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Two-dimensional electron gas in a magnetic field and point potentials

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This paper presents a brief review of the electron properties in two-dimensional systems which contain zero-range scatterers and which are subjected to a magnetic field. The electron spectrum is described for a periodic arrangement of point scatterers and rational magnetic flux per unit cell. Delocalized states on the Landau levels are constructed for the case of positional disorder. The electron localization in a one-dimensional disordered set of scatterers is studied. Application to the study of electron transmission through quantum dots and ballistic channels is reviewed.

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1. Introduction

Fifty years ago Ilya Mikhailovich Lifshits published a paper, entitled «On the Theory of Degenerate Regular Perturbations. I. Discrete Spectrum», in the Journal of Experimental and Theoretical Physics. This paper was a starting point of a series of papers in which I. M. Lifshits developed his famous theory of degenerate perturbations and successfully used it in studying the vibrations of disordered lattices. The theory of degenerate perturbations was found to be a very effective tool for investigation of the local modes that appear outside of an initial spectrum. I. M. Lifshits emphasized that an «important class of perturbations, which can be reduced to the form of a degenerate perturbation, corresponds to local perturbations with a very short range». For an electron described by the Schrödinger equation, the local perturbation corresponds to a zero range or point potential,which was first introduced by E. Fermi [2,3] for three dimensional ($3D$) systems. Today the point potential model is one of the very popular models in solid state physics and nuclear physics (see, e.g., the monographs in Ref. 4 and 5 and the more recent mathematical review papers [6,7]). This model is especially useful for studying the electronic properties of $2D$ systems in a magnetic field. These systems are ideal objects for application of the local perturbation theory because of a very rarefied initial spectrum. It consists of a zero measure set of the infinitely degenerate, discrete Landau levels [8,9] that leave empty almost all of the energy axis for the new states which occure as a result of degenerate perturbation and which are described by the Lifshits theory (see Eqs. (2) and (3) below).

This paper is actually a mini-review dealing with the electron properties of $2D$ systems with point potentials and magnetic field. In Sec. 2 we describe the main points of the Lifshits theory of degenerate perturbations in the modern formulation, the definition of a point potential, and modification of the main equations of the degenerate perturbation theory for the case of an electron moving in an external field and in a field of a set of point potentials. In Sec. 3 we recall the well-known basic results concerning electron dynamics in the presence of a magnetic field, obtain the scattering amplitude of a point potential in a magnetic field, and describe qualitatively the structure of electron spectrum in a magnetic field and in a field of a single point potential and of a set of point potentials. The Bloch-type electron eigenstates in a field of an ordered set of point potentials and in a so-called rational magnetic field are described in Sec. 4, in which

the dispersion laws and energy-flux diagram (Hofstadter-type butterfly [10]) are obtained. Two-dimensional systems containing a disordered set of point potentials are considered in Sec. 5. Here we describe the delocalized states on the Landau levels (Subsec. 5.1), the density of states (DOS) for some of the exactly solvable models (Subsec. 5.2), and the electron localization in the field of a one-dimensionally disordered set of point potentials (Subsec. 5.3). In Sec. 6 we deal with the application of point potentials to the theory of mesoscopic objects such as quantum dots (Subsec. 6.1) and ballistic conducting channels (Subsec. 6.2). In Sec. 7 we give a brief list of some of the unsolved problems and conclude this mini-review.

2. Degenerate perturbations and point potentials

Let us consider a quantum system described by a Hamiltonian $H + V$. The degenerate perturbation V is defined as [1]

$$V = \sum_j V_j |v_j\rangle\langle v_j| , \tag{1}$$

where $\{|v_j\rangle\}$ is a set of orthogonal and normalized states. The perturbed eigenstates $|\psi\rangle$ with eigenenergies E lying outside the initial spectrum of the unperturbed Hamiltonian H are the sums of scattered waves

$$|\psi\rangle = \sum_j \eta_j G(E)|v_j\rangle , \eta_j = V_j\langle v_j|\psi\rangle ,$$
$$G(E) = (E - H)^{-1} . \tag{2}$$

As was shown in Ref. 1, the eigenenergies E and the corresponding sets of coefficients η_j can be found from the set of equations

$$\sum_j \Lambda_{ij}(E)\eta_j = 0 . \tag{3}$$

Here

$$\Lambda_{ij}(E) = \frac{\delta_{ij}}{T_j(E)} - (1 - \delta_{ij})G_{ij}(E) ,$$
$$G_{ij}(E) = \langle v_j|G(E)|v_j\rangle , \tag{4}$$

where $T_j(E)$ is the scattering amplitude

$$\frac{1}{T_j(E)} = \frac{1}{V_j} - G_{jj}(E) . \tag{5}$$

Equations (2) and (3) are the principal ingredients in the Lifshits theory of local perturbations. In the Russian scientific literature they are often called the Lifshits equations (see, e.g., Ref. 11). The main point in this theory is the expression of the perturbation in a form (1). Because of this form, the perturbation (1) is the sum of the projection operators, such that the j th operator projects the state $|\psi\rangle$ onto a number η_j [see Eq. (2)]. Thus the perturbed states (2) lying outside the initial spectrum can be mapped onto the discrete set of coefficients η_j , which map the initial Schrödinger equation $(H + V)|\psi\rangle = E|\psi\rangle$ on the Lifshits equations (3). Note that these equations, in spite of their visible simplicity, are complex equaction, because all their coefficients contain the spectral parameter (unknown eigenenergy E) in a complicated, nonlinear way.

The Green's operator $G_V(E) = (E - H_V)^{-1}$ of the perturbed Hamiltonian $H_V = H + V$ can be expressed in terms of the Green's operator $G(E)$ of the unperturbed Hamiltonian and the same matrix $\Lambda(E)$ (4)

$$G_V = G + \sum_{i,j} G|v_i\rangle\left(\Lambda^{-1}\right)_{ij}\langle v_j|G . \tag{6}$$

Using expression (5) for the scattering amplitude, we can rewrite the matrix $\Lambda(E)$ [Eq. (4)] in the form

$$\Lambda(E) = A + Q(E) ,$$

where

$$A_{ij} = \frac{\delta_{ij}}{V_j} ,$$

and

$$Q_{ij}(E) = -G_{ij}(E) . \tag{7}$$

The diagonal matrix A depends on the perturbation (1) only, while the matrix $Q(E)$ reflects the properties of the unperturbed Hamiltonian H.

Let us consider now a single electron which is described by the Schrödinger equation with an unperturbed Hamiltonian H which includes the interaction with an external (e.g., magnetic) field. In this case an important class of perturbations, which can be reduced to the form of degenerate pecturbations, i.e. which can be described by equations such as Eqs. (1)–(6), is formed by zero-range limits of finite-range local potentials, which we call point potentials. A $2D$ point potential can be treated as an attractive zero-range potential with a single bound state with a fixed eigenenergy $-E_b$. There are several ways to introduce such a potential. The first one is based on the idea [12] of regarding a point potential as a delta-function with an infinitely small amplitude. More precisely, the Fourier image of a point potential must be equal to some constant within a circle centered at the origin, and zero otherwise. The corresponding limiting procedure (the constant tends to zero and the

radius tends to infinity, keeping the scattering data fixed) should then be used. The second approach [5] is based on the Krein theory of self-adjoint extensions [13]. Finally, a point potential can be described also with the help of a direct zero-range limit in coordinate space [14]. The resulting $2D$ point potential represents a certain generalized function which is less singular than the $2D$ Dirac delta function. The character of this singularity is attributable to the logarithmic singularity of the $2D$ Green's function in coordinate representation with coinciding arguments. A detailed exposition of a general theory and numerous applications of point potentials in all three dimensions can be found in Refs. 4 and 5.

