi \cdot t$ of equation (1) for increasing t.

Proof. Fix φ and consider system (4), (5). By Lemma 1, the sequence $\{t_j(\varphi),\ j \in \mathbb{Z}\}$ has equipotentially almost periodic differences. By Lemma 30 [3], the sequence $\{B(\sigma_j(\varphi)),\ j \in \mathbb{Z}\}$ is almost periodic. By Theorem 2 [9], system (4), (5) is exponential dichotomous on the axis $t \in \mathbb{R}$ and dimensionalities of stable $S(\varphi \cdot t)$ and unstable $U(\varphi \cdot t)$ manifolds do not depend on t. Limiting systems of system (4), (5) are exponentially dichotomous and the dimensionalities of stable and unstable manifolds for limiting systems remain the same. Hence, the dimensionalities of stable $S(\varphi)$ and unstable $U(\varphi)$ manifolds of system (1) – (3) are the same for all $\varphi \in \mathbb{T}_m$.

Now we prove that the projectors $P(\varphi)$ and $Q(\varphi)$ are piecewise continuous. We first prove that $S(\varphi)$ is closed ($U(\varphi)$ is considered similarly). Let $x_k \in S(\varphi_k)$, $\|x_k\| = 1, \varphi_k \to \bar{\varphi}$, $x_k \to \bar{x}$, $k \to \infty$. Assume that $\bar{\varphi} \notin \Gamma$. In order to show that $\bar{x} \in S(\bar{\varphi})$, let us consider system (4), (5) for $\varphi = \varphi_k$, $k = 1, 2, \ldots$. Using the uniqueness of solutions for system (4) and the transversality of intersections $\sigma(t, \varphi)$ with Γ, we conclude that the theorem about continuous dependence on parameters [3] is valid for the impulsive system (4), (5). Hence, for $T > 0$ and for $\varepsilon > 0$, there exists $N = N(T, \varepsilon)$ such that the inequality $j \geq N$ implies

$$\|x(t, x_j, \varphi_j) - x(t, \bar{x}, \bar{\varphi})\| < \varepsilon \tag{10}$$

for $t \in [0, T]$ satisfying $|t - t_i(\bar{\varphi})| > \varepsilon$. The sequence $\{x_k\}$ is bounded. By (8), there exists a compact set $\mathcal{K} \subset \mathbb{R}^n \times \mathbb{T}_m$ such that $x(t, x_j, \varphi_j) \in \mathcal{K}$ for $t \geq 0$. By (10), one has $x(t, \bar{x}, \bar{\varphi}) \in \mathcal{K}$ for $t \geq 0$. If the solution $x(t, \bar{x}, \bar{\varphi})$ of the exponential dichotomous system (4), (5) is bounded for $t \geq 0$, then it belongs to $S(\bar{\varphi})$. Hence, $\bar{x} \in S(\bar{\varphi})$.

If $\bar{\varphi} \in \Gamma$, we consider left-side and right-side sequences $\varphi_k \to \bar{\varphi}$ and prove similarly that $S(\bar{\varphi})$ is closed in $\bar{\varphi} - 0$ and $\bar{\varphi} + 0$.

Analogously to the proof of Lemma 7 [10], one can prove that the projector $P(\varphi)$ is continuous for $\varphi \notin \Gamma$. This completes the proof of the theorem.

It follows from Theorem 1 that the subspace $U(\varphi)$ has a unique negative continuation such that

$$\|x(t, \varphi, x_0)\| \leq K_2 \exp(\alpha t) \|x_0\|, \quad \varphi \in \mathbb{T}_m, \ x_0 \in U(\varphi), \ t \leq 0.$$

Hence, $X(t, \varphi) Q(\varphi)$ is well defined for all $t \leq 0$, and we can define the Green function for system (1) – (3)

$$G(t, \tau, \varphi) = \begin{cases} X(t - \tau, \varphi \cdot \tau) P(\varphi \cdot \tau), & t \geq \tau, \\ -X(t - \tau, \varphi \cdot \tau) Q(\varphi \cdot \tau), & \tau \geq t. \end{cases} \tag{11}$$

For $t \neq \tau$, the Green function $G(t, \tau, \varphi)$ satisfies system (4), (5):

$$\frac{dG(t, \tau, \varphi)}{dt} = A(\varphi \cdot t) G(t, \tau, \varphi), \quad t \neq t_i(\varphi), \tag{12}$$

$$\Delta G(t, \tau, \varphi)\Big|_{t = t_i(\varphi)} = B(\sigma_i(\varphi)) G(t_i(\varphi), \tau, \varphi). \tag{13}$$