The main equations of degenerate perturbations theory, (2)–(4), and (6), remain valid for the case of an electron moving in a field of a point potential. However, some minor modifications must be made (see details, e.g., in Ref. 15). First, instead of the set of states $|v_j\rangle$ we must now use the set of states $|\mathbf{r}_j\rangle$ localized at the points $\{\mathbf{r}_j\}$, where the point potentials are placed. We can therefore write the scattered wave [Eq. (2)] with the eigenenergy E as follows:

$$\psi(r) = \sum_j \eta_j G(\mathbf{r}, \mathbf{r}_j; E), \qquad (8)$$

where $G(\mathbf{r}, \mathbf{r}'; E) = \langle \mathbf{r}|G(E)|\mathbf{r}'\rangle$ is the Green's function of the unperturbed Hamiltonian in coordinate representation. Second, the scattering amplitude in this case is

$$\frac{1}{T_j(E)} = \lim_{\mathbf{r},\mathbf{r}' \to \mathbf{r}_j} \left[G_0(\mathbf{r}, \mathbf{r}'; -E_{bj}) - G(\mathbf{r}, \mathbf{r}'; E)\right] =$$

$$= G_0^{\mathrm{reg}}(\mathbf{r}_j, \mathbf{r}_j; -E_{bj}) - G^{\mathrm{reg}}(\mathbf{r}_j, \mathbf{r}_j; E). \qquad (9)$$

Here $G_0(\mathbf{r}, \mathbf{r}'; E)$ is the Green's function of the free Hamiltonian $H_0 = -\Delta$ (Δ is the $2D$ Laplace operator), E_{bj} is the binding energy of the point potential placed at the point $\mathbf{r}_j$, and finally $G^{\mathrm{reg}}(\mathbf{r}_j, \mathbf{r}_j; E)$ denotes the regular part of the Green's function. The scattering amplitude (9), together with an obvious equality

$$G_{ij}(E) \equiv G^{\mathrm{reg}}(\mathbf{r}_i, \mathbf{r}_j; E) \qquad (10)$$

(in fact, only the diagonal elements are needed in the regularization procedure), defines the matrices $\Lambda(E)$ in Eq. (4), and $Q(E)$ in Eq. (7). The latter matrix Q in this case is called the Krein matrix [6,7,13]. The diagonal matrix A has the elements $\delta_{ij} G_0^{\mathrm{reg}}(\mathbf{r}_j, \mathbf{r}_j; -E_{bj})$. Equation (6) for the perturbed Green's function has the form

$$G_V(\mathbf{r}, \mathbf{r}'; E) = G(\mathbf{r}, \mathbf{r}'; E) +$$

$$+ \sum_{i,j} G(\mathbf{r}, \mathbf{r}_i; E)(\Lambda^{-1})_{ij} G(\mathbf{r}_j, \mathbf{r}'; E). \qquad (11)$$

Thus, Eqs. (3), (4), and (8)–(11) give a complete description of the electron spectral properties in a system with point potentials and an external field. The central role in such a description is played by the matrix $\Lambda(E)$: the zeros of its determinant are the eigenenergies E_n of the perturbed Hamiltonian $H + V$, its eigenvectors $\{\eta_j(E_n)\}$, which correspond to the zero eigenvalue (3), determine the wave functions (8) with eigenenergies lying outside the spectrum of the unperturbed Hamiltonian H, and finally the inverse matrix $(\Lambda)^{-1}$ determines the Green's function (11) of the perturbed system $H + V$.

3. Combination of point potentials and magnetic field

The motion of electron with charge $-e$ and mass M in a $2D$ system, subject to a perpendicular magnetic field $\mathbf{B} = (0, 0, -B)$, is described by the Schrödinger equation

$$-(\nabla + 2\pi i\varphi \mathbf{A})^2 \psi(\mathbf{r}) = E\psi(\mathbf{r}). \qquad (12)$$

We use the set of units, where $\hbar = 2M = 1$ and all the lengths are measured in some scale d. In such units all quantities are dimensionless. In Eq. (12) $\mathbf{A}$ is a vector potential and φ is dimensionless flux

$$\varphi = \frac{\Phi}{\Phi_0}, \quad \Phi = Bd^2 \qquad (13)$$

($\Phi_0 = hc/e$ are the normal flux quanta). In what follows we choose the Landau gauge of a vector potential

$$\mathbf{A} = (y, 0, 0).$$

The spectrum of the system consists of a set of the Landau levels [8,9]

$$E_{nL} = \left(n + \frac{1}{2}\right) E_L, \quad n = 0, 1, ..., \qquad (14)$$

where $E_L = 4\pi\varphi$ is the distance between Landau levels. The normalized eigenfunctions are the Landau functions

$$L_{n,k}(x, y) = (2^{n+1}\pi^{3/2} n! l)^{-1/2} \times$$

$$\times e^{-ikx - 1/2(y/l - kl)^2} H_n(y/l - kl), \qquad (15)$$

$$n = 0, 1, ..., -\infty < k < \infty,$$

($l = (2\pi\varphi)^{-1/2}$ is dimensionless magnetic length, and

H_n is the Hermite polynomial [16]). Another possible set of eigenfunctions consists of the states $L_{n,m}(x, y)$, which coincide with the gauge transformed states with a fixed angular momentum m in the symmetric gauge. For a fixed number n of the Landau level the angular momentum m is an integer and satisfies the inequality $-\infty < m \le n$. The Green's function in coordinate representation is

$$G(\mathbf{r}, \mathbf{r}'; E) = -\frac{\Gamma(-a)}{4\pi} e^{-i\pi\varphi(x-x')(y+y')} g(\mathbf{r} - \mathbf{r}'; E), \quad (16)$$

where

$$g(\mathbf{r}; E) \equiv F(\xi) = \xi^{-1/2} W_{1/2+a,0}(\xi), \quad \xi = \pi\varphi r^2. \quad (17)$$

Here the gamma Γ and Whittaker W functions [16] are used and a is a new energy parameter

$$a = -\frac{1}{2} + \frac{E}{E_L}.$$

The scattering amplitude of a point potential in a magnetic field in accordance with Eqs. (9) and (16) is

$$\frac{1}{T_f(E)} = -\frac{1}{4\pi}\left(\psi(-a) + \ln\frac{E_L}{E_b}\right), \quad (18)$$

where $\psi(-a)$ is the digamma function [16]. This expression, which was obtained with the help of each one of the three approaches mentioned above [14,15,17–20], describes now an electron spectrum in a magnetic field and in a field of a single point potential. A point potential scatters only s-states [21]. Therefore, the unperturbed wave functions $L_{n,m}(x, y)$, which correspond in the symmetric gauge to the states with a fixed nonzero angular momentum m, vanish at the origin and therefore are not modified by a point potential. They correspond to the Landau eigenenergies E_{nL} which are zeros of the scattering amplitude (i.e., poles of the digamma function in Eq. (18)). New eigenenergies E_n, which correspond to the scattered s-states, are determined by the poles of the scattering amplitude, Eq. (18). Because of the attractive character of a point potential, the values E_n are shifted down with respect to the Landau levels. Each perturbed s-wave function $\psi_n(\mathbf{r})$ with eigenenergy E_n coincide with the unperturbed Green's function $G(\mathbf{r}, 0; E_n)$ (16) and therefore has a logarithmic singularity at the origin. The magnetic field dependence of the lowest five shifted eigenenergies is shown [15] in Fig. 1.

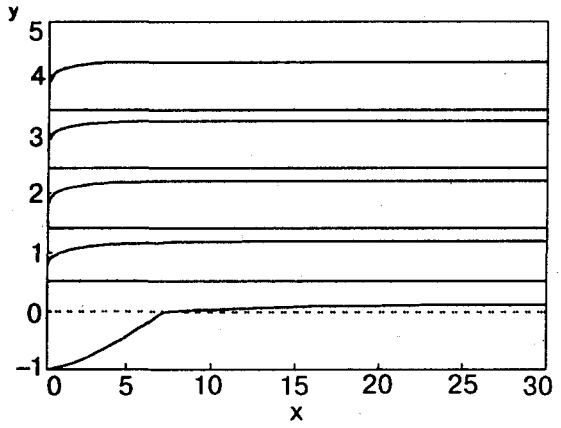

Fig. 1. The s-levels shifted by a single point potential in a magnetic field in the scale $y = E/E_L = a + 1/2$ for the positive energies; $y = E/E_b$ for the negative energies; $x = E_L/E_b$. Thus, $y = -1$ corresponds to a bound state without a magnetic field and $y = n + 1/2, n \ge 0$ corresponds to the nth Landau level.

Such a picture of the spectrum perturbed by a single point potential, is valid in the ultimate zero-range limit only. The finiteness of the potential radius, $\rho \ne 0$, leads to an additional shift of the eigenenergy E_n and to an additional splitting of the Landau level E_{nL} [22]. For a cylindrically symmetric potential the states can be classified with respect to the angular momentum m, which they would have in the symmetric gauge. The corresponding shifted eigenenergies $E_{n,m}$ do not depend on the sign of the momentum m,

$$\varepsilon_{n,m} \equiv E_{nL} - E_{n,m} \propto (\rho/l)^{2|m|} / |\ln(\rho^2 E_b)|,$$

and therefore the eigenenergies $E_{n,m}$ with $|m| \le n$, remain twofold degenerated [23]. Note that the finite radius of the potential leads not only to the energy shift, but also modifies the eigenfunctions with $m \ne 0$, so that their intermediate asymptotic relations contain terms proportional to $r^{-|m|}$,

$$\psi_m(\mathbf{r}) \approx r^{|m|} + \frac{r_m^{2|m|}}{r^{|m|}} \quad \rho \ll r,$$

with $r_m \propto \rho / |\ln(\rho^2 E_b)|^{1/2|m|}$ (Ref. 24).

The main qualitative features of the electron spectrum in a magnetic field and in a field of a set of point potentials can be established solely with the help of the fact that the radius of a point potential equals zero. Indeed, consider an electron moving in a field of a set of zero-range potentials which are placed at the points $\{\mathbf{r}_j\}$ and let d be the averaged distance between scatterers (in standard dimensional variables). If the electron wave function vanishes at a point where a scatterer is placed, the electron does not «feel» this scatterer. Therefore, if one can construct a linear combination $\psi(\mathbf{r})$ of Landau functions (15) with a fixed Landau level number n, which vanishes at all points

$\{\mathbf{r}_j\}$, then this combination will be an exact eigenfunction of an electron *in the presence of this set of point potentials* with the eigenenergy E_{nL} [Eq. (14)], Refs. 22, 25.

To determine whether such a combination exists, recall that the number of states per scatterer per Landau level for a given density of scatterers equals exactly the dimensionless flux φ [Eq. (13)]. On the other hand, we have exactly one condition $\psi(\mathbf{r}) = 0$ per scatterer per Landau level which the wave function must satisfy. If the magnetic field B is sufficiently strong, $\varphi > 1$, then the condition $\psi(\mathbf{r}_j) = 0$ can be satisfied at all points $\{\mathbf{r}_j\}$. This means that only one eigenstate per single scatterer is shifted from each Landau level. The other $\varphi - 1$ states remain on the Landau level [22, 25]. In a small enough field, $\varphi < 1$, all of the states are shifted from the Landau level.

Thus the electron spectrum is naturally divided into two components. The first component consists of the eigenvalues lying outside the Landau levels and is described by the Lifshits equations (3). This component exists in an arbitrary magnetic field. In disordered systems the corresponding eigenstates are localized. (In a recent paper [26] this statement was rigorously proved for a model reduced to the lowest Landau level). The number of these states per point potential per Landau level equals unity for a strong field, $\varphi > 1$, and equals φ in the opposite case $\varphi < 1$. In the periodic case for the rational flux $\varphi = Q_1/Q$ per unit cell with area d^2, these states are Bloch type. In complete accordance with the group theory predictions, [27,28] they form $\min\{Q_1, Q\}$ dispersive subbands below each Landau level with the number of states per point potential equal Q^{-1} per subband [17,29,30]. Each of these subbands in turn is Q-fold degenerate.

The second component consists of the Landau levels themselves. This component exists only in a sufficiently strong magnetic field $\varphi > 1$. The states that realize this component are linear combinations of the Landau functions with a fixed number of levels, which vanish at all points where the point potentials are located. For a spatially uniform (on the average) set of point potentials these states could always be chosen to be delocalized. Indeed, if these states are chosen to be localized, then their localization centers are uniformly distributed. On the other hand, since their eigenenergies are the same, we can rearrange them to be delocalized states [22]. The density of these states is

$$\rho_{\text{sing}}(E) = (\varphi - 1) \sum_n \delta(E - E_{nL}) . \qquad (19)$$

4. Bloch states outside the Landau levels

Let us now consider a set of identical point potentials with binding energy E_b and assume that the positions $\mathbf{r}_j$ form (for simplicity) a square lattice with a unit lattice constant. Thus, $\mathbf{r}_j = (m_j, n_j)$ with integers m_j, n_j. This system represents a particular case of a $2D$ periodic system in a magnetic field. In an irrational magnetic field φ the spectrum of the system is of the devil-staircase type [31], while for a rational field $\varphi = Q_1/Q$ the spectrum is of a Bloch-type [27,28,31]. In this case two components $T_{(1,0)}$ and $T_{(0,Q)}$ of the magnetic translation operator [27]

$$(T_{\mathbf{R}}\psi)(\mathbf{r}) = e^{2\pi i\varphi \mathbf{r}\mathbf{A}(\mathbf{R})}\psi(\mathbf{r} + \mathbf{R})$$

($\mathbf{A}$ is the vector potential) commute with one another and with the perturbed Hamiltonian $H + V$. Therefore, the electron wave functions can be choosen as eigenfunctions of all translation operators $T_{\mathbf{R}}$ with $\mathbf{R} = (m, Qn)$ (m and n are integers), with eigenvalues $e^{i\mathbf{q}\mathbf{R}}$. The quasimomentum vector $\mathbf{q} = (q_1, q_2)$ is defined in a rectangle $0 < q_1, Qq_2 < 2\pi$ (the magnetic Brillouin zone) and the eigenfunctions $\psi_{\mathbf{q}}(\mathbf{r})$ have Bloch-like form

$$\psi_{\mathbf{q}}(\mathbf{r}) = u_{\mathbf{q}}(\mathbf{r})\, e^{i\mathbf{q}\mathbf{r}}$$

with quasi-Bloch amplitudes

$$u_{\mathbf{q}}(\mathbf{r} + \mathbf{R}) = u_{\mathbf{q}}(\mathbf{r})\, e^{-2\pi i\varphi \mathbf{r}\mathbf{a}(\mathbf{R})} .$$

Nevertheless even for rational fluxes the spectrum has a very complicated fractal structure (in spite of simple Bloch-like wave functions as solutions) which is defined by the arithmetical nature of the flux φ [31,10].

For a field which corresponds to a rational flux, the point character of the periodic potential enables one to study the spectrum of the system in more detail. The first results were obtained in Refs. 7, 17, 29, and 30. Using the symmetry of the problem, the authors expanded the Hamiltonian of the system into a direct integral over possible values of quasi-momenta $\mathbf{q}$ and studied the discrete spectra corresponding to each sub-Hamiltonian with a fixed $\mathbf{q}$. An exact expression for the Green's function of the system was obtained and analized for an arbitrary rational flux φ. This enabled the authors to establish, on the mathematical level of rigor, the main qualitative properties of the spectrum, which were mentioned at the end of the last section.

The next step was carried out in Refs.19, 20, and 32, where the Lifshits equations (3) in a magnetic

field were obtained and studied. Here the symmetry of the equations (3) (after mapping) was used. The matrix $(\Lambda)^{m',n'}_{m,n}$, Eq. (4), depends only on the difference $m - m'$. Therefore, the mapped wave function $\eta_j \equiv \eta_{m,n}$ (2) can be found in the form

$$\eta_{m,n} = e^{iq_1 m}\zeta_n \tag{20}$$

with ζ_n satisfying a one-dimensional difference equation

$$\frac{\ln(E_c/E_b)}{\Gamma(-a)}\zeta_{n_0} = \sum_{n=-\infty}^{\infty} F_{n_0,n}\zeta_n . \tag{21}$$

Here

$$F_{n_0,n} = \sum_{m=-\infty}^{\infty} F(\xi_{m,n-n_0})\, e^{im[q_1+\pi\varphi(n+n_0)]},$$

where $F(0) = -\psi(-a)/\Gamma(-a)$, and $F(\xi)$ for $\xi \neq 0$ is defined by Eq. (17) with $\xi_{m,n} = \pi\varphi(m^2 + n^2)$. Equation (21) is used below in Subsec. 5.3. Further, the matrix $(\Lambda)^{m',n'}_{m,n}$ is invariant with respect to a simultaneous shift $n \to n + Q$, $n' \to n' + Q$. This results in the final Bloch form of a mapped wave function

$$\eta_{m,n} = e^{i(q_1 m + q_2 n)}c(n)$$

with the Q-periodic discrete Bloch amplitudes $c(n) = c(n + Q)$, which satisfy the equations [19,20,32]

$$\frac{\ln(E_c/E_b)}{\Gamma(-a)}c(\lambda) = \sum_{\mu=0}^{Q-1} S_{\lambda\mu}c(\mu), \quad 0 < \lambda, \mu < Q - 1. \tag{22}$$

The coefficients in this equation are

$$S_{\lambda\mu} = \sum_{N,m=-\infty}^{\infty} F(\xi_{m,NQ+\mu-\lambda}) \times$$

$$\times\, e^{i[mq_1 + (NQ+\mu-\lambda)q_2] + im\pi[NQ+\mu+\lambda]Q_1/Q}. \tag{23}$$

In each energy interval between adjacent Landau levels equation (22) has, for a fixed q, exactly Q solutions. For a strong field $Q_1 > Q$ these solutions determine Q dispersion laws $E_i(q)$, which correspond to Q sets of coefficients $\eta^{(i)}_{m,n}(q)$, $i = 0, ..., Q - 1$. These coefficients, together with Eq. (2), determine Q dispersive subbands, which accumulate exactly one state per scatterer per Landau level. Some of the dispersion surfaces are demonstrated in Fig.2 [33].