If system (1) – (3) has exponential dichotomy, then the Green function $G(t, \tau, \varphi)$ is bounded by an exponent:

$$\|G(t, \tau, \varphi)\| \leq K_3 \exp(-\alpha |t - \tau|), \quad t, \tau \in \mathbb{R}, \quad K_3, \alpha > 0. \tag{14}$$

The linear inhomogeneous system

$$\frac{dx}{dt} = A(\varphi \cdot t)x + f(t), \quad t \neq t_i(\varphi),$$

$$\Delta x\Big|_{t=t_i(\varphi)} = B(\sigma_i(\varphi))x + g_i$$

has the following unique bounded solution:

$$u(t,\varphi) = \int_{-\infty}^{\infty} G(t,\tau,\varphi)f(\tau)d\tau + \sum_{i\in\mathbb{Z}} G(t,t_i(\varphi),\varphi)g(\sigma_i(\varphi)). \tag{15}$$

Theorem 2. *Suppose that*

1) the manifold Γ is smooth of class C^s;

2) $A(\varphi), B(\varphi) \in C^s(\mathbb{T}_m)$;

3) solutions of equation (1) intersect the manifold Γ transversally;

4) system (4), (5) is exponentially dichotomous.

Then the projector $P(\varphi)$ has continuous partial derivatives of order s with respect to φ on the set $\mathbb{T}_m \setminus \Gamma$.

Proof. Let $\Delta\varphi_i$ be an increment of the i-th coordinate of φ and $\varphi + \Delta\varphi_i = (\varphi_1, \ldots \ldots, \varphi_i + \Delta\varphi_i, \ldots, \varphi_n)$. Let us consider the difference $R = G(t,\tau,\varphi + \Delta\varphi_i) - G(t,\tau,\varphi)$, where $\varphi \in \mathbb{T}_m \setminus \Gamma$. It satisfies the following system:

$$\frac{dR}{dt} = A(\sigma(t,\varphi))R + (A(\sigma(t,\varphi + \Delta\varphi_i)) - A(\sigma(t,\varphi))G(t,\tau,\varphi + \Delta\varphi_i),$$

$$\Delta R\Big|_{t=t_j^1} = B(\sigma_j^1)R - B(\sigma_j^1)G(t_j^1,\tau,\varphi + \Delta\varphi_i),$$

$$\Delta R\Big|_{t=t_j^2} = B(\sigma_j^2)G(t_j^2,\tau,\varphi + \Delta\varphi_i),$$

where $t_j^1 = t_j(\varphi), t_j^2 = t_j(\varphi + \Delta\varphi_i), \sigma_j^1 = \sigma(t_j(\varphi),\varphi), \sigma_j^2 = \sigma(t_j(\varphi + \Delta\varphi_i),\varphi + \Delta\varphi_i)$. By (15), one has

$$G(0,\tau,\varphi + \Delta\varphi_i) - G(0,\tau,\varphi) = \int_{-\infty}^{\infty} G(0,s,\varphi)(A(\sigma(s,\varphi + \Delta\varphi_i)) -$$

$$-A(\sigma(s,\varphi)))G(s,\tau,\varphi + \Delta\varphi_i)ds +$$

$$+\sum_{j\in\mathbb{Z}} G(0,t_j^1,\varphi)(B(\sigma_j^2) - B(\sigma_j^1))G(t_j^1,\tau,\varphi + \Delta\varphi_i) +$$

$$+\sum_{j\in\mathbb{Z}} (G(0,t_j^2,\varphi) - G(0,t_j^1,\varphi))B(\sigma_j^2))G(t_j^1,\tau,\varphi + \Delta\varphi_i) +$$

$$+\sum_{j\in\mathbb{Z}} G(0,t_j^2,\varphi)B(\sigma_j^2)(G(t_j^1,\tau,\varphi + \Delta\varphi_i) - G(t_j^2,\tau,\varphi + \Delta\varphi_i)). \tag{16}$$

Intersections of solutions $\sigma(t,\varphi)$ with the compact manifold Γ are transversal. Therefore,

$$t_{j+1}(\varphi) - t_j(\varphi) \geq \theta > 0, \tag{17}$$

$$\|t_j(\varphi)\|_s \leq C, \quad j \in \mathbb{Z}, \tag{18}$$

where θ and C are positive constants independent of $j \in \mathbb{Z}$, $\varphi \in \mathbb{T}_m$.