For a weak magnetic field $Q_1 < Q$ Eq. (22) can be reduced to a finite set of Q_1 equation [34], which determine Q_1 dispersive subbands $E_i(q)$, $i = 0, ..., Q_1 - 1$. The other $Q - Q_1$ nontrivial solutions $\eta^{(i)}_{m,n}(q)$, $i = Q_1, ..., Q - 1$ of Eq. (22), when substituted into Eq. (2), lead to the wave functions $\psi^{(i)}(r)$ which are identically equal to zero [35]. All these results are in complete agreement with the qualitative picture of the spectrum described in the previous section.

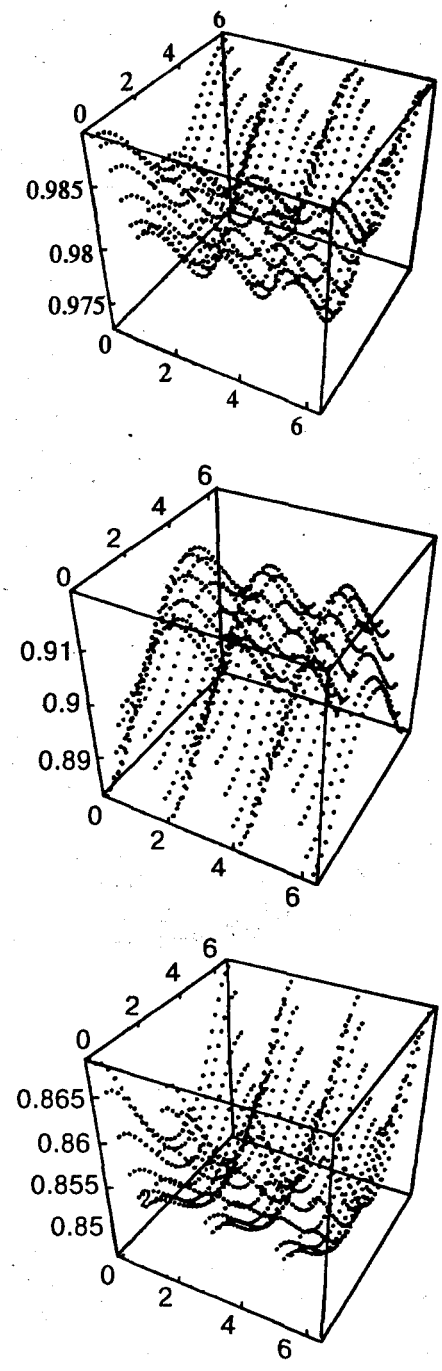

Fig. 2. The dispersion relations $a_i(q)$, $i = 1,2,3$ of the spectrum for flux $\varphi = 5/3$ describing three subbands which are situated between the first two Landau levels.

Equation (22) is very complicated even for a numerical solution because, as was mentioned above, it is not a standard eigenvalue equation: the energy parameter a appears on both sides of this equation in a nonlinear way. Therefore, let us first discuss different approximate methods of the solution. The dispersive bands are the result of spreading of a single scatterer levels discussed in the last section. In the limiting cases of a very strong or very weak scatterer, when the parameter $|\ln(E_L/E_b)|$ is large, these levels are close to the initial Landau levels. In this case one can substitute $E = E_{nL}$ on the right side of Eq. (22) and linearize the left side of this equation with respect to the difference $E - E_{nL}$. Such a procedure significantly simplifies the solution of Eq. (22), reducing it to a standart eigenvalue problem. In Ref. 36, where this approximate sheme was proposed, the linearized version of Eq. (22) was solved numerically for the energy region lying between the zeroth and the fourth Landau levels and for rational fluxes with denominators $Q \leq 9$. All the corresponding dispersion relations were obtained and the energy–flux diagram was constructed. Calculations in a weak magnetic field confirmed the existence of Q_1 dispersive subbands. Numerical results obtained in Ref. 36 indicated that for some rational values of the magnetic field dispersive subbands touch the parent Landau level at some special points q. The condition for touching the nth Landau level is $\varphi \leq n + 1$. The touching of a given Landau level is evidently related to the symmetry nature of the problem and is not related to the strength of the point potential. Note also that the closer is the energy to the Landau level, the more exact is the approximation applied, so we maintain that numerical results of Ref. 36 concerning this touching are exact. These statements are also used below in subsec. 5.3.

Another approximation can be applied in the strong field limit. Consider Eq.(21) near the nth Landau level. If the Larmor radius is larger than the lattice constant, i.e.,

$$\pi\varphi > n\,, \tag{24}$$

then, because of an exponential decrease of the Whittaker function [16], one should account only for the nearest neighbors in this equation. Neglecting coupling with non-nearest neighbors, we obtain the well-known Harper equation [37]

$$\zeta_{n+1} + \zeta_{n-1} + 2\zeta_n \cos(2\pi n\varphi + q_1) = \varepsilon\zeta_n\,, \tag{25}$$

where

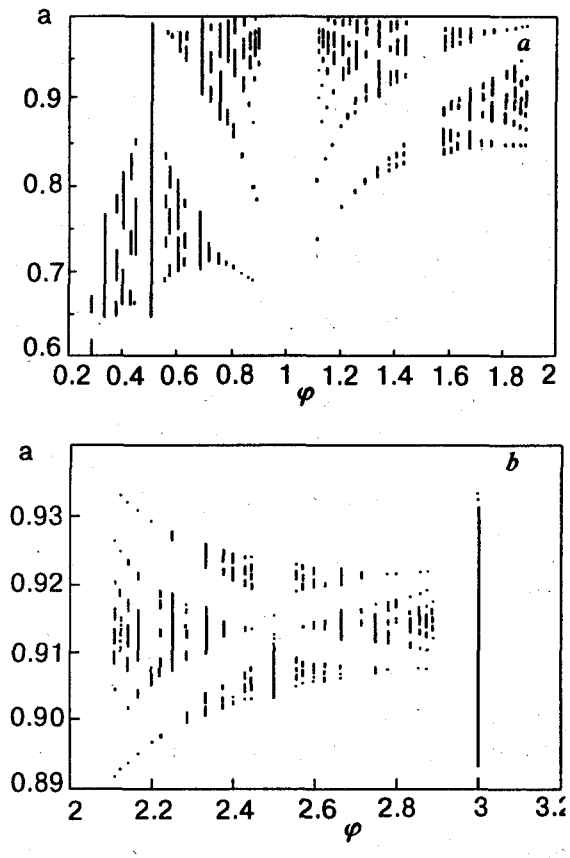

Fig. 3. The energy-flux diagram between the first two Landau levels, $0 \leq a \leq 1$, for the rational fluxes $0 < \varphi < 3$ with denominators up to 9: $0 < \varphi < 2$, where the exact equations are used (*a*); $2 < \varphi < 3$, where the strong-field approximation (the Harper equation) is used; only diagonal values of the quasi-wave vector $\mathrm{q} = \pi t(1/Q, 1)$, $0 < t < 1$, are accounted for (*b*).

$$\varepsilon = \frac{\sqrt{\alpha}}{W_{1/2+a,0}(\alpha)} \frac{\psi(-a) + \ln(E_L/E_b)}{\Gamma(-a)}\,, \quad \alpha \equiv \pi\varphi\,, \tag{26}$$

and the single-valued branches of the gamma- and digamma- functions, which correspond to the vicinity of the nth Landau level, are chosen.

It is interesting to note that Harper equation appeared for the first time in the context of electron theory of metals in the opposite limiting case. For a weak magnetic field one can start from the dispersion law $E(\mathrm{p})$ of the system without a magnetic field, and use the Peierls substitution, which replaces the quasi-momentum p by the operator $[(\hbar/i)\nabla - (e/c)\mathrm{A}]$ [38]. Then the strong-coupling approximation immediately leads to the Harper equation [39].

Substituting into Eq. (25) $\zeta_n = c(n, \mathrm{q}) \exp(inq_2)$ with the Q-periodic functions $c(n)$, we obtain in the general case $Q \geq 3$ (the cases $Q = 1,2$ are trivial) the following difference second-order equation:

$$\exp(iQq_2)\, c(n+1, \mathbf{q}) + \exp(-iQq_2)\, c(n-1, \mathbf{q}) + 2c(n, \mathbf{q}) \cos(2\pi n\varphi + q_1) = \varepsilon c(n, \mathbf{q}).$$

Thus, in the «ε» scale the dispersion laws $\varepsilon_i(\mathbf{q})$ and the eigenvectors $c_i(n, \mathbf{q})$ corresponding to the spectral region (24) are *universal* (within the accuracy of the approximation) in the sense that they do not depend either on the Landau level number or on the integer part of the dimensionless flux φ [40]. The integer part of the flux and the concrete position of the eigenenergy between Landau levels enter when one restores the dispersion function $E_i(\mathbf{q})$ from Eq. (26) in the natural energy scale.

Detailed numerical calculations carried out in Refs. 33 and 40 show that the results of this strong-field approximation are very close to those obtained from the exact equations (22) for all possible values of q except those corresponding to the touching of the Landau level. This is quite natural since touching is an exact one, which cannot be captured by such an approximation. On the contrary, the approximation discussed in the previous paragraph, becomes exact in the nearest neighborhood of the Landau level. The numerical results obtained with the help of the latter approximation completely coincide in these neighborhoods with the exact ones. In turn, such an approximation fails far away from the Landau levels. Therefore, to obtain correct results one has to use different approximations in different regions of the (E, φ) plane. The part of the energy-flux diagram lying between the zeroth and the first Landau levels is shown in Fig. 3.

5. Disordered set of point potentials

We now move on and discuss systems containing a disordered set of point potentials. This disorder can be realized with random positions or strengths of point potentials. Below we will discuss some features of the electron spectrum in such systems: disorder, independent extended states (subsec. 5.1), peculiarities of the DOS (subsec. 5.2), localization properties of a $2D$ system with $1D$ disorder (subsec. 5.3).

5.1. Extended states on the Landau levels

As was mentioned at the end of the previous section, our goal here is to construct a linear combination of Landau functions (15) with a fixed level number n, which vanish on a given set of points (where the point potentials are located). We begin by studying the states on the lowest Landau level E_{0L}. An arbitrary linear combination of Landau functions $L_{0,k}$ has the form

$$\psi_0(\mathbf{r}) = \int_{-\infty}^{\infty} L_{nk}(\mathbf{r}) f_n(k)\, dk. \qquad (27)$$

Evidently, this integral in equation (27) can be written as [17,41,42]

$$\psi_0(x, y) = \exp(-y^2/2l^2) F_0(z), \qquad (28)$$

where $z = x + iy$. The function $F_0(z)$ is defined by

$$F_0(z) = \int_{-\infty}^{\infty} \exp(-ikz - k^2 l^2/2) f_0(k)\, dk.$$

If this integral converges, then it defines an entire function [43] of the complex variable $z = x + iy$. Each entire function is characterized by its order and type. If the modulus of an entire function grows at infinity as $\exp(\tau |z|^r)$, then this function is of order r and of type τ (see exact definition in Ref. 43). We do not, in fact, need the Fourier coefficients $f_0(k)$, since any entire function $F_0(z)$ which, when substituted into Eq. (27), makes $\psi_0(x, y)$ vanish on the sites of the impurities will do. Therefore it is sufficient to construct an entire function $F_0(z)$, which vanishes at all points z_j where the point potentials are located. To this end, let us introduce the Weierstrass product that pertains to the complex sequence $\{z_j\}$ [17,41,42]. For a constant density of impurities, the sequence of complex points $\{z_j\}$ is of genus 2 (i.e., the sum of the inverse squares $1/|z_j|^2$ converges), and hence (assuming there is no scatterer at the origin) the Weierstrass product

$$W(z) = \prod_j \left(1 - \frac{z}{z_j}\right) \exp\left(\frac{z}{z_j} + \frac{z^2}{2z_j^2}\right) \qquad (29)$$

is well defined and defines an entire function of order $r = 2$. If the distribution of point potentials satisfies some uniformity condition (Lindelöf criterion [43]), then $W(z)$ is also of finite type τ. To control the rate of growth of $\psi_0(x, y)$ on the real axis, we multiply $W(z)$ by an exponent e^{-sz^2} with $s > \tau$. We thus obtain an entire function

$$F_0(z) = e^{-sz^2} W(z) \qquad (30)$$

on the order of 2 and of type less equal than $\tau + s$ which falls off with $|x|$ as a Gaussian. Following Eq. (28), it can be easily verified that if the magnetic field is strong enough, i.e.,

$$\frac{1}{2l^2} > s + \tau ,$$

the function $\psi_0(x, y)$ also will fall off as a Gaussian along the imaginary axis. This implies the Gaussian falloff of the wave function at infinity. Hence $\psi_0(x, y)$ [Eq. (28)] with $F_0(z)$ [Eq. (30)] is an acceptable wave function.

To proceed further, we shall consider the simplest site disorder case in which the point potentials with random strengths are located at the sites of a square lattice with constant 1. In this case the Weierstrass product $W(z)$ in Eq. (30) must be replaced by the Weierstrass σ-function

$$\sigma(z) = zW(z) ,$$

where the points z_j in Eq. (29) are all the sites of a square lattice (except the origin). Substituting instead of the Weierstrass σ-function its expression in terms of the Jacobi θ_1-function [16] and using the fact that the type τ of the σ-function equals $\pi/2$ [43], we obtain [17,41,42]

$$\psi_0(\mathrm{r}) = \exp(-y^2/l^2)\theta_1(\pi z) . \qquad (31)$$

Further study shows that if the magnetic field is strong enough $\varphi \geq 1$, then this function represents an exact wave function with eigenenergy E_{oL}. In the case $\varphi = 1$, this wave function is delocalized in both directions, but it decreases with $|y|$ if $\varphi > 1$. The wave function (31) is regular (despite the singular nature of the point potential in its limiting form) and is independent of the strength of the disorder. The lattice symmetry and gauge invariance allow one to construct more general wave functions which are delocalized in both directions for an arbitrary field which satisfy the condition $\varphi \geq 1$ [44]. The simplest example is given by the formula

$$\Psi_{0q}^{(1)}(x, y) = \sum_Y \exp\left(iqY - i\frac{xY}{l^2}\right)\psi_0(x, y - Y) . \qquad (32)$$

This is a quasi-Bloch wave function with respect to y, and an extended function of x. (By the term quasi-Bloch we mean that it has an x-dependent wave number equal to $q - x/l^2$, in which q is a Bloch-type wave number with $-\pi < q \leq \pi$.

This approach can be generalized to an arbitrary index n of the Landau level. In general, the integral in Eq. (27) can be written as follows [44]:

$$\psi_n(x, y) = \exp\left(-\frac{y^2}{l^2}\right)\left\{H_n\left[\frac{y}{l} - il\frac{d}{dz}\right]F_n(z)\right\}\Bigg|_{z=x+iy} ,$$

where in the expansion of the differential operator

$$H_n\left[\frac{y}{l} - il\frac{d}{dz}\right] ,$$

y is assumed to be a constant (independent of z). Unfortunately, here it is not so simple to construct an appropriate analog of the wave function (31) or (32), because of an indirect relation between the point potential positions and the zeros of the entire function $F_n(z)$. For example, it is sufficient to choose $F_n(z)$ in the last equation as

$$F_n(z) = \left[F_0(z)\right]^{n+1} .$$

The resulting function $\psi_n(x, y)$ again vanishes at all points z_j, but it represents a true wave function for essentially higher fields. For the site disorder case (the square lattice considered above), the function $\psi_n(x, y)$ does not grow at infinity when $\varphi \geq n + 1$ (Ref. 44), while the true wave functions exist in a weaker field $\varphi \geq 1$.