Now we compute limits. Let $t_j(\varphi) > \tau, t_j(\varphi + \Delta\varphi_i) > \tau$. Then

$$\lim_{\Delta\varphi_i \to 0} \frac{1}{\Delta\varphi_i}\Big(G(t_j(\varphi+\Delta\varphi_i), \tau, \varphi+\Delta\varphi_i) - (G(t_j(\varphi), \tau, \varphi+\Delta\varphi_i)\Big) =$$

$$= \lim_{\Delta\varphi_i \to 0} \frac{1}{\Delta\varphi_i}\Big(X(t_j(\varphi+\Delta\varphi_i) - \tau, \sigma(\tau, \varphi+\Delta\varphi_i)) -$$

$$-X(t_j(\varphi) - \tau, \sigma(\tau, \varphi+\Delta\varphi_i))\Big)P(\sigma(\tau, \varphi+\Delta\varphi_i)) =$$

$$= \frac{\partial X(t_j(\varphi) - \tau, \sigma(\tau,\varphi))}{\partial t} P(\sigma(\tau,\varphi)) \frac{\partial t_j(\varphi)}{\partial \varphi_i} =$$

$$= A(\sigma_j) G(t_j(\varphi), \tau, \varphi) \frac{\partial t_j(\varphi)}{\partial \varphi_i}. \tag{19}$$

The case $t_j(\varphi) < \tau, t_j(\varphi + \Delta\varphi) < \tau$ in (19) is considered analogously.

We consider the next limit for $t_j(\varphi) > t_j(\varphi + \Delta\varphi_i) > 0:$

$$\lim_{\Delta\varphi_i \to 0} \frac{1}{\Delta\varphi_i}\Big(G(0, t_j(\varphi+\Delta\varphi_i), \varphi) - G(0, t_j(\varphi), \varphi)\Big) =$$

$$= \lim_{\Delta\varphi_i \to 0} \frac{1}{\Delta\varphi_i}\Big(X(-t_j(\varphi), \sigma(t_j(\varphi), \varphi)) Q(\sigma(t_j(\varphi), \varphi)) -$$

$$-X(-t_j(\varphi+\Delta\varphi_i), \sigma(t_j(\varphi+\Delta\varphi_i), \varphi)) Q(\sigma(t_j(\varphi+\Delta\varphi_i), \varphi))\Big) =$$

$$= \lim_{\Delta\varphi_i \to 0} \frac{1}{\Delta\varphi_i} X(-t_j(\varphi), \sigma(t_j(\varphi), \varphi)) Q(\sigma(t_j(\varphi), \varphi)) \times$$

$$\times \Big(I - X(t_j(\varphi) - t_j(\varphi+\Delta\varphi_i), \sigma(t_j(\varphi+\Delta\varphi_i), \varphi))\Big) =$$

$$= -G(0, t_j(\varphi), \varphi) A(\sigma_j(\varphi)) \frac{\partial t_j(\varphi)}{\partial \varphi_i}. \tag{20}$$

Here, we have used the fact that $X(t,\varphi)Q(\varphi) = Q(\varphi \cdot t) X(t,\varphi)$ and

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t}\Big(X(\Delta t, \varphi) - I\Big) = A(\varphi), \quad \varphi \in \mathbb{T}_m \setminus \Gamma.$$

With the use of the definition of the function $G(t,\tau,\varphi)$, the cases $t_j(\varphi+\Delta\varphi_i) \geq t_j(\varphi) > 0$ and $t_j(\varphi) < 0$, $t_j(\varphi+\Delta\varphi_i) < 0$ in (20) are considered similarly.

Taking into account (19) and (20), we get

$$\frac{\partial G(0,\tau,\varphi)}{\partial\varphi_i} = \int\limits_{-\infty}^{\infty} G(0,s,\varphi)\frac{\partial A(\sigma(s,\varphi))}{\partial\sigma}\frac{\partial\sigma(s,\varphi)}{\partial\varphi_i}G(s,\tau,\varphi)ds+$$

$$+\sum_{j\in\mathbb{Z}} G(0,t_j(\varphi),\varphi)\frac{\partial B(\sigma_j(\varphi))}{\partial\sigma}\frac{\partial\sigma_j(\varphi)}{\partial\varphi_i}G(t_j(\varphi),\tau,\varphi)+$$

$$+\sum_{j\in\mathbb{Z}} G(0,t_j(\varphi),\varphi)(A(\sigma_j)B(\sigma_j) - B(\sigma_j)A(\sigma_j))G(t_j(\varphi),\tau,\varphi)\frac{\partial t_j(\varphi)}{\partial\varphi_i}. \qquad (21)$$

The matrix $(\partial A(\sigma(t,\varphi))/\partial\sigma)(\partial\sigma(s,\varphi)/\partial\varphi_i)$ has the elements

$$\sum_{j=1}^{m}\frac{\partial a_{kl}(\sigma(s,\varphi))}{\partial\sigma_j}\frac{\partial\sigma_j(s,\varphi)}{\partial\varphi_i},$$

where $A(\varphi) = \{a_{kl}\}$, $\sigma = (\sigma_1,\ldots,\sigma_m)$.