5.2. Density of states

The condensation of states on the Landau levels, in a strong field $\varphi > 1$, which was predicted earlier [25], implies that the most singular part of the DOS has the form (19) (recall that we fixed a finite-dimensional average distance d between point potentials). In the periodic case of a (square) lattice $\mathrm{r}_j = (m_j, n_j)$ of identical point potentials this result was rigorously obtained in Ref.29. However, the states on the Landau levels are sensitive to the positions of the point potentials, not to their strength. Therefore, Eq. (19) is also valid for any site disorder model [45]. For the position disorder models with uniformly distributed zero-range scatterers, the prediction (19) was confirmed in Ref. 46 for the lowest Landau level and in Ref. 47 for the higher Landau levels. Note that in the two papers [46,47] the projection on the corresponding Landau level was applied. This enables one to use not the point potential, but the conventional $2D$ Dirac delta function as a model of zero-range scatterer.

Let us consider the DOS between Landau levels, which was studied for different types of site-disorder models (square lattice of point potentials). Consider first the so-called Maryland model, which was proposed for a $1D$ case [48] and then generalized to the multidimensional case [49,50]. In the point potential

version of this model [51] the random binding energy E_{bj} of the point potential placed at the site $\mathbf{r}_j$ is

$$E_{bj} = E_b \tan(\pi \mathbf{p}\mathbf{r}_j - \omega) .$$

Here the constant vector $\mathbf{p}$ is chosen in such a way that $\mathbf{p}\mathbf{r}_j$ is not a rational number for any lattice vector $\mathbf{r}_j$ and a random phase ω is uniformly distributed over the interval $(0, 2\pi)$. For a fixed realization (i.e., for a fixed phase ω) this model is described by an almost periodic Hamiltonian, in which the values of the binding energies could be very large. If $\mathbf{p}\mathbf{r}_j$ is «sufficiently irrational», i.e., if it cannot be well approximated by a rational number for any j, then all intervals between the Landau level are densely filled by nondegenerate localized states, which are sums of scattered waves (2) with amplitudes η_j which satisfy some kind of Lifshits equation [51].

The next two site-disorder models [45] deal with the case in which the binding energy is

$$E_{bj} = E_b \exp(4\pi t_j) .$$

Further details depend on the statistics of the random exponents t_j.

The Lloyd model. All t_j are independent random quantities with the same Lorentz distribution

$$p(t) = \frac{\theta}{\pi(t^2 + \theta^2)} .$$

Here the expression for the average DOS is obtained in a closed form [45]. Its analysis shows that the DOS is an analytic function of energy and differs from zero for all energies lying between Landau levels. In the limiting cases of very small concentration of point potentials (very strong magnetic field) the DOS between Landau levels is proportional to the concentration of point potentials (the inverse magnetic field). (In the latter case the relative position of the energy with respect to the neighboring Landau levels must be fixed).

Gaussian distribution. All t_j are independent random quantities with the same Gaussian distribution

$$p(t) = (2\pi\omega)^{-1/2} \exp(-t^2/2\omega) . \qquad (35)$$

Here the asymptotic expression for the DOS in the same limiting cases as in the previous paragraph was obtained. The resulting DOS has sharp peaks near the levels E_n which are shifted from the Landau levels E_{nL} in the presence of a single point potential. In the nearest neighborhood of the levels E_n these peaks are Gaussian, but far from E_n they become asymmetric (due to the asymmetric position of E_n with respect to the Landau level E_{nL}). More detailed investigation shows that for higher Landau levels such a form of peaks is valid also in the case of correlated, identically distributed exponents t_j [Eq. (33)].

All these results concern the case of true point potential. But we have already mentioned that a single scatterer with a finite radius splits the Landau levels. Disorder leads to the spreading of these sublevels. If the concentration of scatterers is small and magnetic field is strong enough $\varphi \geq 1$, then the sublevels which are most remote from the parent Landau level are spread into the resolved impurity subbands [23]. Each of these subbands accommodates one state (if $-m > n$) or two states (if $|m| < n$) per scatteter (recall that n is the number of the Landau level, and m is an angular momentum). Such an oscillating fine structure of the DOS should manifest itself in the oscillations in the low-temperature specific heat, in the magnetic susceptibility, and perhaps in the transport properties when the flux φ changes by 1 or 2 (Ref. 23).

5. 3. Electron localization in a 1D disordered system

Let us now consider a site-disorder model where the binding energies are identical along the x direction and are random in the y direction, i.e.,

$$E_{bj} = E_{bn} , \quad \{j\} = (m, n) .$$

We assume that the random binding energy E_{bn} takes two possible values: E_1 with the probability $1 - c$ and E_2 with the probability c. In what follows we will be interested in the case of weak disorder so that these two binding energies almost coincide. Due to the presence of disorder and with the specific gauge, we can use the representation (20) for the scattering coefficients, which reduced the initial $2D$ system to a purely $1D$ disordered model, which conserves some features of the initial $2D$ problem: it depends explicitly on q_1 and on the magnetic field.

Consider first the unperturbed ordered system with identical binding energies $E_{bj} = E_b$. Fix some rational flux $\varphi = Q_1/Q$ and Landau level number, and choose some dispersive subband (i.e., some concrete segment on the energy-flux diagram). Each value of energy from this subband corresponds to some specific line in the rectangle $0 < q_1$, $Qq_2 < 2\pi$ on the q-plane. In turn, each point of this rectangle corresponds to the eigenstate with quasi-momentum $\mathbf{q}$. Now switch on a weak disorder. The disorder shifts and smears the subband boundaries and, because of the $1D$ nature of the problem, causes all

states to be localized states (in the y direction). The boundaries which do not coincide with some of the Landau levels are the fluctuation boundaries [52]: they depend on the possible strengths of the point potentials. The states in the vicinities of these boundaries are strongly localized. The states lying deeply in the subband are weakly localized. They can be approximately classified by a quasi-momentum q and by the localization length, $\xi(q)$, $\xi q \ll 1$. The edges which coincide with the Landau levels, are the stable boundaries: they do not depend on the possible strengths of point potentials. The localization length near such a boundary diverges

$$\xi(E) \propto (E_{nL} - E)^{-\nu}$$

with a critical exponent ν.

These statements are now substantiated within a quantitative description. In the strong field limit, when the Larmor radius is of the order of or less than the lattice constant, each of the Green's functions that enters in the representation (8) essentially differs from zero only when the point r is close to the corresponding site (m, n). Therefore

1) *the localization properties of the wave function (8) coincide with the localization properties of the set of scattering coefficients* $\{\eta_{mn}\}$ [Eq. (20)],

and

2) the scattering coefficients ζ_n satisfy the *random* Harper equation

$$\zeta_{n+1} + 2\zeta_n \cos(2\pi n\varphi + q_1) + \zeta_{n-1} = \varepsilon\zeta_n + v z_n \zeta_n .$$

Here ε is defined by Eq. (26), where $\ln(E_L/E_b)$ is replaced by its average value. The random variable z_n takes values $-c$ with the probability $1 - c$, and $1 - c$ with the probability c. The intensity v of the effective «random potential» vz_n is

$$v = \frac{\sqrt{\pi\varphi}}{W_{1/2+a,0}(\pi\varphi)} \frac{\ln(E_1/E_2)}{\Gamma(-a)} .$$

In the weak disorder limit, $|\ln(E_1/E_2)| \ll 1$, each solution of the ordered Harper equation (which satisfies a fixed initial condition) is slightly modified because of the appearance of a slowly increasing multiplier. The corresponding eigenfunctions are slowly attenuated Bloch solutions. The pertinent inverse localization length $\xi_i^{-1}(q)$ (measured in units of d^{-1}) coincides with the doubled Lyapunov exponent [52]. In the weak scattering approximation it was calculated exactly [40]:

$$\xi_i^{-1}(q) = \frac{1}{Q}\sum_{p=0}^{Q-1}\left\{(1-c)\left(1 + \frac{Z_i(p, q)}{v^2c^2}\right)^{-1} + c\left(1 + \frac{Z_i(p, q)}{v^2(1-c)^2}\right)^{-1}\right\},$$

$$Z_i(p, q) = 4\,\mathrm{Im}^2\left[\frac{c_i(p+1, q)}{c_i(p, q)}\exp(iq_2)\right].$$

The weak scattering regime corresponds to a very large localization length. We define the condition of the weak scattering as

$$\min_p Z_i(p, q)/v^2 > N \gg 1 . \qquad (34)$$

This inequality determines a domain inside the square $0 < q_1, Qq_2 < 2\pi$, such that for each q in this domain, the localization length (measured in the lattice constant) $\xi_i(q)$ will be larger than N. The dispersion relation $\varepsilon_i(q)$ maps this domain onto an energy interval on the ε axis and, after inversion using Eq. (26), onto a set of intervals on the a axis. Each of these intervals corresponds to a fixed value of the rational flux and a fixed Landau level (i.e., it is located below this level but above the previous one). The set of all these intervals vs. flux φ forms a Hofstadter-type butterfly for the localization length. The points of this diagram correspond to those eigenstates which can be classified as slowly attenuated Bloch states. A part of this butterfly is shown in Fig. 4. Note that some subbands, which are present in Fig. 3, are completely absent in Fig. 4. This means that all states

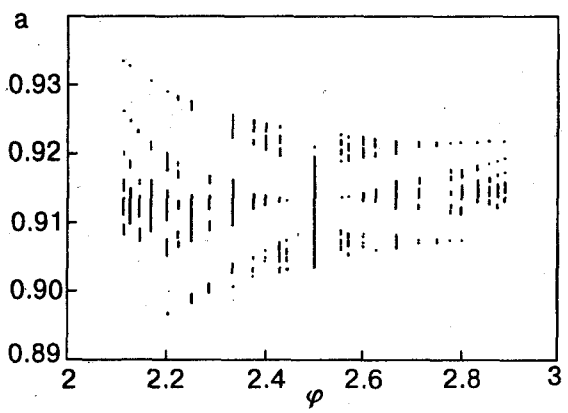

Fig. 4. The energy-flux diagram for the delocalized (with the chosen accuracy) states between the first two Landau levels $0 < a \le 1$. The rational fluxes $2 < \varphi < 3$ with denominators up to 9 are used. Some subbands, which are shown in Fig. 3b, are absent here: the corresponding states are localized.

of these subbands do not satisfy the weak scattering condition (34), i.e., they are localized.

Let us now consider the localization of states near the stable boundaries. Different types of touching scenarios (possible rational fluxes and possible points in the q rectangle) were analyzed in Ref. 36. It was shown [19] that in the case of short, correlated $1D$ disorder the localization length diverges with the critical exponent $\nu = 1$. Exact expressions for the localization length near the Landau levels, which were obtained for the model considered in this subsection, confirm this result [40]. In particular, for the touching of the first Landau level for $\varphi = 2$ the localization length has the form

$$\xi \propto \frac{(\langle \ln (E_L/E_b) \rangle)^2}{(1-a)c(1-c)\ln^2(E_1/E_2)}, \quad F = \text{const},$$

and diverges as $a \to 1$ (the energy tends to E_{1L}) and when the disorder vanishes ($c \to 0, 1$, or $E_2 - E_1 \to 0$).

6. Mesoscopic systems with a point potential

The electron dynamics in the presence of a single short-range impurity in a magnetic field is related to a number of physical problems (magnetotransport phenomena in bulk semiconductors, the quantum Hall effect, the conductance of a microconstriction, tunneling through a quantum dot, etc.). Because of the short-range character of the impurity, it is natural to model it by a point potential. Such a simplification often enables one either to obtain an exact solution of the problem, or at least to obtain a solution in a closed form.

6.1. Point potential model of a quantum dot

Let us first discuss the problem of electron transmission through a quantum dot. On the basis of the model proposed and studied in Ref. 53 the dot itself is described by a $2D$ Hamiltonian H_2 which accounts for a magnetic field (12) and a confinement potential $V_c = E_c^2 r^2/4$, $r^2 = x^2 + y^2$, with some characteristic confinement energy E_c. One-dimensional channel is described by a $1D$ free Hamiltonian $H_1 = -d^2/dz^2$. The unperturbed Hamiltonian of the noninteracting system dot-channel is a direct sum $H_2 \oplus H_1$:

$$H = \begin{pmatrix} H_2 & 0 \\ 0 & H_1 \end{pmatrix}.$$

The unperturbed wave function is a two-component vector

$$\Psi_0(\mathrm{r}, z) = \begin{pmatrix} \psi_2(\mathrm{r}) \\ \psi_1(z) \end{pmatrix}.$$

The perturbation V located at the origin describes the zero-range interaction between the dot and the channel

$$V = \begin{pmatrix} T_2^{-1}(E) & \beta \\ \beta & T_1^{-1}(E) \end{pmatrix}.$$

Here $T_2(E)$ is the $2D$ point potential scattering amplitude (18), where E_L is replaced by $(E_L^2 + E_c^2)^{1/2}$, and $T_1(E)$ is the $1D$ point potential scattering amplitude. The $1D$ point potential, however, is the usual repulsive Dirac delta function $k_0\delta(z)$, $k_0 > 0$ (repulsion models the potential barriers that separate the real quantum dot from the leads, which are modelled here by the channel). Therefore, in accordance with the general rule (9), the scattering amplitude $T_1(E)$ is

$$\frac{1}{T_1(E)} = \frac{i}{2\sqrt{E}} + \frac{1}{k_0}.$$

The constant β describes mixing between the dot and the channel. The case $\beta = 0$ corresponds to two independent subsystems — the dot, which contains the $2D$ point potential, and the channel with the $1D$ point potential. The nonzero value of the parameter β provides a mixing of the two subsystems and thus the dot actually influences the electron transmission along the channel.

The transmission coefficient $T(E)$, which is calculated by a standard method, has the form

$$\frac{1}{T(E)} = 1 + \frac{1}{4E(k_0^{-1} - \beta^2 T_2(E))^2}.$$

At energies $E_n^{\min}$ satisfying the resonant condition $k_0^{-1} = \beta^2 T_2(E)$ the channel is closed: the transmission vanishes. On the other hand, at energies $E_n^{\max}$, which coincide with the poles of the scattering amplitude $T_2(E)$, the transmission is perfect $T(E) = 1$. Note that these energies are simply Landau levels in an effective magnetic field which is shifted by the $2D$ point potential. In this effective field the interlevel distance is not E_L but $(E_L^2 + E_c^2)^{1/2}$. The channel conductance calculated by the Landauer formula has sharp peaks at the energies $E_n^{\max}$.

A similar idea was used for studying the electron spectrum in a periodic array of quantum dots subject to a magnetic field [54]. Consider a lattice of points R. Let H_{R} be the potential of the dot located at the point R [i.e., with the confinement potential

$V_c(\mathbf{r}-\mathbf{R})$]. The array of dots is described by the Hamiltonian $H = H_0 + U$. Here the unperturbed Hamiltonian $H_0 = \oplus H_{\mathbf{R}}$ is the direct sum of the single-dot Hamiltonians $H_{\mathbf{R}}$ over all sites of the lattice $\{\mathbf{R}\}$. This means that the unperturbed wave function $\Psi_0(\mathbf{r})$ is the vector column

$$\Psi_0(\mathbf{r}) = \begin{pmatrix} \cdots \\ \psi_{\mathbf{R}}(\mathbf{r}) \\ \cdots \end{pmatrix},$$

and each one of its components $\psi_{\mathbf{R}}(\mathbf{r})$ satisfies the Schrödinger equation with the Hamiltonian $H_{\mathbf{R}}$. The mixing of the states in the different dots is realized by adding the perturbation U which in coordinate representation has the form

$$U^{\mathbf{r},\mathbf{r}'}_{\mathbf{R},\mathbf{R}'} = t_{\mathbf{R},\mathbf{R}'} u(\mathbf{r}-\mathbf{R}) u^*(\mathbf{r}'-\mathbf{R}') .$$

Here the matrix t has the lattice symmetry

$$t_{\mathbf{R}+\mathbf{R}_0,\mathbf{R}'+\mathbf{R}_0} = t_{\mathbf{R},\mathbf{R}'} ,$$

which contains only nondiagonal elements, and $u(\mathbf{r})$ is a localized function.