The derivative $\partial G(0,\tau,\varphi)/\partial\varphi_i$ exists if the integral and series in (21) are convergent. Using (14) and (18), we estimate

$$\int\limits_{-\infty}^{\infty}\left\| G(0,s,\varphi)\frac{\partial A(\sigma(s,\varphi))}{\partial\sigma}\frac{\partial\sigma(s,\varphi)}{\partial\varphi_i}G(s,\tau,\varphi)\right\| ds \leq$$

$$\leq \int\limits_{-\infty}^{\infty} K_3^2 M e^{-\alpha|s|-\alpha|\tau-s|}ds \leq K_3^2 M\left(\frac{1}{\alpha}+|\tau|\right)e^{-\alpha|s|},$$

where $\|A(\varphi)\|_s \leq M$, $\|B(\varphi)\|_s \leq M$, and $\|\partial\sigma(s,\varphi)/\partial\varphi_i\| = \|\partial(\omega t+\varphi)/\partial\varphi_i\| = 1$.

By (14), (17) and (18), we get

$$\sum_{j\in\mathbb{Z}}\left\| G(0,t_j(\varphi),\varphi)\frac{\partial B(\sigma(t_j,\varphi))}{\partial\sigma}\frac{\partial\sigma(t_j(\varphi),\varphi)}{\partial\varphi_i}G(t_j(\varphi),\tau,\varphi)\right\| \leq$$

$$\leq K_3^2 MC\left(\frac{2}{1-e^{-2\alpha\theta}}+\frac{|\tau|}{\theta}\right)e^{-\alpha|\tau|}.$$

The last sum in (21) is estimated similarly.

Therefore,

$$\left\|\frac{\partial G(0,\tau,\varphi)}{\partial\varphi_i}\right\| \leq (K_4+K_5|\tau|)e^{-\alpha|\tau|} \leq K_6 e^{-\alpha_1|\tau|}, \qquad (22)$$

where $\alpha_1 = \alpha-\varepsilon$, ε is an arbitrarity small positive value, and $K_4, K_5, K_6 = K_6(\varepsilon)$ are positive constants independent of $\varphi \in \mathbb{T}_m$.

Now we consider the second derivative

$$\frac{\partial^2 G(0,\tau,\varphi)}{\partial\varphi_k\partial\varphi_l} = \int\limits_{-\infty}^{\infty} \frac{\partial}{\partial\varphi_k}\left[G(0,s,\varphi)\frac{\partial A(\sigma(s,\varphi))}{\partial\varphi_l}G(s,\tau,\varphi)\right]ds+$$

$$+\sum_{j\in\mathbb{Z}}\left(G(0,s,\varphi)\frac{\partial A(\sigma(s,\varphi))}{\partial\varphi_l}G(s,\tau,\varphi)\right)\frac{\partial t_j(\varphi)}{\partial\varphi_k}\Bigg|_{s=t_j(\varphi)+0} -$$

$$-\sum_{j\in\mathbb{Z}}\left(G(0,s,\varphi)\frac{\partial A(\sigma(s,\varphi))}{\partial\varphi_l}G(s,\tau,\varphi)\right)\frac{\partial t_j(\varphi)}{\partial\varphi_k}\Bigg|_{s=t_j(\varphi)-0} +$$

$$+\sum_{j\in\mathbb{Z}}\frac{\partial}{\partial\varphi_k}\left[G(0,t_j(\varphi),\varphi)\left(\frac{\partial B(\sigma_j)}{\partial\varphi_l}G(t_j(\varphi),\tau,\varphi)+\right.\right.$$

$$\left.\left.+(B(\sigma_j)A(\sigma_j)-A(\sigma_j)B(\sigma_j))G(t_j(\varphi),\tau,\varphi)\frac{\partial t_j(\varphi)}{\partial\varphi_l}\right)\right]. \tag{23}$$