The perturbation U in this model is not a degenerate perturbation. But like the degenerate perturbation (1), which acts in the subspace formed by the set of states $\{|v_j\rangle\}$, the perturbation U acts in the subspace formed by the set of states $\{|u_{\mathbf{R}}\rangle\}$. The spectrum of a single dot is rather rarified $E_{n,m} = (n + 1/2)E_L + (m + 1/2)E_c$ with integers $n, m \geq 0$. As a result, the set of the perturbed eigenstates $\Psi(\mathbf{r})$ with eigenenergies lying outside the initial spectrum is sufficiently rich and can be found from the set of Lifshits equations (2)–(4). The index j must be replaced by $\mathbf{R}$,

$$\eta_{\mathbf{R}} = \int u(\mathbf{r}-\mathbf{R})\psi_{\mathbf{R}}(\mathbf{r})d\mathbf{R} ,$$

$t_{\mathbf{R},\mathbf{R}'}$ stanfs for G_{ij} and

$$\frac{1}{T_{\mathbf{R}}(E)} = \int u^*(\mathbf{r}-\mathbf{R})G_{\mathbf{R}}(\mathbf{r},\mathbf{r}')\, u(\mathbf{r}'-\mathbf{R})d\mathbf{r}d\mathbf{r}' .$$

These formulas are valid for an arbitrary localized function $u(\mathbf{r})$. A further simplification can be made by using the point potential as $u(\mathbf{r})$. In this case and for a rational flux per plaque an explicit equation for the Bloch-type spectrum can be obtained [54].

An explicit spectrum of the model can be obtained because of its internal simplicity, which is attributable to the use of degenerate-like perturbation, on the one hand, and to the construction of the unperturbed Hamiltonian as a direct product, on the other. Due to the latter fact, the interaction between different dots is described independently from the dynamics of a single dot. In a sense, the model considered is an ordered analog of the famous Anderson model [55], while the case of the point potential lattice, which is considered in Sec. 4, is an ordered analog of the much more complicated Lifshits model [56,57].

6.2. Point scatterer in a microconstriction

Here we consider electron scattering by a single point potential in a magnetic field and some additional field. We start from the case of the crossed magnetic and electric fields. The modification of the spectrum of the system due to the presence of an impurity is defined by the poles of the scattering amplitude $T(E)$ [Eq. (9)], where $G(\mathbf{r},\mathbf{r}';E)$ now stands for the Green's function in the presence of the two fields. As a result, in contrast with the case of zero electric field, where exactly one bound state exists below each Landau level (see Fig. 1), there are now n novel, nondegenerate, quasi bound states with energies close to the nth Landau level [58].

The next group of problems is related to electron transmission through a saddle-point potential

$$V(x, y) = -\frac{x^2}{\kappa^4} + y^2 \qquad (35)$$

[here κ is the asymmetry parameter] in the presence of a short-range impurity and magnetic field. Without impurity and a magnetic field the electron wave functions are saddle-point potential waveguide modes,

$$\Psi^{\pm}_{E,n}(x, y) = \Phi_n(y)E(-\varepsilon_n, \pm\frac{x\sqrt{2}}{\kappa}) .$$

Here $\Phi_n(y)$ is a harmonic oscillator wave function, which corresponds to the y-dependent part of the potential [Eq. (35)] and energy $E_n = 2n + 1$. The x-dependent factor is the Weber-function [16] $E(-\varepsilon, \xi)$, and $\varepsilon_n = \kappa^2(E - 2n - 1)/2$. The mode $\Psi^{+}_{E,n}(x, y)$ describes the initial wave which comes from the left ($x = -\infty$) and which contains the reflected and transitted waves with the corresponding transmission coefficient $T_n = (1 + \exp(-2\pi\varepsilon_n))^{-1}$ (this result was initially obtained in the WKB approximation in Ref. 59) and reflectivity $R_n = (1 + \exp(2\pi\varepsilon_n))^{-1}$. The energy E_n is the threshold for mode n. When $E < E_n$, the mode n is mainly reflected, while in the opposite case this mode is mainly transmitted. The crossover from reflection to transmission turns out to

be in the energy band $|E - E_n| \leq 2\kappa^{-2}$. The saddle-point potential (35) models a ballistic microjunction, to which the two-terminal conductance in terms of the mode transmission coefficients is given by [60]

$$G_0 = \sum_{n=0}^{\infty} T(\varepsilon_n) .$$

The presence of a strong point potential, which is located in the central section of the constriction (i.e., near the origin), modifies the spectrum. Exactly one bound state appears below the threshold E_n of each transverse quantization mode. The bound state with the energy far from the threshold decays mostly into the continuous spectra of the above-barrier modes (mode mixing). In this case the point potential correction to the conductance is due to the resonant reflection of the above-barrier modes, which have a downward dip $\Delta G = -1$ on the plot of G vs. the Fermi energy E. The bound state close to threshold decays mostly into the continuous spectrum of the threshold mode (tunneling). Here the point potential correction to the conductance is due to resonant transmission of the threshold mode which is seen as a resonant tunneling peak [61], $\Delta G = +1$.

Let us consider the electron scattering by an isolated point potential in a magnetic field. In the limiting case of a narrow constriction $\alpha \geq 1$ an explicit expression for the transmission coefficient as a function of the point potential position and magnetic field could be obtained [62]. Its analysis shows that the transmission coefficient has a sharp peak (the Breit-Wigner resonance) as a function of energy, and that it oscillates as a function of the point potential position. [62]. One more aspect of electron transmission through a symmetric [$\kappa = 1$] saddle-point potential in a magnetic field is related to the levitation of delocalized states in the quantum Hall effect regime. In the network model of the quantum Hall effect the delocalization of states results from the tunneling of an electron through the saddle points of a smooth random potential, which are connected by equipotential lines [63]. The mixing of Landau levels changes the transmission coefficient of a saddle point, on the average, in such a way that it becomes smaller than 1/2 for the energy at the Landau level [64]. This means that to achieve the 1/2 average transmittivity, the energy should be shifted upwards, which is equivalent to levitation. However, in a smooth potential the Landau level mixing is generally weak since it is associated with a large momentum transfer. Short-range potentials are much more effective in this respect. The average transmission through a saddle point in a strong magnetic field in the presence of a random short-range scatterer was studied in Ref. 65. It was shown that a small portion of the short-range random impurities located in the vicinity of the saddle points reduce the transmission at a given energy even if the mixing of the Landau levels by a smooth potential is disregarded. The resulting upward shift of the energy position of the delocalized state increases with decreasing magnetic field as B^{-4} [65].

Conclusions

In summary, we presented a brief review of the results concerning the electron dynamics in a magnetic field and in a field of a point or a short-range potential. We should emphasize that the possibilities contained in the point-potential model for the physics of $2D$ systems in a magnetic field are not exhausted yet. This is a unique model which makes it possible to obtain exact results starting from first principles. In Sec. 4 we presented the results of a calculation of the electron spectrum and wave functions using solely the zero-range property of the scattering potential. It is very intriguing to obtain some transport characteristics, e.g., the Hall conductivity or ac conductivity of the $2D$ periodic system of point potentials. Unexpected application of point potentials is attributable to the superconductivity. The results of Subsec. 5.1 make it possible construct explicitly an Abrikosov trial function for an arbitrary position of the vortices in type-II superconductors (this idea was communicated to us by the late Arkadii Aronov). Investigation in this direction is now in progress.

We shall conclude this article with some non-academic words. We believe that the best way to honor the memory of an outstanding scientist is to show that his ideas are alive, that they are at work, and that they are being developed. We attempted to demonstrate this point on a single subject of local perturbations which has been proposed by Ilya Mikhailovich back in 1947. The fact that the topics related to this subject, such as electron localization in a magnetic field, the quantum Hall effect, the physics of quantum dots, and others, are at the forefront of contemporary solid state physics speaks for itself. All the authors, two of whom (M. Ya. Azbel' and S. A. Gredeskul) are privileged to be former pupils of Ilya Mikhailovich Lifshits, would like to express their gratitude and to dedicate this article with love and respect, to the memory of this brilliant scientist and person.

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