The integrand in (23) is estimated by the exponent $\exp(-\alpha_1|s|-\alpha|\tau-s|)$, and the j-terms in all series are estimated by $\exp(-\alpha_1|t_j|-\alpha|\tau-t_j|)$. Hence, repeating estimates for $\partial G/\partial\varphi_k$, we get

$$\left\|\frac{\partial^2 G(0,\tau,\varphi)}{\partial\varphi_k\partial\varphi_l}\right\| \leq K_7 e^{-\alpha_1|\tau|}, \tag{24}$$

where $\alpha_1 = \alpha-\varepsilon, \varepsilon > 0$, and $K_7 = K_7(\varepsilon)$. For $\tau = 0$, we get the second derivative of the projector $P(\varphi)$.

To estimate higher-order derivatives of $G(0,\tau,\varphi)$ and $P(\varphi)$ (up to the s-th order), we continue the above approach. By successive differentiation of (23) and by the estimates for the i-th derivative of the integrand by $\exp(-\alpha_1|s|-\alpha|\tau-s|)$, and for the j-th terms in all series by $\exp(-\alpha_1|t_j|-\alpha|\tau-t_j|)$, we conclude that the integral and all series are convergent. Thus, we prove the existence of derivatives (up to the s-th order) of the projector $P(\varphi)$ and the Green function $G(t,\tau,\varphi)$. The theorem is proved.

2. Invariant sets of nonlinear systems. Let us consider a nonlinear impulsive system

$$\frac{d\varphi}{dt} = \omega, \tag{25}$$

$$\frac{dx}{dt} = A(\varphi)x + f(\varphi,x), \quad \varphi\in\mathbb{T}_m\setminus\Gamma, \tag{26}$$

$$\Delta x\Big|_{\varphi\in\Gamma} = B(\varphi)x + g(\varphi,x), \tag{27}$$

where $x\in\mathbb{R}^n$, $\varphi\in\mathbb{T}_m$, $\mathbb{T}_m$ is m-dimensional torus, and Γ is a smooth compact submanifold of $\mathbb{T}_m$ of codimension 1. The functions $f(\varphi,x)$, $g(\varphi,x)$ are supposed to be periodic in φ_i and piecewise smooth in (φ,x).

We consider the problem of existence of a piecewise continuous (piecewise smooth) invariant set of system (25) – (27), i.e., a function $x = u(\varphi)$ continuous (smooth) in $\mathbb{T}_m \setminus \Gamma$ with discontinuities of the first kind on Γ such that $x(t, \varphi) = u(\varphi \cdot t)$ is a solution of system (25) – (27). Using the piecewise smoothness of the projector $P(\varphi)$ and the Green function $G(t, \tau, \varphi)$, analogously to Theorem 2 [5], we can prove the following theorem:

Theorem 3. *Assume that system (1) – (3) is exponentially dichotomous and the following inequalities hold:*

$$\|f(\varphi, 0)\|_0 \leq M_0, \quad \|g(\varphi, 0)\|_0 \leq M_0,$$

$$\|f(\varphi, x) - f(\varphi, y)\|_0 + \|g(\varphi, x) - g(\varphi, y)\|_0 \leq N\|x - y\|,$$

where $M_0 > 0$ and the constant $N > 0$ satisfies the inequality

$$2NK_3\left(\frac{1}{\alpha} + \frac{1}{1 - e^{-\alpha\theta}}\right) < 1.$$

Then system (25) – (27) has an invariant set $x = u(\varphi), \varphi \in \mathbb{T}_m$, where the function $u(\varphi)$ is continuous for $\varphi \in \mathbb{T}_m \setminus \Gamma$ and has discontinuities of the first kind on the set $\varphi \in \Gamma$. The trajectories on the invariant set $x = u(\varphi)$ are quasi-periodic with the first-order discontinuities.

Assume that the functions $A(\varphi), B(\varphi), f(\varphi, x), g(\varphi, y)$ have continuous partial derivatives with respect to φ, x up to the s-th order inclusively and we have the inequality

$$\|f(\varphi, x) - f(\varphi, y)\|_s + \|g(\varphi, x) - g(\varphi, y)\|_s \leq N\|x - y\|, \tag{28}$$

$$\|f(\varphi, 0)\|_s \leq M_0, \quad \|g(\varphi, 0)\|_s \leq M_0 \tag{29}$$

with a sufficiently small constant N. Then the function $u(\varphi)$ has continuous partial derivatives with respect to φ up to the s-th order for $\varphi \in \mathbb{T}_m \setminus \Gamma$.

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Received 10.01.98